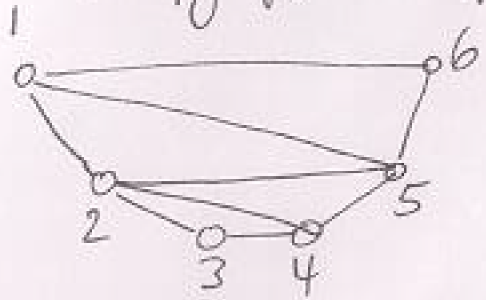
More structure for n-gons = $C(n,2)$:

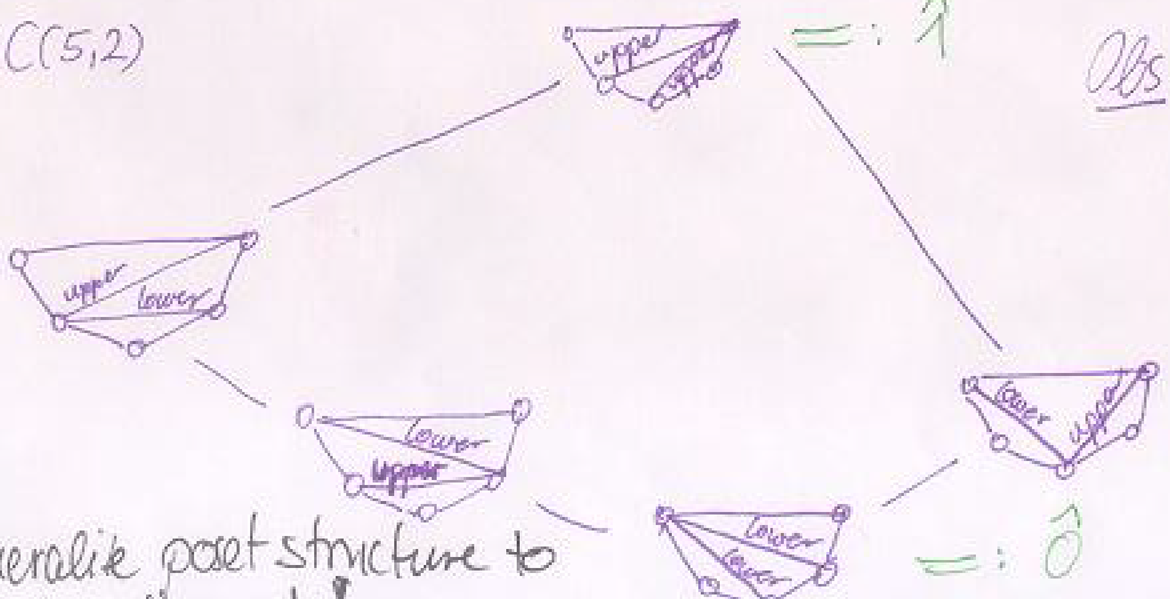
Idea: Classify flips in upflips and downflips.



$T = \{156, 125, 245, 234\}$

Def: upflip : replace lower diag. by upper diag.
downflip : vice versa.

Example: $C(5,2)$



Obs: (i) bounded poset
(ii) Graph of triang's
=
Hasse-diagram of poset
(iii) l.p.: Graph of triang's connected.

Goal: Generalize poset structure to dim d!

posets:

L6

· **Def**: • A partially ordered set $(P,\leq)$ is a set P with a binary relation "$\leq$" s.t.

(i) $x \leq x \quad \forall x \in P$

(ii) $\left.\begin{array}{l} x \leq y \\ y \leq x \end{array}\right\} \Rightarrow x = y$

(iii) $x \leq y,\ y \leq z \Rightarrow x \leq z$

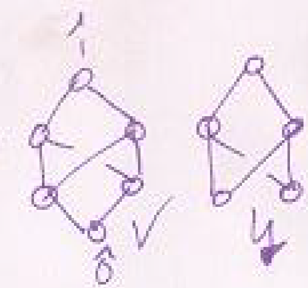
• $x \leq y,\ x \neq y \Leftrightarrow: x < y$.

