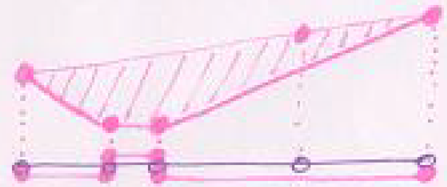
Main actor:

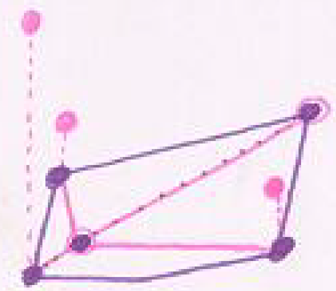
Def.: A $\left\{\begin{matrix}\text{subdiv.}\\ \text{triang.}\end{matrix}\right\}$ T of $\mathcal{A}$ is regular if there exist heights $\alpha(a)$ for all $a \in \mathcal{A} \subset \mathbb{R}^d$ such that T is the set of lower facets of $\mathcal{A}^\alpha := \left\{ \binom{a}{\alpha(a)} \in \mathbb{R}^{d+1} : a \in \mathcal{A} \right\}$

Example:

dim 1:



dim 2:



Notation: $T(\mathcal{A}, \alpha) :=$ regular subdivision induced by $\alpha \in \mathbb{R}^{\mathcal{A}}$

$n := |\mathcal{A}|$

W.l.o.g.: $\alpha(a) \geq 0 \quad \forall \alpha, a$ considered in this lecture.

Preview:

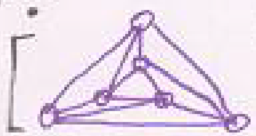
What is nice about subdivisions in dim ≤ 2?

1. Easy to visualize
2. Graph of triang's is connected (Lecture 2)

Even better for dim $=1$ and n-gon:

3. All triang's are regular (exercise)
4. Graph of triang's is the graph of a polytope (dim 1: $(n-2)$-cube, n-gon: associahedron) [Haiman, Lee]

What structure is responsible for 4.?

dim 2 ? No.! [] [tomorrow, Lecture 4]

convex pos.? No.!

regularity? Yes! [today]

THM.: [Gelfand, Kapranov, Zelevinsky 1989]
The graph of all regular triang's of a d-dim. point configuration with n points $\mathcal{A}$ is the graph of an $(n-d-1)$-dim. polytope, the secondary polytope of $\mathcal{A}$.

COR.: (i) Graph of all reg. triang's is $(n-d-1)$-connected [Balinsky's thm, Ziegler book]
(ii) Can do linear programming to optimize lin. functionals for reg. triang's.

How does the secondary polytope look like?

A strange polytope:

<u>Def.</u>: Let T be a triang. of $\mathcal{A}$. Then

$$\phi_{\mathcal{A}}(T) := \sum_{a\in\mathcal{A}} \sum_{\sigma\in T:\, a\in\sigma} \mathrm{vol}(\sigma)\, e_a \in \mathbb{R}^{\mathcal{A}}, \quad e_a: \text{coord. unit vector in dir. } a,$$

is the <u>GKZ-vector</u> of T.

<u>Def.</u>: $\Sigma(\mathcal{A}) := \mathrm{conv}\{\phi_{\mathcal{A}}(T) : T \text{ triang of } \mathcal{A}\}$ is the <u>secondary polytope of $\mathcal{A}$</u>.

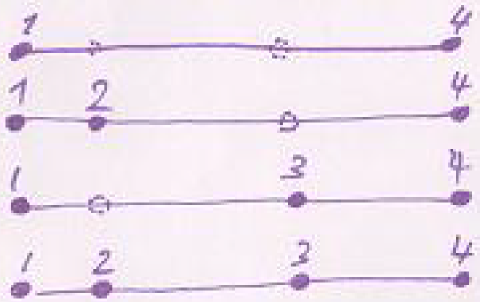
<u>Rem.</u>: Regularity not required in definition.

Example:

dim 1:

$\mathcal{A}$

triang's:



$$\phi_{\mathcal{A}}(T_1) = \begin{pmatrix}5\\0\\0\\5\end{pmatrix}$$

$$\phi_{\mathcal{A}}(T_2) = \begin{pmatrix}1\\1+4\\0\\4\end{pmatrix} = \begin{pmatrix}1\\5\\0\\4\end{pmatrix}$$

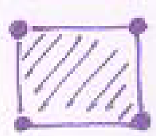
$$\phi_{\mathcal{A}}(T_3) = \begin{pmatrix}3\\0\\3+2\\2\end{pmatrix} = \begin{pmatrix}3\\0\\5\\2\end{pmatrix}$$

$$\phi_{\mathcal{A}}(T_4) = \begin{pmatrix}1\\1+2\\2+2\\2\end{pmatrix} = \begin{pmatrix}1\\3\\4\\2\end{pmatrix}$$

$$\Rightarrow \quad \Sigma(\mathcal{A}) = \mathrm{conv}\left\{\begin{pmatrix}5\\0\\0\\5\end{pmatrix}, \begin{pmatrix}1\\5\\0\\4\end{pmatrix}, \begin{pmatrix}3\\0\\5\\2\end{pmatrix}, \begin{pmatrix}1\\3\\4\\2\end{pmatrix}\right\}$$

