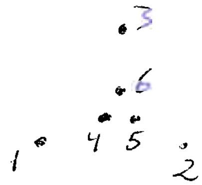
The Gale transform of M.O.A.E.

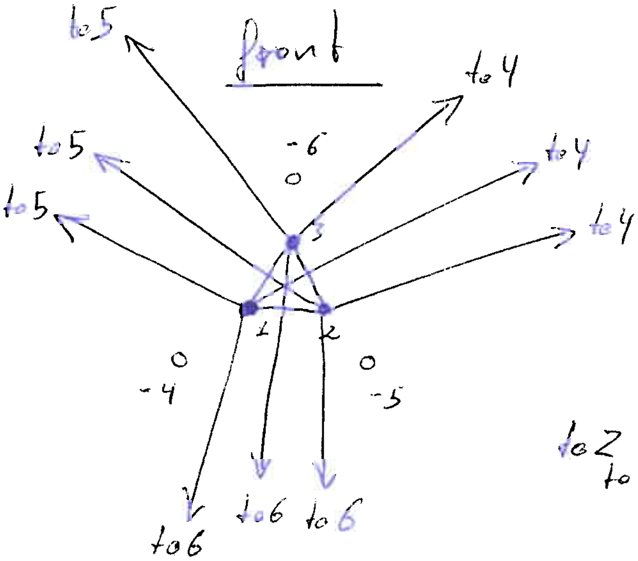
$$
A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 4 & 0 & 0 & & 2 & 1 & 1 \\ 0 & 4 & 0 & & 1 & 2 & 1 \\ 0 & 0 & 4 & & 1 & 1 & 2 \end{pmatrix} \end{array}
\qquad
B^T = \begin{array}{c} \begin{matrix} 1^* & 2^* & 3^* & & 4^* & 5^* & 6^* \end{matrix} \\ \begin{pmatrix} 2 & 1 & 1 & & -4 & 0 & 0 \\ 1 & 2 & 1 & & 0 & -4 & 0 \\ 1 & 1 & 2 & & 0 & 0 & -4 \end{pmatrix} \end{array}
$$



$\downarrow$

"affine Gale diagram":

front



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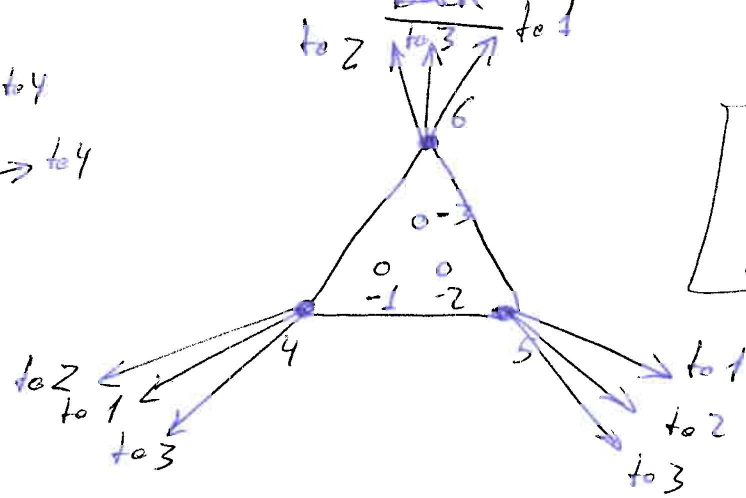
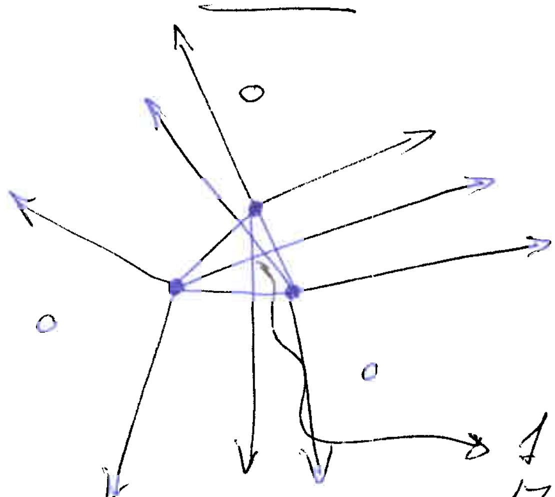


16 chambers $\Downarrow$ 16 regular triangs

What happens if we perturb the coordinates slightly?

front



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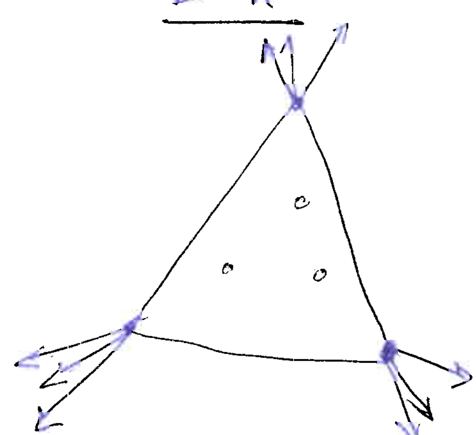


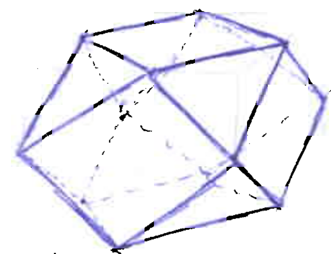
$\to$ 1 new chamber appears.