• A covering relation in $(P,\leq)$ is a relation $x \lessdot y$ s.t. $\nexists\, z : x < z < y$

A Hasse-diagram of P is a graph $G_P = (P, E)$, directed, with $(x,y) \in E \Leftrightarrow x \lessdot y$

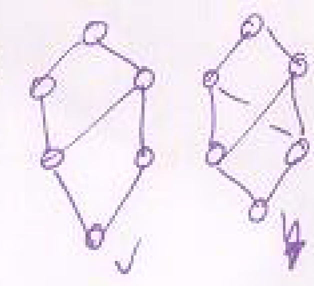
• Given an acyclic {relation "<" / directed graph} G, the transitive closure of G is the poset $(V(G), \leq)$ with

$x \leq y :\Leftrightarrow$ {a chain of relations $x \lessdot z_1 \lessdot \dots \lessdot z_k \lessdot y$. / $\exists$ directed path from x to y in G.}

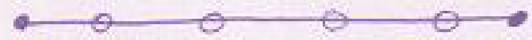
Example: 

• P bounded if $\exists\, \hat{0}, \hat{1} \in P$ with $\hat{0} \leq x \leq \hat{1} \quad \forall x \in P$.

• P is a lattice if there is a $\underbrace{\text{unique smallest upper}}_{x \vee y}$ and a $\underbrace{\text{unique largest lower bound}}_{x \wedge y}$ for any pair of elements x, y.

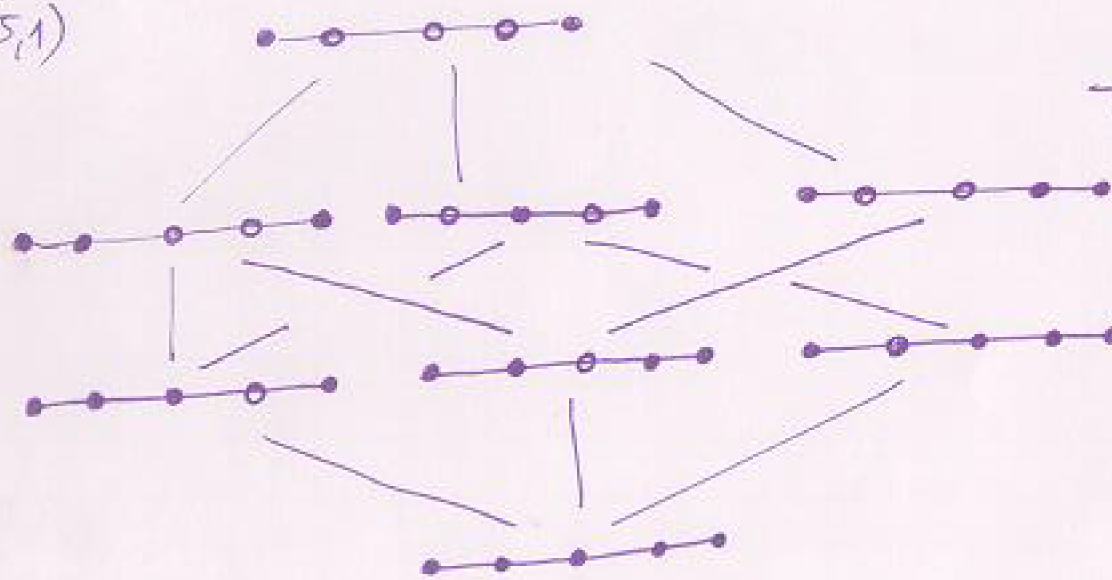
Example: 

The projective view of up- and downflips:

$C(n,1)$:  flips $\hat{=}$ add/remove point in interior

Def.: An upflip in $C(n,1)$ removes a point,
a downflip in $C(n,1)$ adds a point.

Example: $C(5,1)$

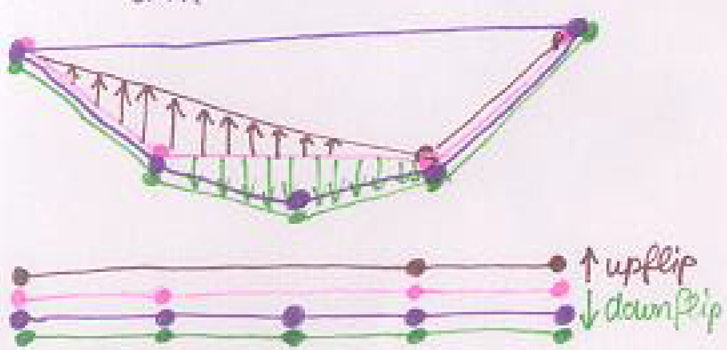


→ bounded poset!

