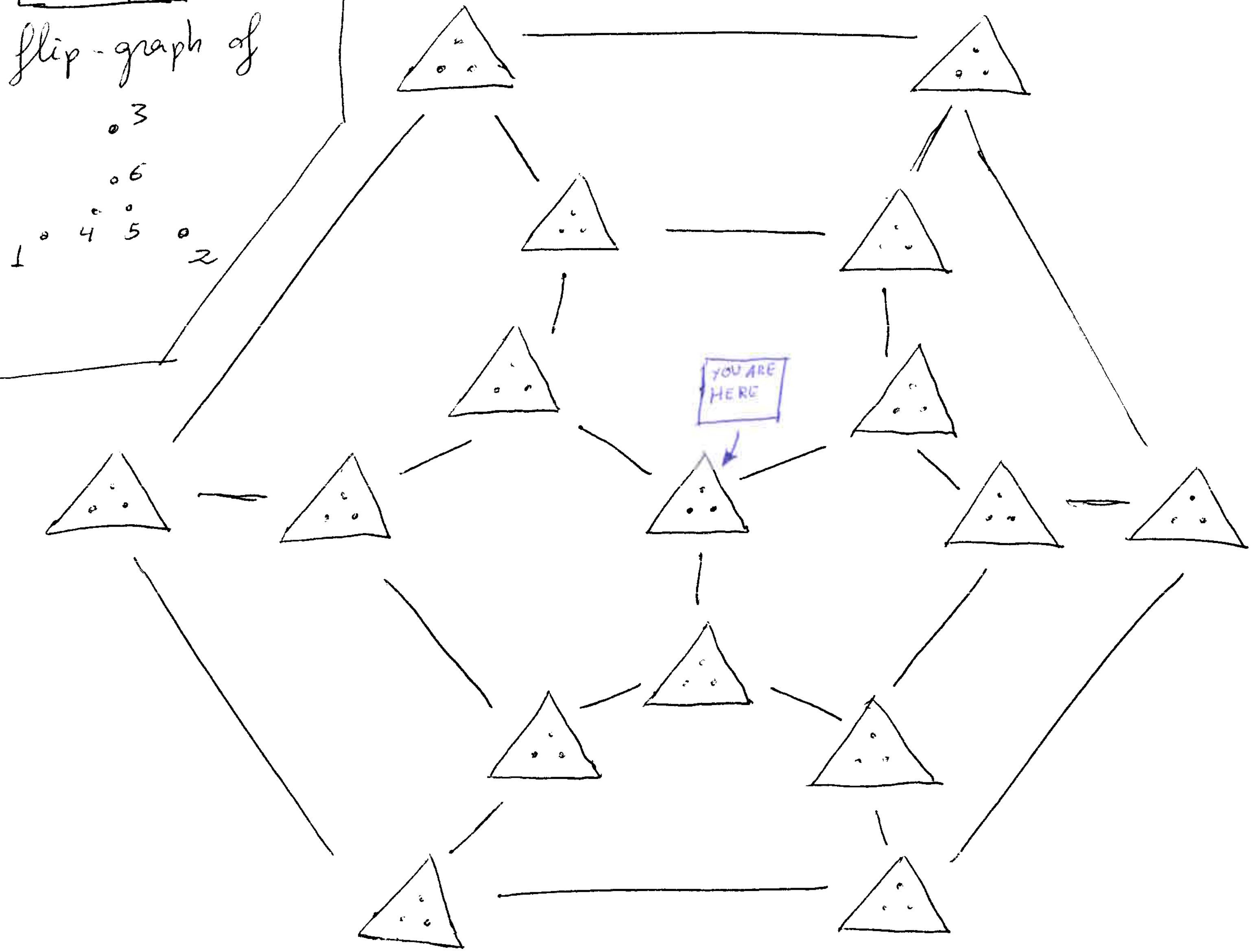
$\boxed{\text{Example}}$: the flip-graph of



"THE MOTHER OF ALL EXAMPLES"

NON-REGULAR TRIANGULATIONS

How to look for flips in a triangulation.

Prop 1: Any $(d+2)$ points which affinely span $\mathbb{R}^d$ contain a unique circuit.

Pf: The points have a unique affine dependence equation. The subset of points involved in that equation is the unique circuit in the statement $\square$

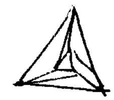
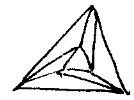
Prop 2: Every flip in a triangulation (other than the ones that insert a point) happens at the unique circuit contained in some pair of adjacent d-simplices

Pf: Let $C=(C^+, C^-)$ be the circuit at which the flip happens. "The flip inserts a point" $\equiv$ "C^+ has one element"

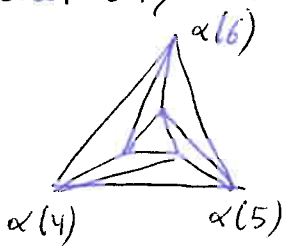
Hence, we assume C^+ to have at least two elements a and b. Then $C\setminus\{a\}$ and $C\setminus\{b\}$ are two adjacent simplices in T_C^+, and the "link condition for flips" implies they can be extended to two adjacent simplices of T. $\square$

Corollary: To look for flips in a given triangulation T you do not need to check all the circuits of A. Only look at pairs of adjacent (full dimensional) simplices. plus the flips that insert points. There is one for each unused pt.

