— FUNDAMENTAL NOTIONS —

① Polyhedral subdivisions
② Constructions of subdivisions
③ Flips

} OUTLINE

① POLYHEDRAL SUBDIVISIONS:

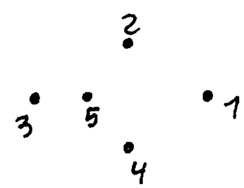
Let $\mathcal{A}=\{a_1,\dots,a_n\}\subseteq\mathbb{R}^d$ be a point configuration (there may be multiple points)

*OBS.: All this can be done as well for vector configurations.

- A FACE of $\mathcal{A}$ is a subset where a certain linear functional is maximized

* EXAMPLE:

$\mathcal{A}:=\{(2,0),(0,2),(-2,0),(0,-2),(-1,0)\}$



▷ Faces: $\{\underbrace{12345}_{\text{Total}},\ \underbrace{12,\ 23,\ 34,\ 14}_{\text{Edges}},\ \underbrace{1,\ 2,\ 3,\ 4}_{\text{Vertices}},\ \underbrace{\emptyset}_{\text{Empty}}\}$

Note that $B:=\{12345\}\subseteq\mathcal{A}$ is a triangle; its faces are

$\{1235,\ 12,\ 23,\ 135,\ 1,\ 2,\ 3,\ \emptyset\}$

13 is not a face!

- A SIMPLEX of $\mathcal{A}$ is an affinely independent subset of $\mathcal{A}$.

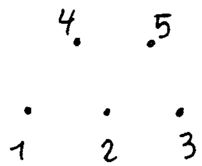
* For example, $\{1,2,4\}\subset\mathcal{A}$ is a simplex, but $\{1,3,5\}$ is not.

- A POLYHEDRAL SUBDIVISION of $\mathcal{A}$ is a collection $\mathcal{S}$ of subsets of $\mathcal{A}$, called CELLS, such that:

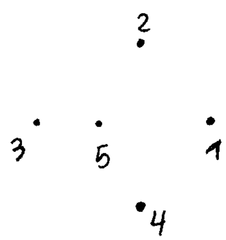
  ▷ (UP) $\bigcup_{\sigma \in \mathcal{S}} \mathrm{conv}(\sigma) = \mathrm{conv}(\mathcal{A})$ UNION PROPERTY

  ▷ (IP) $\mathrm{conv}(\sigma \cap \tau) = \mathrm{conv}(\sigma) \cap \mathrm{conv}(\tau)$
  and $\sigma \cap \tau$ is a face of both σ and τ, $\forall \sigma, \tau \in \mathcal{S}$ INTERSECTION PROPERTY

* Both parts of (IP) are needed:



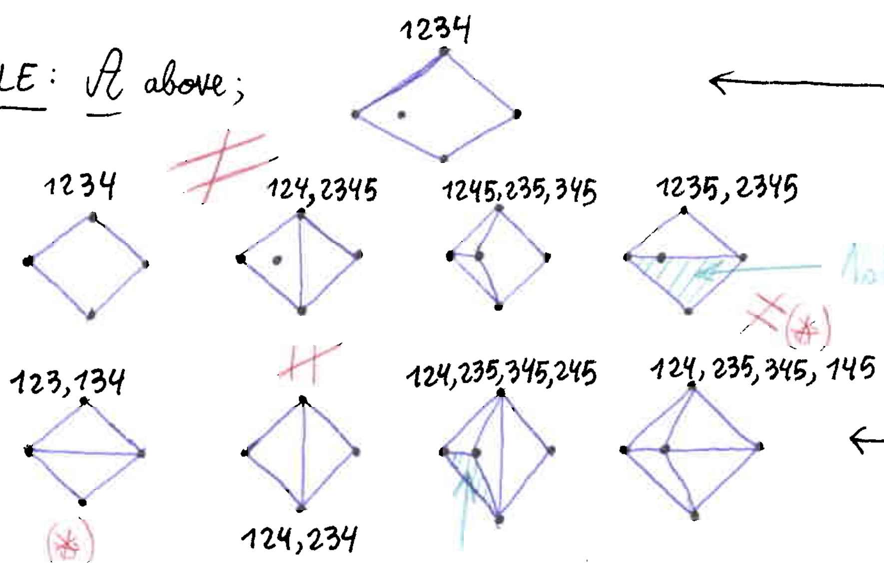
{1234, 1235} does not fulfill the first part
does " " second "



{123, 1345} does fulfill the first part
does not " " second "

- A TRIANGULATION of $\mathcal{A}$ is a subdivision whose cells are all simplices.

