Lecture 1

Content: Multilinear algebra, simple vectors, external derivati

Ex 1 Prove that $\forall\, \alpha \in \Lambda^h(V)$, $\beta \in \Lambda^k(V)$, $\beta\wedge\alpha = (-1)^{hk}\alpha\wedge\beta$

Sol

$$\alpha\wedge\beta\,(v_1,\dots,v_k) = \frac{1}{k!\,h!}\sum_{\sigma\in S_{h+k}} \operatorname{sgn}(\sigma)\, \alpha(v_{\sigma(1)},\dots,v_{\sigma(h)})\,\beta(v_{\sigma(h+1)},\dots,v_{\sigma(h+k)}).$$

Consider the permutation σ' s.t.

$\sigma'(k+1) = 1$

$\vdots$

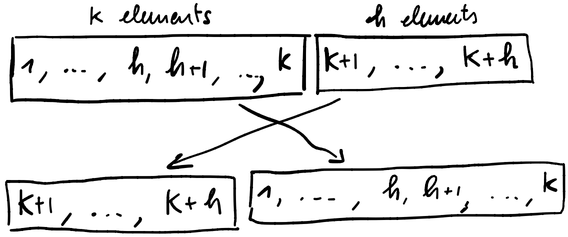
$\sigma'(k+h) = h$

$\sigma'(1) = h$

$\vdots$

$\sigma'(k) = h+k$

σ :



We have $\operatorname{sgn}\sigma = (-1)^{hk}$.

Moreover,

$$\beta\wedge\alpha(v_1,\dots,v_{h+k}) = \frac{1}{h!\,k!}\sum_{\sigma\in S_{h+k}} \operatorname{sgn}(\sigma)\,\alpha(v_{\sigma(k+1)},\dots,v_{\sigma(k+h)})\,\beta(v_{\sigma(1)},\dots,v_{\sigma(k)})$$

$$= \frac{1}{k!\,h!}\sum_{\sigma\in S_{h+k}} \operatorname{sgn}(\sigma\circ\sigma')\,\alpha(v_{\sigma(\sigma'(k+1))},\dots,v_{\sigma(\sigma'(k+h))})\,\beta(v_{\sigma(\sigma'(1))},\dots,v_{\sigma(\sigma'(k))})$$

$$= \frac{1}{k!\,h!}\sum_{\sigma\in S_{h+k}} (-1)^{hk}\operatorname{sgn}(\sigma)\,\alpha(v_{\sigma(1)},\dots,v_{\sigma(h)})\,\beta(v_{\sigma(h+1)},\dots,v_{\sigma(h+k)})$$

$$= (-1)^{hk}\,\alpha\wedge\beta\,(v_1,\dots,v_{h+k})$$

Rmk If h is odd, $\alpha\wedge\alpha = (-1)^{h^2}\alpha\wedge\alpha = -\alpha\wedge\alpha$.

Hence $\alpha \wedge \alpha = 0 \quad \forall \alpha \in \Lambda^{k}(V)$.

Ex 2 Let $v \in \Lambda_m(V)$, $\omega \in \Lambda^n(V)$ and, for $i = 1,2$
$v_i \in \Lambda_{m_i}(V)$, $\omega_i \in \Lambda^{n_i}(V)$.

Prove that

(a) $$v \llcorner (\omega_1 \wedge \omega_2) = (v \llcorner \omega_1) \llcorner \omega_2$$

(b) $$(v_1 \wedge v_2) \lrcorner \omega = v_1 \lrcorner (v_2 \lrcorner \omega)$$

Sol

(a) Take $\alpha \in \Lambda^{m-n_1-n_2}(V)$. Then,

$$\begin{aligned}
\langle v \llcorner (\omega_1 \wedge \omega_2), \alpha \rangle &= \langle v, (\omega_1 \wedge \omega_2) \wedge \alpha \rangle \\
&= \langle v, \omega_1 \wedge (\omega_2 \wedge \alpha) \rangle \\
&= \langle v \llcorner \omega_1, \omega_2 \wedge \alpha \rangle \\
&= \langle (v \llcorner \omega_1) \llcorner \omega_2, \alpha \rangle .
\end{aligned}$$

(b) Similar: $\forall u \in \Lambda_{n-m_1-m_2}(V)$,

$$\begin{aligned}
\langle (v_1 \wedge v_2) \lrcorner \omega, u \rangle &= \langle \omega, u \wedge (v_1 \wedge v_2) \rangle \\
&= \langle \omega, (u \wedge v_1) \wedge v_2 \rangle \\
&= \langle v_2 \lrcorner \omega, u \wedge v_1 \rangle \\
&= \langle v_1 \lrcorner (v_2 \lrcorner \omega), u \rangle .
\end{aligned}$$

Ex 3 Let $\omega_1, \dots, \omega_k \in \Lambda^1(V)$, $v_1, \dots, v_k \in \Lambda_1(V)$. Then,

$$\omega_1 \wedge \dots \wedge \omega_k \, (v_1, \dots, v_k) = \det \big(\omega_i(v_j)\big)_{i,j}$$

Sol

We prove the claim by induction on k.