<u>Obs.</u>: (i) $x_1 + x_2 + x_3 + x_4 = 10 \quad \forall \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \Sigma(\mathcal{A})$. Why? $10 = 2\cdot \mathrm{vol}(\mathrm{conv}\,\mathcal{A})$! $\Rightarrow \mathrm{rank}\, \Sigma(\mathcal{A}) \le 3$

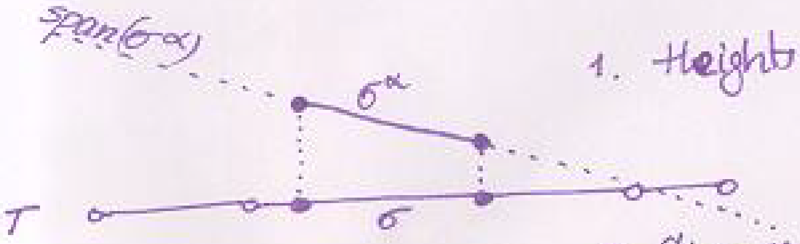
(ii) $\mathrm{rank}\, \Sigma(\mathcal{A}) = 2$ (exercise).

(iii) GKZ-thm. $\Rightarrow \Sigma(\mathcal{A}) =$ 

Why does $\Sigma(\mathcal{A})$ look like this?

The secondary fan:

When is $T = T(\mathcal{A},\alpha)$? When does T refine $T(\mathcal{A},\alpha)$?
Idea: Find conditions on α regarded as set of n variables!



1. Heights of σ must be in a hyperplane.

2. Given the heights on σ, all other heights must be
strictly above this hyperplane for "$T = T(\mathcal{A},\alpha)$" "Strict Convexity"
weakly above this hyperplane for "T refines $T(\mathcal{A},\alpha)$". "Convexity"
i.e. $\alpha(a) >$ affine function of $a, \alpha(s), s \in \sigma \ \forall a \in \mathcal{A} \setminus \sigma$.
resp. $\alpha(a) \geq$ — " — — " — " — " — — " — — " — — " —— .

Def.:
$C \subseteq \mathbb{R}^n$ is a <u>convex polyhedral cone</u> if it is the intersection of finitely many halfspaces.
<u>relint C</u> is the intersection of the corresponding open halfspaces.

Prop.: $\mathring{\mathcal{C}}_{\mathcal{A}}(T) := \{\alpha \in \mathbb{R}^n : T = T(\mathcal{A},\alpha)\}$ is the relint of a convex polyhedral cone
$\mathcal{C}_{\mathcal{A}}(T) := \{\alpha \in \mathbb{R}^n : T \text{ refines } T(\mathcal{A},\alpha)\}$.

Def.:
$\mathcal{C}_{\mathcal{A}}(T)$ is the <u>secondary cone</u> of T. $\mathring{\mathcal{C}}_{\mathcal{A}}(T)$ is the <u>open secondary cone</u> of T.

Def.:
A <u>polyhedral fan</u> in $\mathbb{R}^n$ is a collection $\mathcal{C}$ of polyhedral cones with
(i) $C \in \mathcal{C} \Rightarrow$ all faces of C are in $\mathcal{C}$
(ii) $C, C' \in \mathcal{C} \Rightarrow C \cap C' \in \mathcal{C}$
(iii) $C \cap C'$ is a face of both C and C'!

Def.:
$\mathcal{C}_{\mathcal{A}} := \{ C_{\mathcal{A}}(T) \mid T \text{ polyhedral subd. of } \mathcal{A} \}$ is the <u>secondary fan</u> of $\mathcal{A}$.

Def.: A polyhedral fan $\mathcal{C}$ in $\mathbb{R}^n$ is complete if
$$\bigcup_{C \in \mathcal{C}} C = \mathbb{R}^n .$$

Prop.:
$\mathcal{C}_{\mathcal{A}}$ is a complete polyhedral fan in $\mathbb{R}^n$. $\square$

What has this fan to do with $\Sigma(\mathcal{A})$?