* EXAMPLE: $\mathcal{A}$ above;



← TRIVIAL SUBDIVISION

Not a simplex

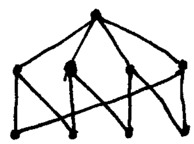
← TRIANGULATIONS

• <u>REFINEMENT</u>: A subdivision $\mathcal{S}$ refines another $\mathcal{S}'$ iff

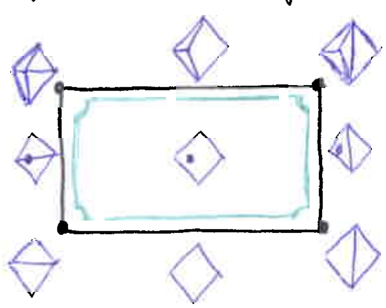
$$\forall \sigma \in \mathcal{S}, \exists \sigma' \in \mathcal{S}' \text{ s.t. } \sigma \subseteq \sigma'$$

↳ This gives a poset structure for the subdivisions above;

<u>REFINEMENT POSET</u>



In this case, it is the face poset of a square:



* <u>OBS</u>: Triangulations $\Longleftrightarrow$ Minimal elements in the refinement poset

It is not a coincidence that the refinement poset was the face poset of a polytope:

— <u>THM.</u>:
For every point configuration $\mathcal{A}$ with n elements and dimension d, there is a polytope $\Sigma(\mathcal{A})$ such that the poset of (non-empty) faces of $\Sigma(\mathcal{A})$ equals the refinement poset of regular subdivisions of $\mathcal{A}$.

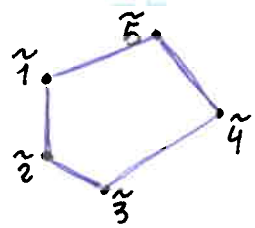
② CONSTRUCTIONS OF SUBDIVISIONS:

- REGULAR SUBDIVISIONS: (Also called "coherent", "convex", "generalized Delaunay")

Given $\mathcal{A}=\{a_1,\dots,a_n\}$, choose $\alpha_1,\dots,\alpha_n\in\mathbb{R}$ and consider the lifted point set $\tilde{\mathcal{A}}=\{(a_1,\alpha_1),\dots,(a_n,\alpha_n)\}\subseteq\mathbb{R}^{d+1}$.

Look at the facets of $\tilde{\mathcal{A}}$ (maximal proper faces), and a facet is a LOWER FACET of $\tilde{\mathcal{A}}$ iff the exterior normal vector has last coordinate negative.

* EXAMPLE:

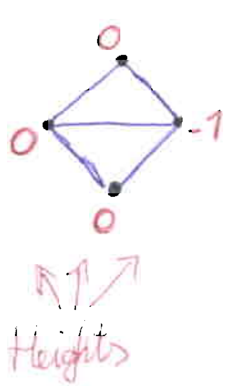


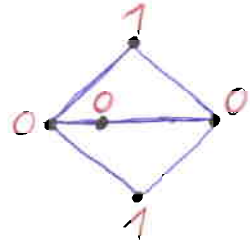
– CLAIM: The lower facets of $\tilde{\mathcal{A}}$ form a subdivision of $\mathcal{A}$.

↳ The REGULAR SUBDIVISIONS are those which can be obtained in this way.

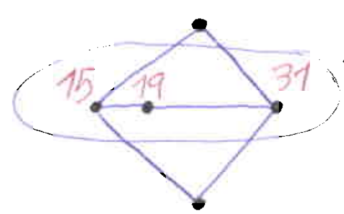
* EXAMPLES:

For the above subdivisions;





or



Makes those three points to lie in a line

⇓

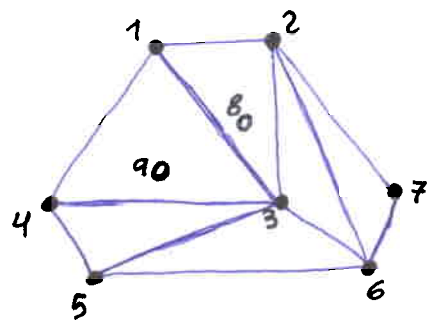
The result is not a triangulation

— PROP.:

If $\alpha_1, \dots, \alpha_n$ are "sufficiently generic", then all "non-general position subsets" of $\mathcal{A}$ are vertical. In particular, all lower facets are simplices and the subdivision is a triangulation.