For $k=1$ it is trivial.

Let it be true for $k \in \mathbb{N}$, we prove it for $k+1$:

$$\omega_1 \wedge \dots \wedge \omega_k \wedge \omega_{k+1} \, (v_1, \dots, v_k, v_{k+1})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \omega_1 \wedge \dots \wedge \omega_k \, (v_{\sigma(1)}, \dots, v_{\sigma(k)}) \, \omega_{k+1} \, (v_{\sigma(k+1)}) \operatorname{sgn} \sigma$$

missing index

$$= \sum_{m=1}^{k+1} \omega_{k+1} (v_m) (-1)^{m+1} \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \; \omega_1 \wedge \dots \wedge \omega_k (v_{\sigma(1)}, \dots, \hat{v}_m, \dots, v_{\sigma(k)})$$

$$= \sum_{m=1}^{k+1} \omega_{k+1} (v_m) (-1)^{m+1} \frac{1}{k!} \sum_{\sigma \in S_k} \omega_1 \wedge \dots \wedge \omega_k (v_1, \dots, \hat{v}_m, \dots, v_k)$$

$$= \sum_{m=1}^{k+1} \omega_{k+1} (v_m) (-1)^{m+1} \frac{1}{k!} \sum_{\sigma \in S_k} \det \big(\omega_i(v_j)\big)_{\substack{i=1,\dots,k \\ j=1,\dots,k+1,\, j \neq m}}$$

$$= \sum_{m=1}^{k+1} \omega_{k+1} (v_m) (-1)^{m+1} \det \big(\omega_i(v_j)\big)_{\substack{\text{no row } k+1 \\ \text{no column } m}}$$

$$= \det \big(\omega_i(v_j)\big)_{i,j}$$

$$\begin{matrix} \omega_1(v_1) & \omega_1(v_2) & \cdots & \omega_1(v_{k+1}) \\ \vdots & & & \\ \omega_{k+1}(v_1) & \omega_{k+1}(v_2) & \cdots & \omega_{k+1}(v_{k+1}) \end{matrix}$$

Rmk If $V = \mathbb{R}^n$, then

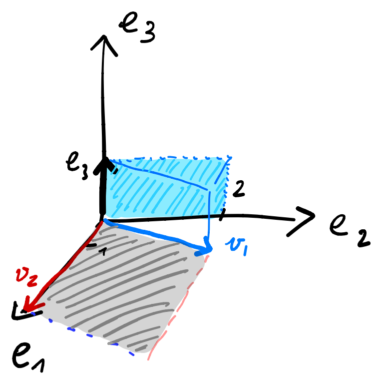
$$\omega_i(v_j) = \langle \omega_i, v_j \rangle.$$

Therefore $\omega_1 \wedge \dots \wedge \omega_k \, (v_1, \dots, v_k) = \det (\langle \omega_i, v_j \rangle)$

and $e_1 \wedge \dots \wedge e_k (v_1, \dots, v_k) = \det (v_j^i)$

= (signed) k-volume of the projection of parallelepiped spanned by $v_1, \dots, v_k$ on the k-plane associated with $e_1 \wedge \dots \wedge e_k$

Example



$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. It holds

bad orient of v_1, v_2 wrt e_1, e_2

$$e_1 \wedge e_2 (v_1, v_2) = \det (v_j^i)_{i,j=1,2} = \det \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = -6$$

$$e_2 \wedge e_3 (v_1, v_2) = 0$$

$$e_2 \wedge e_3 (v_1, e_3) = \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

Ex 4 $v \in \Lambda_k(V)$ is simple if $\exists v_1, \dots, v_k \in V$ s.t. $v = v_1 \wedge \dots \wedge v_k$.