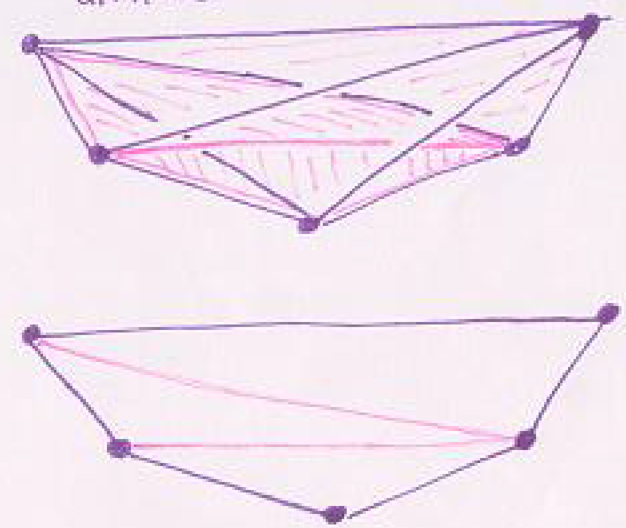
Q: Can we unify definitions of up- and downflips in dim 1 and dim 2?

A: Yes!

dim 1



dim 2



Obs.: (i) A T–T' flip is an upflip if the lifting of T' is weakly higher than the lifting of T. It is a downflip otherwise (in dim 1 and 2).

(ii) If the triangs T and T' are connected by a flip, then exactly one $(d+1)$-simplex fits between their liftings.

(iii) If between the liftings of T and T' there is exactly one $(d+1)$-simplex then T and T' differ by a flip.

(iv) The vertices of this simplex form the circuit in $\mathbb{R}^d$ that supports the flip.

Def.: $$p_{n,d}: \begin{cases} C(n,d) \longrightarrow C(n,d-1) \\ x \longmapsto (x_0 : x_1 : \cdots : x_{d-1})^T \end{cases}$$ "forget last coordinate"

is the canonical projection of $C(n,d)$.

For T triang of $C(n,d)$ let

$$s_T: \begin{cases} C(n,d) \longrightarrow C(n,d+1) \\ v_d(i) \longmapsto v_{d+1}(i), \quad i = 1,\dots,n, \end{cases}$$ affinely extended on simplices of T.

This is the characteristic section of T in $C(n,d+1)$

The first Stasheff-Tamari poset $\mathcal{S}_1(n,d)$:

Def.: $\mathcal{S}_1(n,d)$ is the poset on all triang's of $C(n,d)$ given by the transitive closure of

$T_1 < T_2$:iff $T_1 \xrightarrow[\text{upflip}]{} T_2$, where an upflip is a flip that replaces in T_1 the lower facets of a $(d+1)$-simplex in $C(n,d+1)$ by its upper facets, yielding T_2.

[Edelman, Reiner 1995]

Thm.: [TAMARI, HUANG]

$T_{n-2} := \mathcal{S}_1(n,2)$ is a lattice $\forall n$. "Tamari lattice"

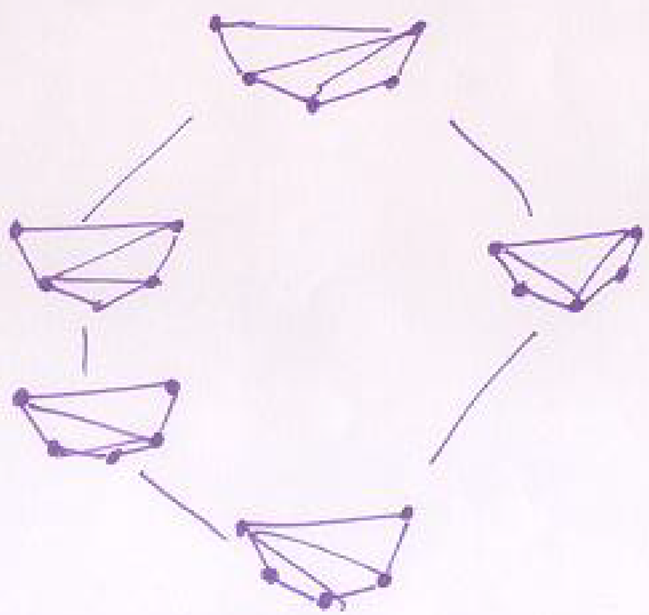
THM.: [EDELMAN, REINER]

$\mathcal{S}_1(n,3)$ is a lattice $\forall n$.

