Exercise Sheet 2

Exercise 1 Check that, for any $f \in C^1(\mathbb{R}^n)$ and any $\omega \in C_c^\infty(\mathbb{R}^n, \bigwedge^k(\mathbb{R}^n))$, it holds

$$f^\# d\omega = d(f^\# \omega). \tag{1}$$

Exercise 2 Given the Stokes' theorem for differential forms, prove:

1. (divergence theorem) for any $F\colon \mathbb{R}^3 \to \mathbb{R}^3$

$$\int_V \operatorname{div} F(x)\, dx = \int_S F(x) \cdot n(x)\, dS(x),$$

where $V \subset \mathbb{R}^3$, and n, dS are respectively the outer normal and the surface measure associated with $S = \partial V$;

2. (curl theorem)

$$\int_S \nabla \times F(x) \cdot n(x)\, dS(x) = \int_\Gamma F(x) \cdot \tau(x)\, d\Gamma(x),$$

where S is a surface in $\mathbb{R}^3$ and $\tau, d\Gamma$ are respectively the tangent vector and the line measure associated with $\Gamma = \partial S$.

Exercise 3 (Exercise 7.2.1 and equation (7.10) in Krantz - Parks) Let $\phi \in C_c^\infty(\mathbb{R}^n, \bigwedge^k(\mathbb{R}^n))$, any $\xi \in C_c^\infty(\mathbb{R}^n, \bigwedge_p(\mathbb{R}^n))$ and any $T \in \mathcal{D}_m(\mathbb{R}^n)$. Defining $D_{x_j} T(\phi) = T(\partial_{x_j} \phi)$, show that it holds

1. $\partial(\partial T) = 0$
2. $\partial T \llcorner \phi = T \llcorner d\phi + (-1)^k \partial(T \llcorner \phi)$
3. $\partial T = -\sum_{j=1}^n D_{x_j} T \llcorner dx_j$ if $m \geq 1$
4. $T = \sum_{i \in I(k,n)} [T \llcorner dx_i] \wedge e_i$
5. $D_{x_j}(T \llcorner \phi) = D_{x_j} T \llcorner \phi + T \llcorner \frac{\partial \phi}{\partial x_j}$
6. $D_{x_j}(T \wedge \xi) = D_{x_j} T \wedge \xi + T \wedge \frac{\partial \xi}{\partial x_j}$
7. $(T \wedge \xi) \llcorner \phi = T \wedge (\xi \llcorner \phi)$ if $m = 0$ and $k \leq p$

8. $\partial(T \wedge \xi) = -T \wedge \mathrm{div}\xi - \sum_{j=1}^{n} D_{x_j} T \wedge (\xi \llcorner dx_j)$ if $m = 0$. Observe that in particular $\partial(\mathcal{L}^n \wedge \xi) = -\mathcal{L}^n \wedge \mathrm{div}\, \xi$

9. $\partial T \wedge e_j = (-1)^n D_{x_j} T$ if $m = n$.

Exercise 4 Let $T = [\![\mathbb{S}^2]\!]$ be the 2-current associated with the unit sphere in $\mathbb{R}^3$. Consider the projection $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi(x_1, x_2, x_3) = (x_1, x_2)$. Show that $\pi_\# T = 0$.