Normal fans of polytopes:

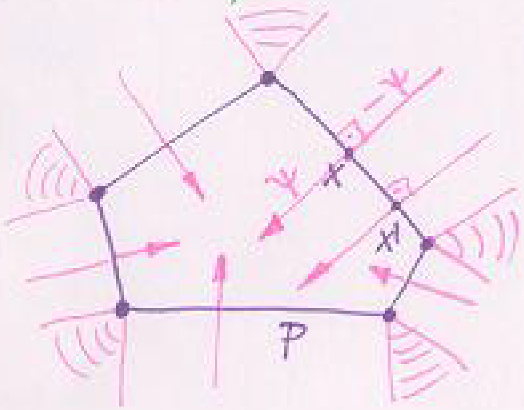
Def.: Let P be a polytope in $\mathbb{R}^n$ and $x \in P$.
The *inner normal cone* of x in P is

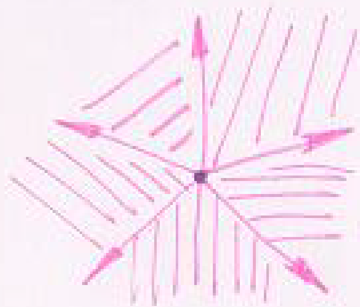
$$\mathcal{N}_P(x) := \{\psi \in \mathbb{R}^n : \langle \psi, x \rangle \leq \langle \psi, y \rangle \ \forall y \in P\}$$

The *inner normal fan* of P is

$$\mathcal{N}_P := \{\mathcal{N}_P(x) : x \in P\}$$

Example:





$\mathcal{N}_P$

Hope: $\mathcal{N}_{\Sigma(\mathcal{A})} = \mathcal{C}_{\mathcal{A}}$.

Obs.: (i) vertices in P $\longleftrightarrow$ full-dim cones in $\mathcal{N}_P$.

(ii) P determined by its vertices $\Leftrightarrow$ $\mathcal{N}_P$ determined by full-dim cones in $\mathcal{N}_P$.

(iii) vertices of $\Sigma(\mathcal{A})$ are of the form $\phi_{\mathcal{A}}(T)$ for triang's T of $\mathcal{A}$.

Need to show:

$$\langle \alpha, \phi_{\mathcal{A}}(T) \rangle \leq \langle \alpha, \phi_{\mathcal{A}}(T') \rangle \quad \forall T' \text{ triang. of } \mathcal{A} \quad \forall \alpha \in \mathcal{C}_{\mathcal{A}}(T).$$

Proof: For all α and all T we have:

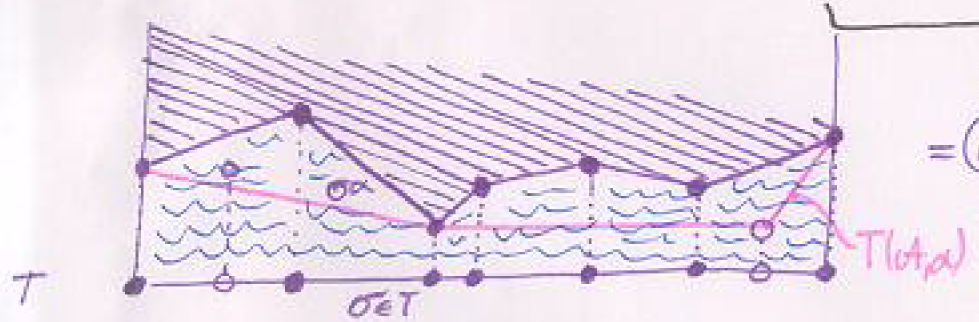
$$\langle \alpha, \phi_{\mathcal{A}}(T) \rangle = \langle \alpha, \sum_{a \in \mathcal{A}} \sum_{\substack{\sigma \in T \\ a \in \sigma}} vol(\sigma) e_a \rangle$$

$$= \sum_{a \in \mathcal{A}} \sum_{\substack{\sigma \in T \\ a \in \sigma}} vol(\sigma) \alpha(a)$$

$$= \sum_{\sigma \in T} vol(\sigma) \sum_{a \in \sigma} \alpha(a)$$

$$= (d+1) \sum_{\sigma \in T} \underbrace{\Big(\underbrace{vol(\sigma)}_{\text{volume of } \sigma} \cdot \underbrace{\frac{1}{d+1} \sum_{a \in \sigma} \alpha(a)}_{\text{barycenter of } \sigma^\alpha} \Big)}_{\text{volume below } \sigma^\alpha}$$

$= (d+1) \cdot$ amount of water below $\{\sigma^\alpha : \sigma \in T\}$



Which T has the smallest amount of water below?

$T(\mathcal{A}, \alpha)$!

Remark: (i) vertices in $\Sigma(\mathcal{A}) \longleftrightarrow$ regular triang's of $\mathcal{A}$

(ii) $\Sigma(\mathcal{A})$ not full-dim.

(iii) $\mathcal{C}_{\mathcal{A}}(T)$ contain linear subspaces (not pointed):

$\Rightarrow$ adding affine functions to α does not change $T(\mathcal{A},\alpha)$,

$\Rightarrow$ $d+1$ degrees of freedom $\Rightarrow$ $\Sigma(\mathcal{A})$ is in $\mathbb{R}^{n-d-1}$.

How can we mod this out?