(Remark: these are the "edges" in the adjacency graph or dual graph of the triangulation. Yes, the same graph that Jesús used to prove the "lower bound thm".)

Two proofs that the triangulations  and  are not regular:

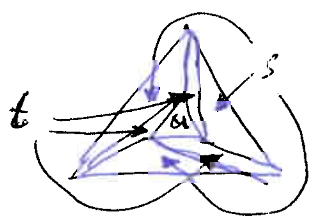
① No valid heights exist. Without loss of generality assume $\alpha(1)=\alpha(2)=\alpha(3)=0$ and then



regular would imply $\alpha(6)>\alpha(5)>\alpha(4)>\alpha(6)$ $\square$

② Compute the GKZ-vectors of these two triangulations:

$(2s+t,\ 2s+t,\ 2s+t,\ s+2t+u,\ s+2t+u,\ s+2t+u)$

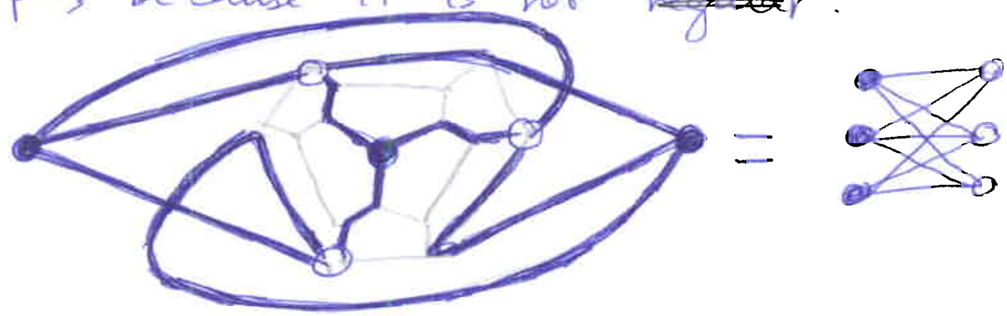


(exactly the same . . .)

And we know that the regular triangulation of a set of heights $\alpha \in \mathbb{R}^n$ is the one that minimizes scalar product of α with the GKZ-vectors. In particular, if two triangulations have the same GKZ-vector then they are both non-regular.

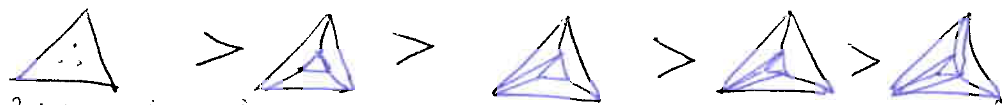
Two proofs that "the mother" has non-regular triangulations:

③ The graph of flips is not the graph of any 3-polytope, because it is not ~~regular~~ planar:



$= K_{3,3}$

④ The poset of subdivisions is not the face poset of any 3-polytope, because it has "too long chains":



$\{123456\} > \{1245, 2356, 1346\} > \{1245, 2356, 146, 136\} > \{\vdots\} > \{\vdots\}$

Remark: strictly speaking, this only shows that it has some non-regular subdivision (one of the subdivisions in the chain must be non-regular, not necessarily the triangulation).

But it is a (non-trivial) fact that every non-regular subdivision can be refined to a non-regular triangulation.

By the way: it is also a (sort of trivial) fact that every subdivision can be refined to a triangulation. That is to say, "triangulations are the minimal elements in the poset of subdivisions".

Other point sets with non-regular triangulations

Proof ④ allows you to derive that the following point sets have non-regular triangulatios "with zero work":

(a) prism + 1 interior point : let n = base of prism.
We have $2n+1$ vertices, secondary has dim $2n-3$.
Cone the interior point to all faces. This gives a non-trivial subdivision of "height" $3n-6 \Rightarrow$ chain of length $3n-5$ in the poset (and $3n-5 > 2n-3$ since $n>2$).

(b) regular icosahedron + center : divide the bdry. into 10 pairs of adjacent triangles :

and same on other side...

This gives a non-trivial subdivision of height $10 \Rightarrow$ chain of length 11 in the poset (and secondary polytope has dimension $13-4=9$)

(c) any 3-polytope with # vertices > # facets, together with an interior point (Exercise).
This happens for : cube, dodecahedron, prisms, rhombic dodecahedron, all simple polytopes except tetrahedron, ...