COR.: $\mathcal{G}_{\mathcal{C}(n,3)}$ is connected.

Rem.: > 60 equivalent comb. structures [Stanley] for $\mathcal{S}_1(n,2) = T_{n-2}$

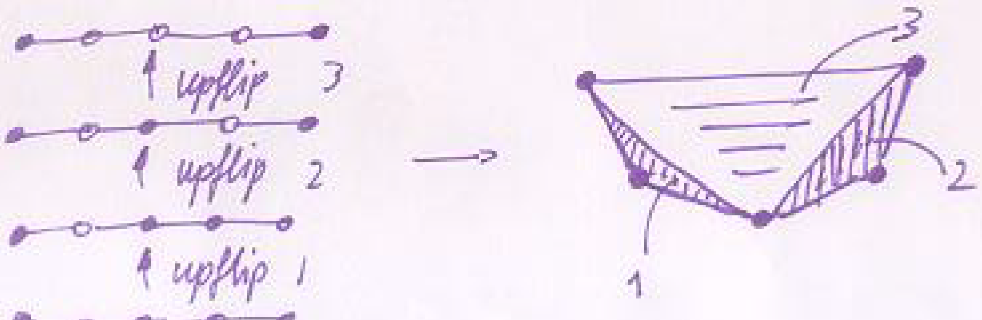
Example:



Lattice ✓

Observations:

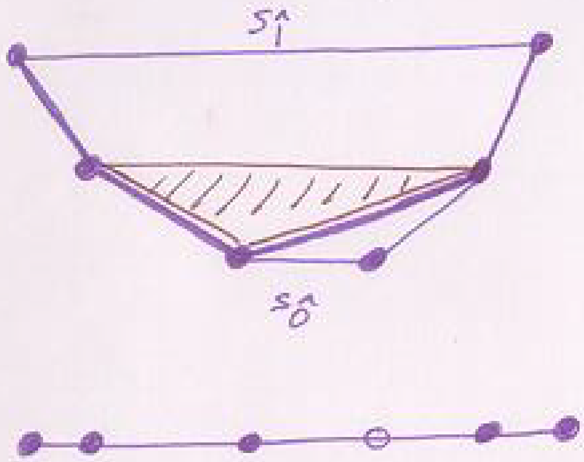
1) Every triangulation can be recovered from its characteristic section $\to$ can identify T with $s_T(C(n,d))$.
2) The set of upper facets of $C(n,d+1)$ is a triang. of $C(n,d)$: $\hat{1}$.
3) — " — " — " lower — " — " — " — " " — " — " — " — : $\hat{0}$.
4) There is no upflip in $\hat{1}$, there is no downflip in $\hat{0}$.
5) If every triang. except $\hat{0}$ has a downflip then $\hat{0}$ is the unique minimum of $\mathcal{S}_1(n,d)$.
If every triang. except $\hat{1}$ has an upflip then $\hat{1}$ is the unique maximum of $\mathcal{S}_1(n,d)$.
6) An upflip at a $(d+1)$-simplex replaces its lower facets with its upper facets.
A downflip at a $(d+1)$-simplex replaces its upper facets with its lower facets.
7) A chain of upflips from $\hat{0}$ to $\hat{1}$ yields a triangulation of $C(n,d+1)$.
8) We can build such a triang. by stacking simplices vertically in the order of the chain.
(like tetris without rotation)



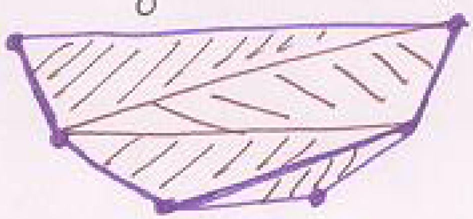
How can we prove that $\mathcal{S}_1(n,d)$ is bounded?

