Gale transforms: Regard $\mathcal{A}$ as a vector configuration $\{\binom{a}{1} : a \in \mathcal{A}\} \in \mathbb{R}^{d+1}$.

L8

Associate to $\mathcal{A}$ the matrix

$$A := \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 1 & & 1 \end{pmatrix}, \quad a_i \in \mathcal{A},\ i = 1, \dots, n, \quad A \in \mathbb{R}^{(d+1)\times n} = \mathbb{R}^{r \times n}$$

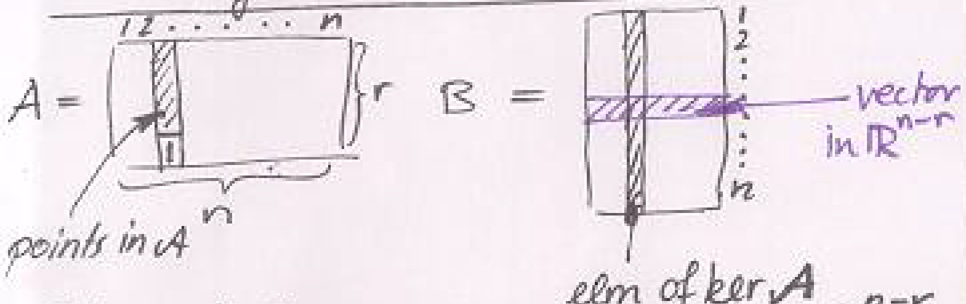
Assumption:

$$\operatorname{rank} A = d+1 =: r.$$

Consider $\mathcal{B} \in \mathbb{R}^{n \times (n-r)}$ with

$$A\mathcal{B} = 0 \in \mathbb{R}^{r \times (n-r)}$$

General



Interprete the rows of $\mathcal{B}$ as vectors in $\mathbb{R}^{n-r}$

$\rightsquigarrow$ Vector configuration $\mathcal{B}$.

Def.: Every $\mathcal{B}$ as above is a Gale transform of $\mathcal{A}$, $\mathcal{B} \in \operatorname{Gale}(\mathcal{A})$.

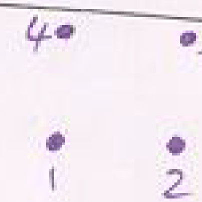
Obs.: $\mathcal{B} \in \operatorname{Gale}(\mathcal{A}) \Rightarrow \mathcal{A} \in \operatorname{Gale}(\mathcal{B})$

because

$$AB = 0 \iff B^T A^T = 0.$$

Example

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



$\subset \mathbb{R}^2$

$r = 3,\ n = 4,\ n - r = 1$

Choose $\mathcal{B} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{matrix} \to b_1 \\ \to b_2 \\ \to b_3 \\ \to b_4 \end{matrix} \Bigg\} \in \mathbb{R}^1$

But: • tedious to calculate

• lots of choices

Idea: Look at properties of a Gale transform!

Circuits and cocircuits:

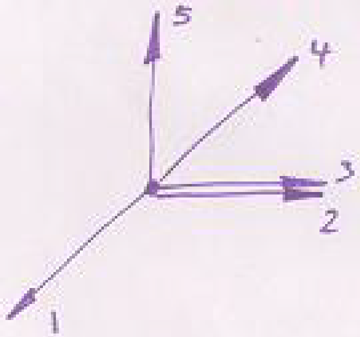
Def.: A circuit signature on a minimally $\left\{\begin{matrix}\text{affinely}\\ \text{linearly}\end{matrix}\right\}$ depend. subset $C \subseteq \mathcal{A}$ of a $\left\{\begin{matrix}\text{point}\\ \text{vector}\end{matrix}\right\}$ configuration $\mathcal{A}$ is a partition

$$C = C_+ \,\dot\cup\, C_-$$

with

$$\sum_{a_i \in C_+} \lambda_i a_i = \sum_{a_j \in C_-} \lambda_j a_j \quad , \quad \left\{\begin{matrix}\lambda_i \geq 0, \ \sum_{a_i \in C_+} \lambda_i = \sum_{a_j \in C_-} \lambda_j \\ \lambda_i \geq 0 \ \ \forall a_i \in C.\end{matrix}\right.$$

Example:



Circuits

C_+	C_-
2	3
1,4	
1,2,5	
1,3,5	
2,5	4
3,5	4

also written as

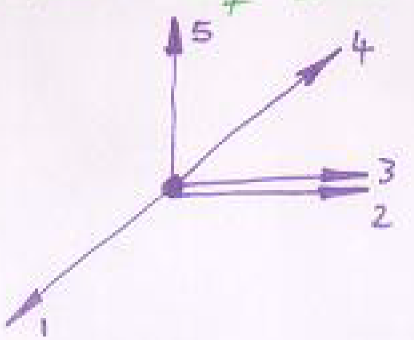
$2\bar{3}$
14
125
135
$25\bar{4}$ or $2\bar{4}5$
$3\bar{4}5$