17 regular triangulations (and 1 non-regular)

Some flip-deficient triangulations (less than $n-d-1$ flips)

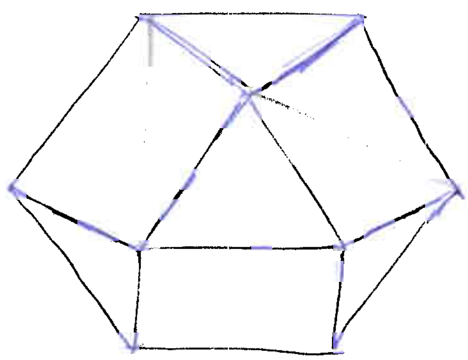
(1) The cube-octahedron:

obtained by cutting all vertices of a cube.

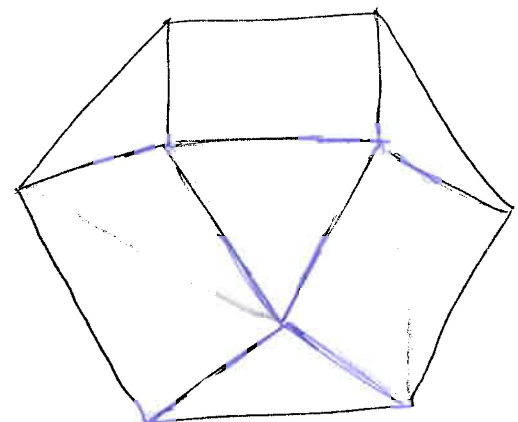


Let $A = \{\text{vertices of the cubeoctahedron}\} \cup \{O\}$ (13 vertices, $n-d-1 = 9$)

(i) Triangulate the boundary in the "most skew" way:



upper half



lower half

Remark: every vertex belongs to two squares. Diagonals are inserted so that every vertex belongs to exactly one diagonal.

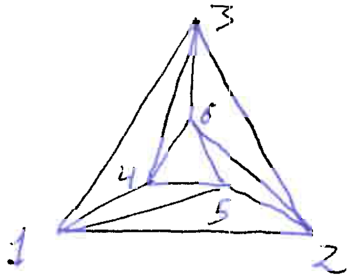
(ii) Cone the triangulation of the boundary to the point in the interior.

$\boxed{\text{This has only 6 flips!!}}$ (hence it is non-regular)

Exercise: give a direct proof that no heights exist giving this triangulation

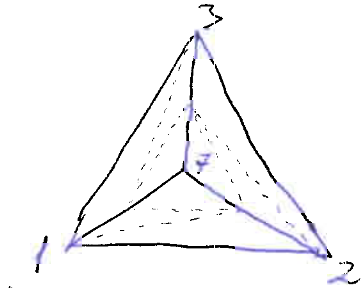
(2) "A Lawrence lift" of the mother of all examples.

(i) Start with MOAE, in a non regular triangulation:



... 3 flips so far

(ii) Cone over a point "7" above that plane.



... same 3 flips; in particular, still not flip-deficient...

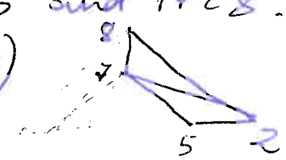
(iii) Cone over a point "8" above point 7:

... same 3 flips!!

Proof: we need to look at pairs of adjacent tetrahedra. That is, to look at interior triangles. Among those incident to MOAE we have already computed the flips, and there are three.

For the others:

- 178 (or 278 or 378): the two adjacent tetrahedra are 1782 and 1783. The circuit is (1238, 7)
  T_c^+ is not contained in our triangulation: 1237 is missing.
- 172 (or 173 or 273): tetrahedra: 1725 and 1728.
  circuit: (72, 58)



BUT: 257 is linked to 1 and 6.
278 is linked to 1 and 3

Link condition no satisfied: no flip arises

REMARK: This example can be perturbed into general position.

What is known about triangulations with small dimension and/or number of points?

$\boxed{d=0}$ n triangulations, the graph of flips is the complete graph, the secondary polytope is the $(n-1)$-simplex, all triangulations regular.

$\boxed{d=1}$ all triangulations regular. If there are no repeated points, there are 2^{n-2} triangulations. The secondary polytope is an $(n-2)$-dimensional cube. combinatorially (not metrically...)