Proof programme:

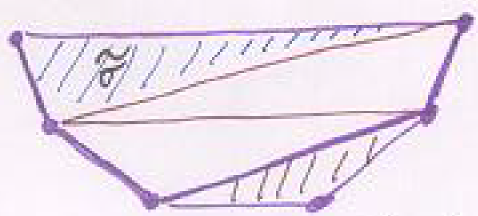
1. For every triang T of $C(n,d)$ $T \neq \hat{1}$ construct an upflip.
2. For this, find a $(d+1)$-simplex in $C(n,d+1)$ with all its lower facets in $s_T(C(n,d))$



3. For this, triangulate $C(n,d+1)$ with $s_T(C(n,d))$ as a subcomplex. !



4. Pick a simplex $\tilde{\sigma}$ above $s_T(C(n,d))$.



5. While $\tilde{\sigma}$ has not all lower facets in $s_T(C(n,d))$, set $\tilde{\sigma}'$ to a simplex adjacent at a lower facet of $\tilde{\sigma}$ and set $\tilde{\sigma}$ to $\tilde{\sigma}'$. !

6. Finitely many simplices $\Rightarrow$ find upflip. □

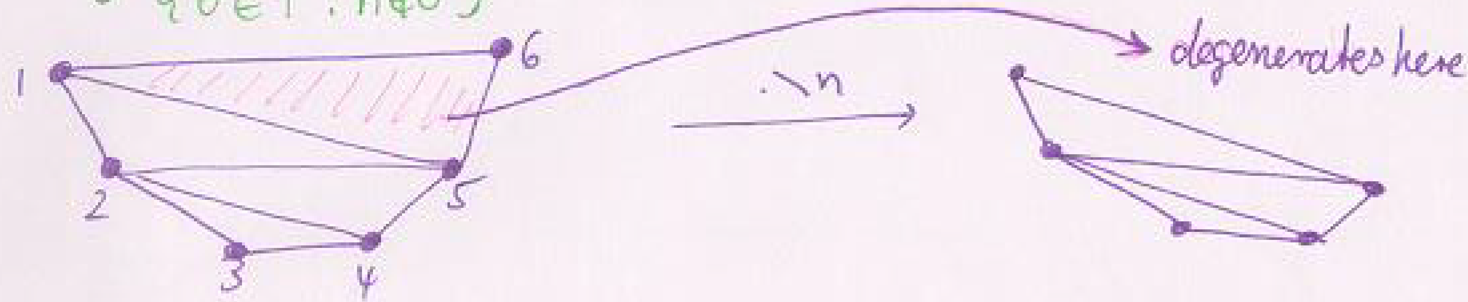
Constructions I:

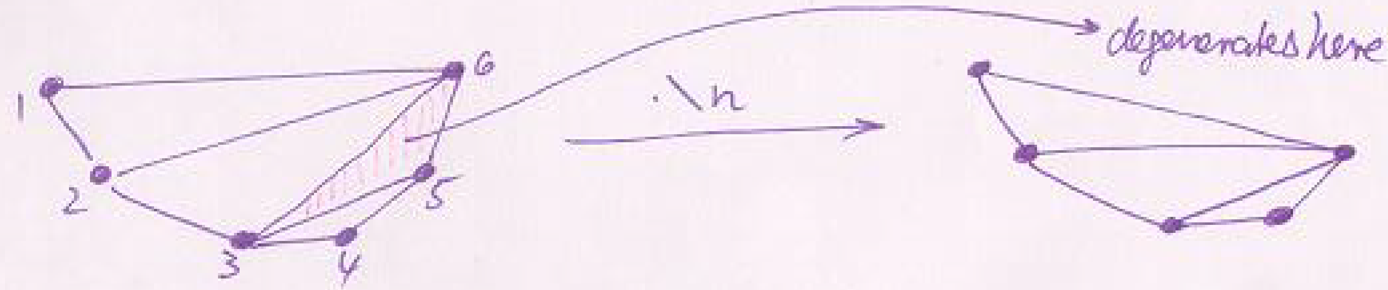
Def.: For a triangulation T of $C(n,d)$ let

$$T\setminus n := \{\sigma\setminus n \cup \{n-1\} : \underset{n\in\sigma}{\sigma\in T}\} \cup \{\sigma\in T : n\notin\sigma\}$$

(only full-dim. simplices kept)

Example:





Obs.: $T\setminus n$ can be obtained from T by continuously sliding vertex n to vertex $n-1$ and removing all degenerated simplices.