▵ General position in dim. d ≡ No $d+1$ points lie in a hyperplane.

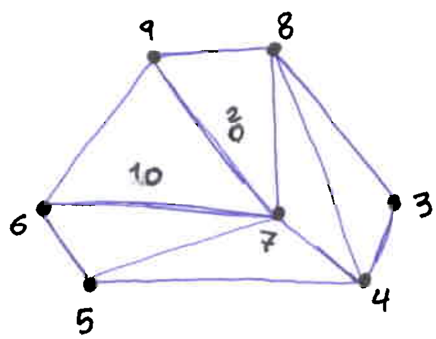
2.1) PLACING TRIANGULATION:



- Order your points (Supp. that $1, 2, \dots, d+1$ form a simplex).
- Add points one by one, coning $i+1$ to the part of the already constructed triangulation which is visible from it.

("Visible" ≡ from the outside ⇒ Do nothing for pt. 8).

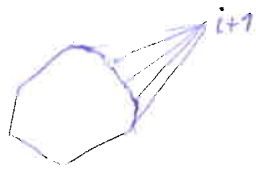
2.2) PUSHING TRIANGULATION:



- Order your points.
- Start with the trivial subdivision.
- Push the points one by one each time, refining what you had.

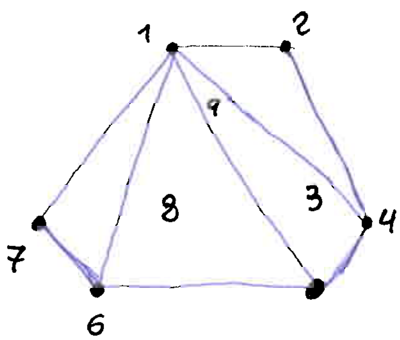
(When pushing 1 and 2 away from you, you just don't see them anymore)

1. The placing triangulation for an ordering equals the pushing triangulation for the reverse order.
2. Pushing triangulations are regular; it is the triangulation you get if you take $\underline{\alpha_i := t^i}$ and make $\underline{t \to +0}$.
3. <u>Combinatorial description of "pushing"</u>:
   After you have constructed the $\underline{i}$-th step, subdivide every cell $\underline{\sigma}$ containing $\underline{i+1}$ in the unique way that makes cell $\underline{\sigma \setminus (i+1)}$ appear.



Is already a subdivision.

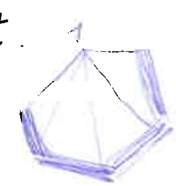
2.3) PULLING TRIANGULATION: ("Moving towards you")



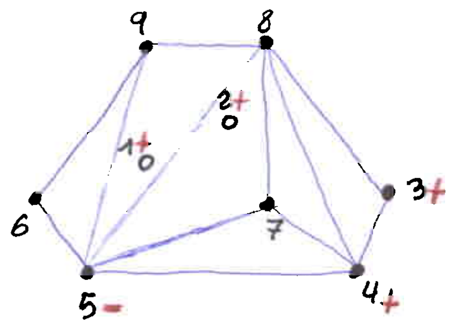
- Order your points.
- Start with the trivial subdivision.
- At each step, pull point $\underline{i+1}$ from the already constructed triang.

(Having $\underline{i}$, the height for $\underline{i+1}$ is gonna be much less)

1. Pulling triangulations are regular, taking $\underline{\alpha_i := -t^i}$, $\underline{t \to +0}$.
2. <u>Combinatorial description of "pulling"</u>:
   After $\underline{i}$-th step, subdivide every cell $\underline{\sigma}$ containing $\underline{i+1}$ by coning it to the facets not containing it.



2.4) LEXICOGRAPHIC TRIANGULATION:



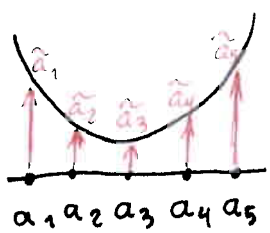
+ = Pushing   − = Pulling

- Order your points and give signs to them.
- Start with the trivial subdivision.
- Perform pushings and pullings in order, depending on the sign.

1. lexicographic triangulations are also regular ($\alpha_i = \pm t^i$).

