Exercise Sheet 1

June 8, 2026

Exercise 1 Prove that, for any $\alpha \in \bigwedge^h(V)$, $\beta \in \bigwedge^k(V)$, it holds $\beta \wedge \alpha = (-1)^{hk}\alpha \wedge \beta$

Exercise 2 Let $v \in \bigwedge_m(V)$, $\omega \in \bigwedge^n(V)$ and, for $i = 1, 2$, $v_i \in \bigwedge_{m_i}(V)$, $\omega_i \in \bigwedge^{n_i}(V)$. Show that

- $v \llcorner (w_1 \wedge w_2) = (v \llcorner w_1) \llcorner w_2$,
- $(v_1 \wedge v_2) \lrcorner w = v_1 \lrcorner (v_2 \lrcorner w)$.

Exercise 3 Let $\omega_1, \ldots, \omega_k \in \bigwedge^1(V)$, and $v_i, \ldots, v_k \in \bigwedge_1(V)$. Then,

$$(\omega_1 \wedge \ldots \wedge \omega_k)(v_1, \ldots, v_k) = \det(\omega_i(v_j))_{i,j}.$$

What is the geometrical meaning of this quantity?

Exercise 4 Recall $v \in \bigwedge_k(V)$ is *simple* if there exist $v_1, \ldots, v_k \in V$ such that $v = v_1 \wedge \ldots \wedge v_k$.

1. Exhibit a non simple vector in $\mathbb{R}^n$
2. Show that any $v \in \bigwedge_{n-1}(\mathbb{R}^n)$ is simple. *Hint: use Hodge duality (see exercise 6)*

Exercise 5 Let $\omega \in C_c^\infty(\mathbb{R}^n; \bigwedge^k(\mathbb{R}^n))$, and write $\omega(x) = \sum_{i \in I(k,n)} \omega_i(x) dx_i$. Define the exterior derivative $d\colon C_c^\infty(\mathbb{R}^n; \bigwedge^k(\mathbb{R}^n)) \to C_c^\infty(\mathbb{R}^n; \bigwedge^{k+1}(\mathbb{R}^n))$ be defined by

$$d\omega(x) = \sum_{i \in I(k,n)} \sum_{h=1}^{n} \frac{\partial \omega_i}{\partial x_h}(x) dx_h \wedge dx_i.$$

Let $\omega' \in C_c^\infty(\mathbb{R}^n; \bigwedge^h(\mathbb{R}^n))$. Show that

1. $d^2 = 0$
2. $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^h \omega \wedge d\omega'$

Exercise 6 Let $* : \bigwedge^k(\mathbb{R}^n) \to \bigwedge_{n-k}(\mathbb{R}^n)$ be the Hodge duality operator defined by

$$*\,\omega = e_1 \wedge \ldots \wedge e_n \llcorner \omega$$

and let, with an abuse of notation, also $* : \bigwedge_h(\mathbb{R}^n) \to \bigwedge^{n-h}(\mathbb{R}^n)$ be defined by

$$*\,v = v \lrcorner dx_1 \wedge \ldots \wedge dx_n.$$

Recall that the Hodge operator is an isomorphism. Prove the following identities for any $v \in C_c^\infty(\mathbb{R}^n, \bigwedge_k(\mathbb{R}^n))$ and any $\alpha \in C_c^\infty(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$, with $h \leq k \leq n$:

1. $** = \mathrm{id}$
2. $*(v \llcorner \alpha) = *(v) \wedge \alpha$
3. $*(d(*v)) = (-1)^{n-k} \operatorname{div} v$
4. $\operatorname{div}(v \llcorner \alpha) = (-1)^k (\operatorname{div} v \llcorner \alpha + v \llcorner d\alpha)$

where $\operatorname{div} v = \sum_{i \in I_{n,k}} \sum_{h=1}^{n} \frac{\partial v_i}{\partial x_h}(x) e_i \lrcorner dx_h$