Def.: A cocircuit signature on the complement of a maximal subset $\bar{C}^* \subseteq \mathcal{A}$ of points on a hyperplane spanned by $\bar{C}^*$ is a partition

$$C^* = \mathcal{A} \setminus \bar{C}^* = C^*_+ \,\dot\cup\, C^*_-$$

such that C^*_+ and C^*_- lie strictly on opposite sides of this hyperplane.

Example:



Cocircuits

H	C^*_+	C^*_-
1	2,3	5
2	1	4,5
3	1	4,5
4	2,3	5
5	1	2,3,4

also written as

$23\bar{5}$
$1\,\overline{45}$
$1\,\overline{234}$

Circuits and cocircuits in the Gale transform

<u>Prop.</u>: $B \in \text{Gale}(A) \Rightarrow$ circuits of A are the cocircuits of B
cocircuits of A are the circuits of B $\square$

<u>Pf</u>: (Lin. Alg).

<u>Example</u>:

<u>Circuits</u>:

$1\bar{2}3\bar{4}$

<u>Cocircuits</u>:

34
14
12
23
$1\bar{3}$
$2\bar{4}$

<u>Circuits</u>:

12
14
23
34
$1\bar{3}$
$2\bar{4}$

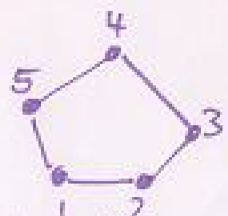
<u>Cocircuits</u>:

$1\bar{2}3\bar{4}$

✓

<u>Application</u>

Draw Gale transform of



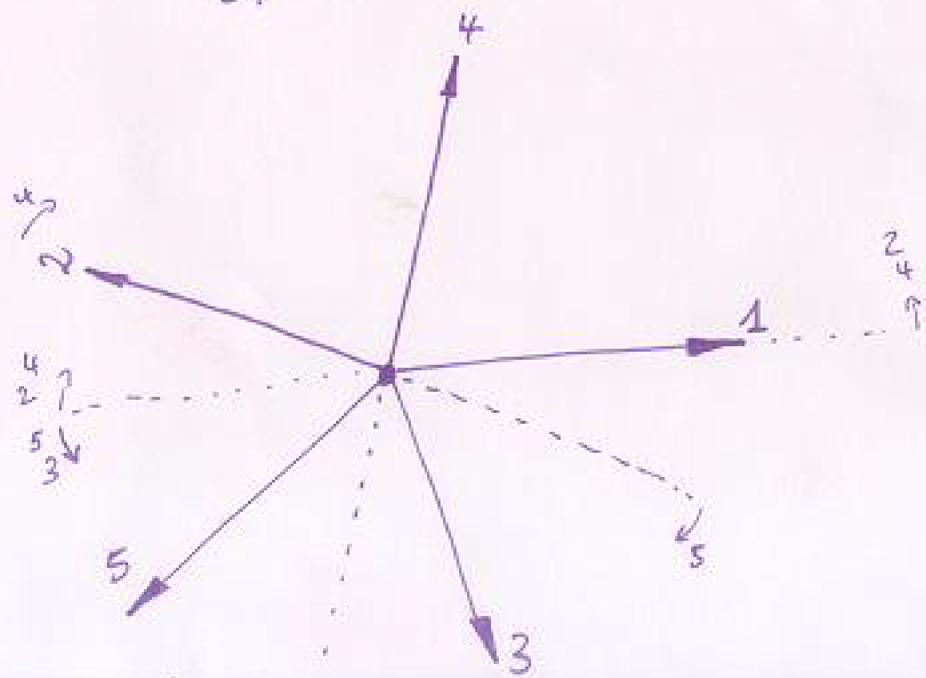
!

<u>Circuits</u>:

$1\bar{2}3\bar{4}$
$1\bar{2}3\bar{5}$
$1\bar{2}4\bar{5}$
$1\bar{3}4\bar{5}$
$2\bar{3}4\bar{5}$

<u>Cocircuits</u>:

123
234
345
$12\bar{4}$
$23\bar{5}$
$\bar{1}34$
$\bar{2}45$
$1\bar{3}5$



The chamber fan:

Def.: Let $\mathcal{B} \in Gale(\mathcal{A})$. Then

$$B(\mathcal{C}_{\mathcal{A}}) := \{ B(\mathcal{C}_{\mathcal{A}}(T)) : T \text{ pol. subdiv. of } \mathcal{A} \}$$

$$:= \{ \{ \alpha B : \alpha \in \mathcal{C}_{\mathcal{A}}(T) \} : T \text{ pol. subdiv. of } \mathcal{A} \}$$

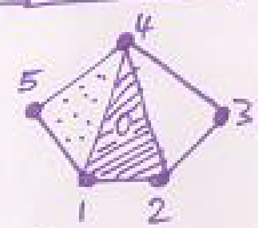
is the chamber fan of $\mathcal{A}$. $B(\mathcal{C}_{\mathcal{A}}(T))$ is the chamber of T.