(a) Exhibit a non simple vector in $\mathbb{R}^n$

(b) Show that any $v \in \Lambda_{n-1}(\mathbb{R}^n)$ is simple

Sol

a) Consider $v = e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2(\mathbb{R}^4)$.
If v was simple, then we could find $v_1, v_2 \in \mathbb{R}^4$ s.t. $v = v_1 \wedge v_2$. But then

$$v \wedge v = (v_1 \wedge v_2) \wedge (v_1 \wedge v_2) = 0$$

$$\shortparallel$$

$$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)$$

$$= e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_3 \wedge e_4 \wedge e_1 \wedge e_2$$

$$= 2 e_1 \wedge e_2 \wedge e_3 \wedge e_4 .$$

b) Let $v \in \Lambda_{n-1}$. Then, $v = * v_0$ for some $v_0 \in \mathbb{R}^n$, since $* : \Lambda_{n-1} \to \Lambda^1 \simeq \mathbb{R}^n$ is an isomorphism.
Complete v_0 to a basis $v_0, v_1, \dots, v_{n-1}$ of $\mathbb{R}^n$.

Let $\alpha \in \Lambda^{n-1}(\mathbb{R}^n)$, and write

$$\alpha = \sum_{i=0}^{n-1} a_i \, \hat{v}_i . \leftarrow v_0 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_n$$

Then, $\langle \alpha, v \rangle = \langle \alpha, * v_0 \rangle = \langle v_0 \wedge \alpha, e_1 \wedge \dots \wedge e_n \rangle$

$$= a_0 \det(\langle v_i, e_j \rangle),$$

hence $\quad v = \dfrac{\det(\langle v_i, e_j\rangle)}{|v_1\wedge\ldots\wedge v_{n-1}|^2}\, v_1\wedge\ldots\wedge v_{n-1}.$

Ex 5 Show that $d^2 = 0$.

Sol for any $\omega\in C_c^\infty(\mathbb{R}^n; \Lambda^k(\mathbb{R}^n))$, $\omega = \sum_{i\in I_{n,k}} \omega_i(x)\, dx_i$,

one has
$$d\omega(x) = \sum_{i\in I_{n,k}} \sum_{h=1}^{n} \frac{\partial \omega_i(x)}{\partial x_h}\, dx_h \wedge dx_i.$$

Therefore
$$dd\omega(x) = \sum_{i\in I_{n,k}} \sum_{h=1}^{n} \sum_{m=1}^{n} \frac{\partial^2 \omega_i(x)}{\partial x_m\, \partial x_h}\, dx_m \wedge dx_h \wedge dx_i$$
$$= -\sum_{i\in I_{n,k}} \sum_{h=1}^{n} \sum_{m=1}^{n} \frac{\partial^2 \omega_i(x)}{\partial x_m\, \partial x_h}\, dx_h \wedge dx_m \wedge dx_i$$

Change $m \leftrightarrow i$ & $\dfrac{\partial^2}{\partial x_i\, \partial x_m} = \dfrac{\partial^2}{\partial x_m\, \partial x_i}$
$$= \sum_{i\in I_{n,k}} \sum_{m=1}^{n} \sum_{h=1}^{n} \frac{\partial^2 \omega_i(x)}{\partial x_m\, \partial x_h}\, dx_m \wedge dx_h \wedge dx_i$$

Ex 7 (Hodge duality and ext derivative)

Prove the following identities for any $v \in C_c^\infty(\mathbb{R}^n, \Lambda_k(\mathbb{R}^n))$ and any $\alpha \in C_c^\infty(\mathbb{R}^n, \Lambda^h(\mathbb{R}^n))$, $h \leq k \leq n$.

a) $** = \mathrm{id}$

b) $*(v \llcorner \alpha) = *v \wedge \alpha$

c) $*(d(*v)) = (-1)^{n-k} \operatorname{div} v$

If $v = \sum_{i \in I_{n,k}} v_i(x)\, e_i$,

$\operatorname{div} v := \sum_{i \in I_{n,k}} \sum_{h=1}^{n} \frac{\partial v_i}{\partial x_h} e_i \llcorner dx_h \in C_c^\infty(\mathbb{R}^n, \Lambda_{k-1}(\mathbb{R}^n))$

Sol

a) Consider e_i for $i \in I(n,k)$: for any $e_j \in \Lambda_{n-k}(\mathbb{R}^n)$,

$$\langle * e_i, e_j \rangle = \langle e_i \lrcorner dx_1 \wedge \dots \wedge dx_n, e_j \rangle$$

$$= \langle dx_1 \wedge \dots \wedge dx_n, e_j \wedge e_i \rangle$$

$$= \begin{cases} 0 & \text{otherwise} \\ \operatorname{sgn}(j,i) & \text{if } \{j,i\} = \{1,\dots,n\} \end{cases}$$

$$= \operatorname{sgn}(i^c, i)\, e_{i^c}$$

element of $I(n,n-k)$ that completes $i \in I(n,k)$

permutation ordering $(\underline{i}^c, \underline{i}) \to (1,\dots,n)$

Similarly

$$\langle dx_j, * dx_i \rangle = \langle dx_j, e_1 \wedge \dots \wedge e_n \llcorner dx_i \rangle$$

$$= \langle dx_i \wedge dx_j, e_1 \wedge \dots \wedge e_n \rangle$$

$$= \operatorname{sgn}(i, i^c)\, dx_{i^c}$$

Therefore $**e_i = e_i$ and $**dx_i = dx_i$.
By linearity one concludes.