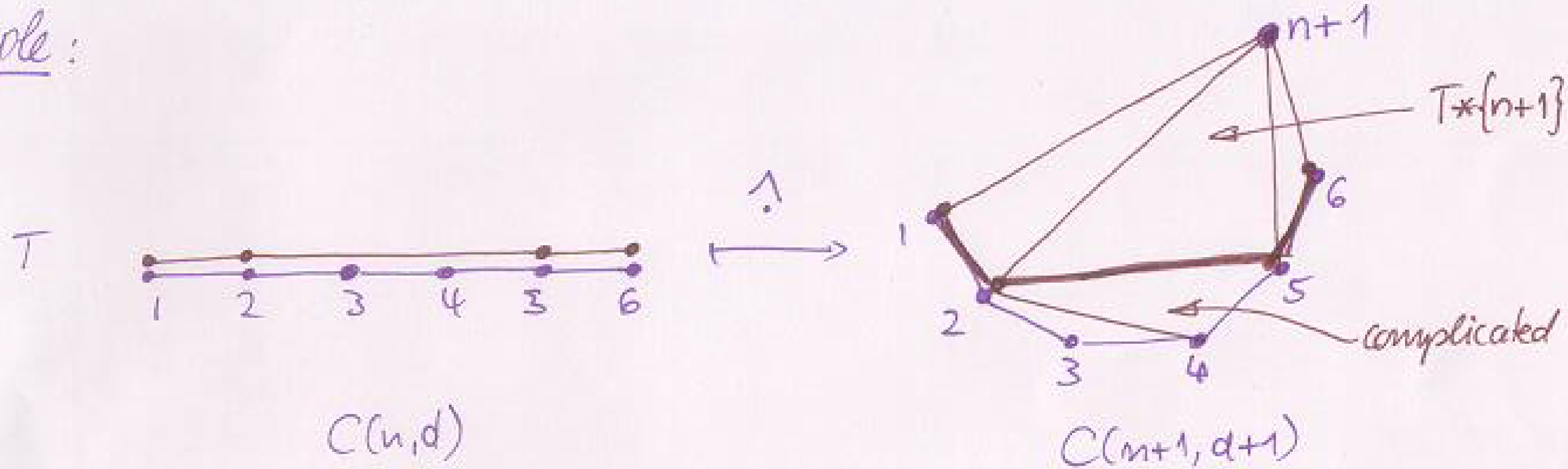
Prop.: $T\setminus n$ is a triang. of $C(n-1,d)$

Constructions 2:

Def.: For a triang. T of $C(n,d)$ let

$$\hat{T} := \{T * \{n+1\}\} \cup \{\text{complicated stuff}\}$$

Example:



Prop.: $\hat{T}$ is a triang. of $C(n+1,d+1)$ containing $S_T(C(n,d))$ as a subcomplex.

Cor.: T triang. of $C(n,d) \Longrightarrow \hat{T} \setminus \{n+1\}$ is a triang. of $C(n,d+1)$ containing $S_T(C(n,d))$ as a subcomplex.

The full theorem:

THM: $\mathcal{S}_1(n,d)$ is bounded. Moreover:
The triang's of $C(n,d+1)$ are in one-to-one correspondence with the maximal chains in $\mathcal{S}_1(n,d)$.

COR: (i) $\# \hat{0} \geq \# T \geq \# \hat{1}$

(ii) for even d, $\# \hat{0} = \# \hat{1}$, so $\# T = \# \hat{0} = \# \hat{1}$.

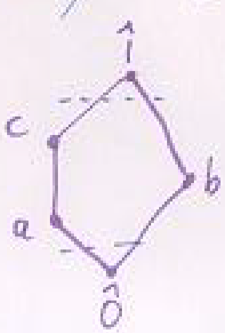
(iii) $\# \hat{0}$ in dim $d+1$ $\geq$ length of a max. chain in dim d $\geq \# \hat{1}$ in dim $d+1$

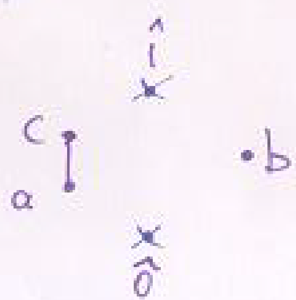
Q: Is $\mathcal{S}_1(n,d)$ a lattice?

A: No: Computer calculated counterexample for $C(9,5)$ and $C(9,4)$.

THM: $\Delta(\overline{\mathcal{S}_1(n,d)})$ ($:=$ simpl. complex of chains without $\hat{0},\hat{1}$) $\simeq S^{n-d-3}$.

Example:





$\cdot b \simeq S^0$.