Remark: (i) αB is a lin. comb. of vectors in $\mathcal{B}$ (rows of $\mathcal{B}$) with coeff. $\alpha_i = \alpha(a_i)$ f. b_i.

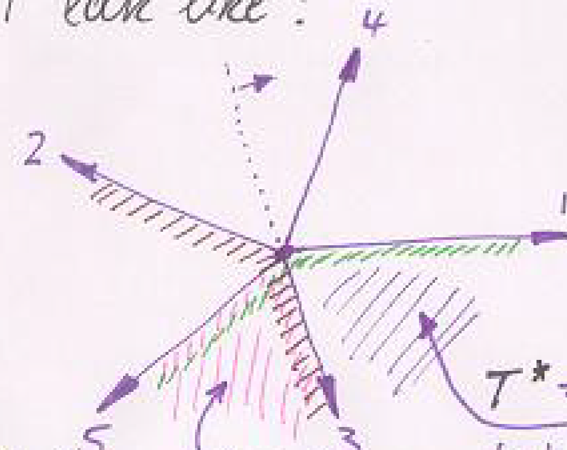
(ii) $\alpha \mathcal{B} \in \mathbb{R}^{n-r}$.

What does the chamber of T look like?

Magic example:

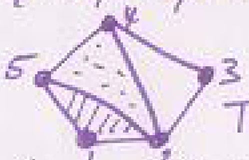


$T = \{124, 145, 234\}$



$T^* = \{35^*, 23^*, 15^*\} = \{\sigma^* := \mathcal{A} \setminus \sigma : \sigma \in T\}$

$B(\mathcal{C}_{\mathcal{A}}(T)) = \text{cone}\{b_1, b_5\} \cap \text{cone}\{b_2 b_3\} \cap \text{cone}\{b_3, b_5\}$



$T' = \{125, 245, 234\}$ $\quad B(\mathcal{C}_{\mathcal{A}}(T')) = \text{cone}\{b_3, b_4\} \cap \text{cone}\{b_1, b_3\} \cap \text{cone}\{b_1, b_5\}$

Why is this always true?

Flip from T to T' at circuit $1\bar{2}4\bar{5}$

Same as crossing cocircuit $1\bar{2}4\bar{5}$ in $\mathcal{B}$!

Equivalent heights leveling off $\sigma \in T$:

Obs.: For all heights α there is for all $\sigma \in T(\mathcal{A}, \alpha)$ another height α_σ with

$$\alpha_\sigma(a) \begin{cases} = 0 & \text{for all } a \in \sigma \\ > 0 & \text{for all } a \in \mathcal{A} \setminus \sigma \end{cases}$$

s.t. $T(\mathcal{A}, \alpha) = T(\mathcal{A}, \alpha_\sigma)$.

$\Rightarrow$ αB and $\alpha_\sigma B$ lie in the same chamber for all $\sigma \in T$

Moreover, $\alpha_\sigma B$ is a positiv comb. of $b_i \in \mathcal{B}$ with $a_i \in \mathcal{A} \setminus \sigma = \sigma^*$, for all $\sigma \in T$.

$\Rightarrow$ αB is in the intersection of all cones spanned by σ^*, $\sigma \in T$.

$\Rightarrow$ $\alpha B \in \bigcap_{\sigma \in T} \operatorname{cone} \sigma^*$, $|\sigma^*| = n - r$.

$\Rightarrow$ Chambers are intersections of simplicial cones in the Gale transform.

THM.: [Gelfand, Kapranov, Zelevinsky 1989]

(i) The chamber fan of $\mathcal{A}$ is the normal fan of a polytope in dim $n-r$.

(ii) Every full-dim. chamber corresponds to a regular triang of $\mathcal{A}$

(iii) Adjacent full-dim. chambers correspond to bistellar flips.

(iv) The face poset of the chamber fan is the opposite of the refinement poset of regular subdivisions of $\mathcal{A}$.

COR.: The Graph of ~~re~~regular triangs of $\mathcal{A}$ is the edge graph of the secondary polytope.