$\boxed{d=2, \text{ points in convex position}}$ all triangs. regular. There are $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$ of them. Secondary polytope is the associahedron.

$\boxed{d=2}$ Non-regular triangulations appear, but the graph of triangulations is connected (and $(n-3)$-connected??). All triangulations have at least $n-3$ flips.

$\boxed{d=3, \text{ points in convex position}}$ all triangulations have at least $n-4$ flips. Graph connected??

$\boxed{n=d+1}$ One triangulation...

$\boxed{n=d+2}$ Two triangulations (a unique circuit). Both regular.

$\boxed{n=d+3}$ Graph of triangulations is a polygon. All triang's regular.

$\boxed{n=d+4}$ Graph is connected and 3-connected. Non-regular triangulations exist (M.O.A.E.)

Deletion and contraction in point/vector configurations

Deletion: if $p \in A$, the <u>deletion of p in A</u> is the point/vector configuration $A \setminus \{p\}$.

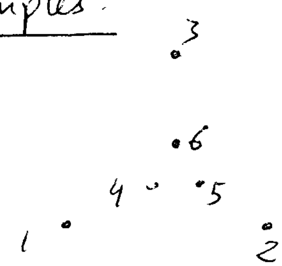
Contraction: if $p \in A$, the <u>contraction of A at p</u> (or of p in A) is the ~~point~~/vector configuration:

- $\{ q - p \mid q \in A \setminus \{p\} \}$ if A is a point conf.
- $\Big\{ \underbrace{\vec{q} - \frac{\vec{p}\cdot\vec{q}}{\|\vec{p}\|\,\|\vec{q}\|}\,\vec{p}}\ \Big|\ \vec{q} \in A \setminus \{\vec{p}\} \Big\}$ if A is a vector conf.

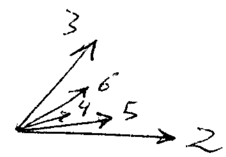
↳ orthogonal projection of $\vec{q}$ to the linear hyperplane orthogonal to $\vec{p}$.

Examples:

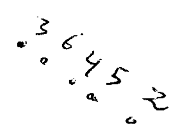
$A =$



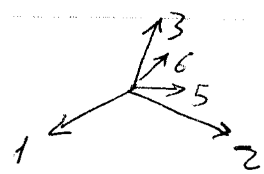
$A/\{1\} =$



$\sim$



$A/\{4\} =$



→ a non-pointed vector configuration ⇒ not equivalent to a point configuration.

Basic property: if T is a triangulation of A and $p \in A$ is used in T then:

$\mathrm{link}_T(p)$ <u>is</u> a triangulation of $A/\{p\}$

Remarks:

① If we are given a triangulation of A, that induces a triangulation of $A/\{p\}$ (the link) but it may or may not induce (be extendable to) a triangulation of $A \setminus \{p\}$.

Example: let $A =$ "twisted prism" $\cup$ $\{$interior point$\}$

If we use the three "almost flat" tetrahedra in the three "almost quadrilateral faces" of the prism, the interior space is a Schönhardt polyhedron.

If we cone the boundary of that Schönhardt to the interior point, we have a triangulation of A in which the deletion of A cannot be completed to a triangulation of $A \setminus \{p\}$

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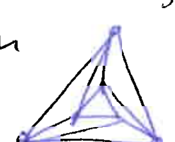
② If we are given a triangulation of $A \setminus \{p\}$, that induces a triangulation of A $\left(\begin{matrix}\text{cone } p \text{ to} \\ \text{what it "sees"}\end{matrix}\right)$ but if we are given a triangulation of $A/\{p\}$, that may or may not induce (be extendable to) a triangulation of A.

Example: Let $A = \{1, 2, 3\} \cup \{4, 5, 6\} \cup \{7\}$

$\{1,2,3\}$ at height 0; $\{4,5,6\}$ at height 1; 7 at height 1000

Then $A/7 =$ MOAE. The triangulation



cannot be extended to a triangulation of A. (Essentially, it produces a non-triangulable Schönhardt).

But: regular triangulations behave well:

Lemma 1: Given a regular triangulation T of A:

((a) $\mathrm{lk}_T(p)$ is a regular triangulation of $A/\{p\}$)

(b) deleting p in T can be extended to a regular triangulation of $A\setminus\{p\}$

Lemma 2:

(a) Given a regular triangulation t of $A/\{p\}$, there is a regular triangulation T of A with $t = \mathrm{lk}_T(p)$.

((b) Given a regular triangulation t of $A\setminus\{p\}$, there is a regular triangulation T of A with $t \subseteq T$.)

Proof(1): restrict to $A/\{p\}$ and $A\setminus\{p\}$ the heights used in A.

Proof(2): (a) use ~~the~~ in A the same heights as in $A/\{p\}$, with height 0 for p itself.

(b) use in A the same heights as in $A\setminus\{p\}$, with very high height for p itself.