b) $*((*v)\wedge\alpha) = e_1\wedge\dots\wedge e_n \,\llcorner\, ((*v)\wedge\alpha)$ $\left\{\begin{array}{l} *(\beta\wedge\alpha) \\ = (*\beta)\llcorner\alpha \end{array}\right.$

$\overset{\text{Ex 2}}{=} (e \,\llcorner\, (*v))\llcorner\alpha = (**v)\llcorner\alpha$

$\overset{a)}{=} v\llcorner\alpha.$

So $*((*v)\wedge\alpha) = v\llcorner\alpha \overset{(a)}{=} *(*(v\llcorner\alpha))$.
Since $*$ is an isomorphism, we deduce the claim.

c) $*(d(*v)) = *\left(d\left(\sum_i v_i(x)\, *e_i\right)\right)$

$= *\left(\sum_i\sum_{h=1}^n \frac{\partial v_i}{\partial x_h}(x)\, dx_h\wedge *e_i\right)$

$\overset{\text{Ex 1}}{=} (-1)^{(n-k)\cdot 1} *\left(\sum_i\sum_{h=1}^n \frac{\partial v_i}{\partial x_h}(x)\, *e_i\wedge dx_h\right)$

$= (-1)^{(n-k)\cdot 1} \sum_i\sum_{h=1}^n \frac{\partial v_i}{\partial x_h}(x) *(*e_i\wedge dx_h)$

$\overset{*(\beta\wedge\alpha) = *\beta\llcorner\alpha}{=} (-1)^{n-k} \sum_i\sum_{h=1}^n \frac{\partial v_i}{\partial x_h}(x)\; e_i\llcorner dx_h$

$= (-1)^{n-k}\,\mathrm{div}\, v$

Ex 8 Let $\omega \in C_c^\infty(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$, $\omega' \in C_c^\infty(\mathbb{R}^n, \Lambda^h(\mathbb{R}^n))$, $v \in C_c^\infty(\mathbb{R}^n, \Lambda_h(\mathbb{R}^n))$

Prove that

a) $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'$

b) $\operatorname{div}(v \llcorner \omega) = (-1)^h \left(\operatorname{div} v \llcorner \omega + v \llcorner d\omega\right)$

Sol

a)
$$d(\omega \wedge \omega') = d\left(\left(\sum \omega_i \, dx_i\right) \wedge \left(\sum \omega'_j \, dx_j\right)\right)$$
$$= d\left(\sum\sum \omega_i \omega'_j \, dx_i \wedge dx_j\right)$$
$$= \sum\sum\sum \frac{\partial(\omega_i \omega'_j)}{\partial x_m} dx_m \wedge \overbrace{dx_i}^{k \text{ form}} \wedge dx_j$$
$$= \left(\sum\sum \frac{\partial \omega_i}{\partial x_m} dx_m \wedge dx_i\right) \wedge \left(\sum \omega'_j \, dx_j\right)$$
$$+ (-1)^k \sum\sum\sum \omega_i \frac{\partial \omega'_j}{\partial x_m} dx_i \wedge dx_m \wedge dx_j$$
$$= d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'.$$

b)
$$\operatorname{div}(v \llcorner \omega) = (-1)^{n-h+k} * \left(d\left(*(v \llcorner \omega)\right)\right)$$
$$= (-1)^{n-h+k} * \left(d\left((*v) \wedge \omega\right)\right)$$
$$= (-1)^{n-h+k} * \left(d(*v) \wedge \omega + (-1)^{n-k} *v \wedge d\omega\right)$$

$$= (-1)^{n-h+k}\left[* d(*v) \lrcorner\, \omega + (-1)^{n-k} * * v \lrcorner\, d\omega \right]$$

$*(\alpha \wedge \beta)$

$= *\alpha \lrcorner\, \beta$

$$= (-1)^{n-h+k}\left[(-1)^{n-k} \operatorname{div} v \lrcorner\, \omega + (-1)^{n-k} v \lrcorner\, d\omega \right]$$

$$= (-1)^{2n-h}\left[\operatorname{div} \lrcorner\, \omega + v \lrcorner\, d\omega \right]$$

$$= (-1)^{h}\left(\operatorname{div} v \lrcorner\, \omega + v \lrcorner\, d\omega \right)$$