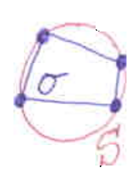
2.5) DELAUNAY SUBDIVISION: (Crucial in computational geometry)

The regular subdivision obtained taking $\alpha_i := a_i \cdot a_i = \|a_i\|^2$.
In other words; lift $\mathcal{A}$ to the paraboloid $x_{d+1} = x_1^2 + \ldots + x_d^2$.



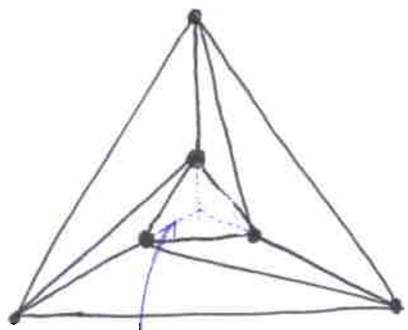
1. It might not be a triangulation; it is if points are sufficiently generic.
2. It has nice properties; for example,

T is the Delaunay subdiv. $\Leftrightarrow$ $\forall \sigma \in T$, $\exists$ sphere S s.t.

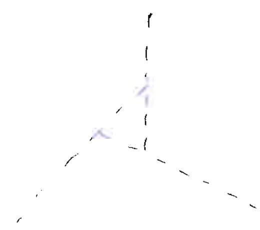


- interior(S) is "empty" (no points of $\mathcal{A}$).
- $\partial(S) \cap \mathcal{A} = \sigma$.

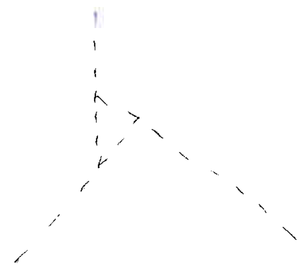
2.6) SMALLEST NON-REGULAR TRIANGULATION:



If those lines are not intersecting at a common point;



or



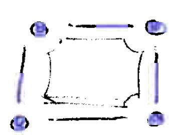
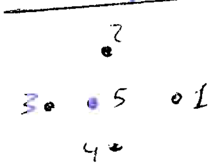
One of them would be regular, the other would not.

③ FLIPS

We know that triangulations are the minimal elements in the poset of subdivisions. One definition of what a flip is, is that "flips correspond to the next-to-minimal elements in the poset. The definition makes sense thanks to:

<u>Proposition</u>: If a subdivision is not a triangulation and is only refined by triangulations, then it is refined by <u>exactly two</u> triangulations.

<u>Definition</u>: We say that those two triangulations "differ by a flip"

<u>Example</u>: in the poset  of subdivisions of 

, the four edges of the square can be regarded as "next-to-minimal" elements, and also as elements connecting two triangulations to one another.

... but let us see a more "concrete" (but not simpler!) definition of what a flip is.

<u>Circuits</u>: a circuit of A is a minimal affinely dependent subset.

If C is a circuit, $C = \{c_1, \dots, c_n\}$ then it has a unique (up to a constant factor) affine dependence.

<u>Pf</u>: For if we had two of them:

$$\left.\begin{aligned} \lambda_1 c_1 + \dots + \lambda_n c_n = 0 \\ \lambda_1 + \dots + \lambda_n = 0 \end{aligned}\right\} \qquad \left.\begin{aligned} \mu_1 c_1 + \dots + \mu_n c_n = 0 \\ \mu_1 + \dots + \mu_n = 0 \end{aligned}\right\}$$

then $\lambda_1 \neq 0 \neq \mu_1$ (otherwise C is not minimal dependent)

and we get
$$\left.\begin{aligned} \frac{\lambda_2}{\lambda_1} c_2 + \dots + \frac{\lambda_n}{\lambda_1} c_n = \frac{\mu_2}{\mu_1} c_2 + \dots + \frac{\mu_n}{\mu_1} c_n \\ \frac{\lambda_2}{\lambda_1} + \dots + \frac{\lambda_n}{\lambda_1} = \frac{\mu_2}{\mu_1} + \dots + \frac{\mu_n}{\mu_1} \end{aligned}\right\}$$

... hence C is not minimal dependent either. $\square$

This dependence <u>canonically</u> partitions C into two subsets $C = C^+ \dot\cup C^-$ defined as

$$C^+ = \{c_i \mid \lambda_i > 0\}, \quad C^- = \{c_i \mid \lambda_i < 0\}$$

coeffs in the unique aff. dependence.

<u>Remark</u>: we can choose which subset we call C^+ and which C^-, because we can change the sign of all λ_i's. What is canonical is the "partition".

Geometrically:

$C^+ \dot\cup C^-$ is the only way to partition C such that the relative interiors intersect:

$$\operatorname{relint}(C^+) \cap \operatorname{relint}(C^-) \neq \emptyset$$

Example:

$c_2 = (0,2)$

$c_1 = (2,0)$

$c_3 = (-1,0)$

$c_4 = (0,-2)$

Then: $\begin{cases} -2c_1 + 3c_2 - 4c_3 + 3c_4 = 0 \\ -2 + 3 - 4 + 3 = 0 \end{cases}$

$\updownarrow$

$$3c_2 + 3c_4 = 2c_1 + 4c_2$$

$\updownarrow$

$$\underbrace{\frac{1}{2} c_2 + \frac{1}{2} c_4}_{\text{convex combination of } c_2 \text{ and } c_4} = \underbrace{\frac{2}{3} c_3 + \frac{1}{3} c_1}_{\text{convex combination of } c_3 \text{ and } c_1}$$

So, there is a common point in the (relative) interior of the segments $[c_2, c_4]$ and $[c_1, c_3]$

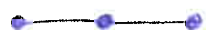
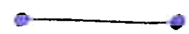
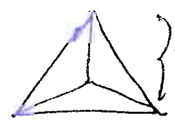
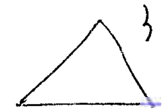
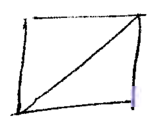
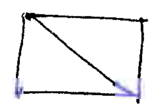
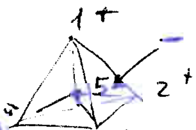
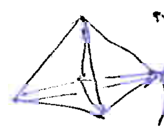
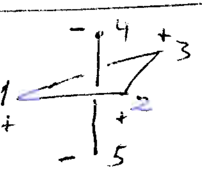
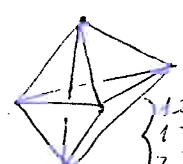
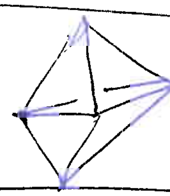
(... and vice versa: from the fact that there is a common point in the relative interiors, you can derive an affine dependence with one sign on C^+ and the other on C^-)

Claim: a circuit has exactly two triangulations.
More precisely, if $C = C^+ \dot\cup C^-$, then its triangulations are

$$T_C^+ = \{ C \setminus \{c_i\} \mid c_i \in C^+ \} \quad \text{and}$$

$$T_C^- = \{ C \setminus \{c_i\} \mid c_i \in C^- \}$$

Examples:

	circuit	T_C^+	T_C^-
d = 1	1 2 3 • • • + − +	 $\{12, 23\}$	 $\{13\}$
d = 2	1^+, 2^+, 3^-, 4^+	 $\{123, 234, 134\}$	 $\{124\}$
	1^-, 2^+, 3^-, 4^+	 $\{123, 134\}$	 $\{124, 234\}$
d = 3	"point inside tetrahedra"  1+, 2+, 3+, 4+, 5 −	 $\{1235, 1245, 1345, 2345\}$	 $\{1234\}$
	"edge crossing triangle"  1+, 2+, 3+, 4 −, 5 −	 $\{1245, 1345, 2345\}$	 $\{1234, 1235\}$
d = 4	⋮	⋮	⋮

Definition of flip ("general position case"). Let $A \subseteq \mathbb{R}^d$

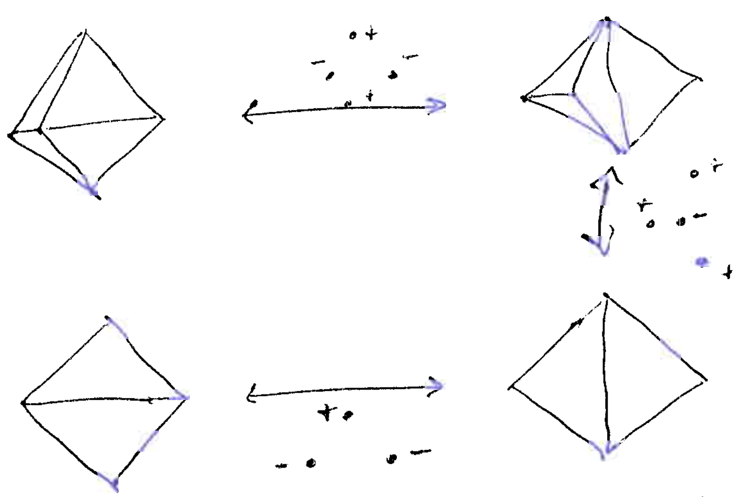
Let T be a triangulation of A. Let C be a circuit of A of dimension d (that is to say, a set of $d+2$ points in general position).

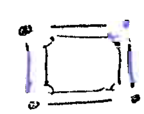
Suppose that T contains one of the two triangulations of C (say T_C^+). Then,

$$T' = (T \setminus T_C^+) \cup T_C^-$$

is another triangulation of A. We say that T and T' differ by a flip.

Example: the following are three flips in triangulation of our working example A:



What about the fourth edge of the square  ?

It is a flip on a lower dimensional circuit.

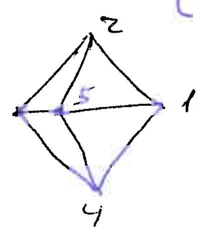
Definition of flip ("including lower dimensional circuits")

Links in simplicial complexes:

Given a simplicial complex K (e.g. a triangulation of a point set) and a simplex $\sigma \in K$ (of any dimension; that is, faces of cells are allowed) the link of σ in K is the set of "things joined to σ in K". That is:

$$lk_K(\sigma) = lk(\sigma, K) = \left\{ \tau \;\middle|\; \begin{array}{l} \tau \cap \sigma = \emptyset \\ \tau \cup \sigma \in K \end{array} \right\}$$

$$\left(\text{or } = \left\{ \tau \setminus \sigma \;\middle|\; \begin{array}{l} \sigma \subseteq \tau \\ \tau \in K \end{array} \right\} \right)$$

Example: in 

$K = \{125, 235, 345, 145\}$

Then: $lk_K(1) = \{25, 54\}$ $\quad lk_K(5) = \{12, 23, 34, 14\}$

$lk_K(15) = lk_K(35) = \{2, 4\}$

$lk_K(125) = \{\emptyset\}$

Then, the general definition of flip is:

DEFINITION OF FLIP

Let T be a triangulation of A. Let C be a circuit of A of any dimension k (that is, $k+2$ pts spanning a subspace of dim k and "in general position" in that subspace).

Suppose that T contains one of the two triangulations of C (say T_c^+) as a subcomplex (that is, as faces of some full-dimensional cells of T).

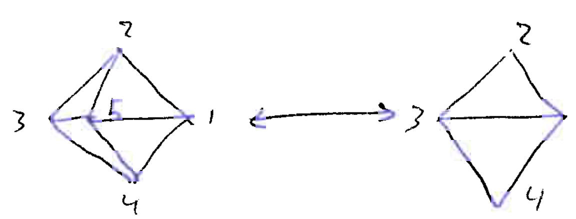
Suppose also that all the simplices of T_c^+ have the same link L in T. Then,

$$T' = (T \setminus (T_c^+ * L)) \cup (T_c^- * L)$$

is another triangulation of A, where

$$T_c^+ * L = \left\{ \sigma \cup \tau \;\middle|\; \begin{array}{l} \sigma \in T_c^+ \\ \tau \in L \end{array} \right\} \quad \left(\text{"join" of two simplicial complexes}\right)$$

We say that T and T' differ by a flip.

Example: in 

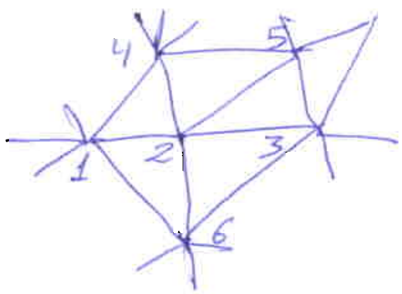
we have:

$C = \{135\}$, $C^+ = \{1,3\}$, $C^- = \{2\}$, $T_c^+ = \{15, 35\}$

$T_c^- = \{13\}$, $lk_T(13) = lk_T(35) = \{2,4\} = L$

$$\boxed{\begin{array}{l} T_c^+ * L = \{152, 154, 352, 354\} \quad (= T) \\[1em] T_c^- * L = \{132, 134\} \quad (= T') \end{array}}$$

<u>Remark</u>: to see why we need the link condition, think of the following example:



T contains the triangulation $T_C^+ = \{12, 23\}$ of the circuit $C = \{1, 2, 3\}$. But removing the simplices 124, 126, 235, 236 does not allow us to insert the triangulation $T_C^- = \{13\}$. If we decide to remove also 245 then we can insert T_C^-, but there is no "canonical" way of completing the rest to be again a triangulation.