Unfriendly spaces of triangulations

July 28, 2003

(A) Triangulations with very few flips. ($d=3$)

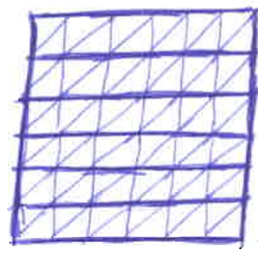
Warm-up exercise

Let $k \in \mathbb{N}$. Let $A \subseteq \mathbb{R}^3$ consist of the following $(2k+1)^2 + 2k$ points:

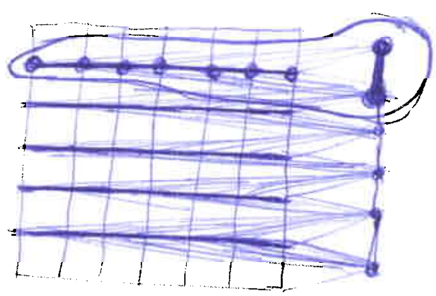
- The $(2k+1)^2$ points $\{(i,j,0) \mid i,j = -k, -k+1, \dots, k-1, k\}$ (a square grid of size $2k$).
- The $2k$ points $\{(0, j, 1),\ j = -k, k+1, \dots, k-1\}$.

Triangulate A as follows:

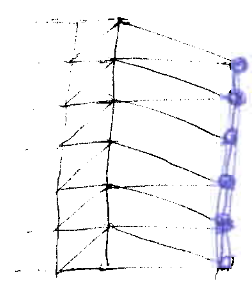
- Refine the square grid using the "positive diagonals":



- Join the j-th "horizontal strip" to the j-th point $(0, i, 1)$
- Fill-in the gaps, joining the j-th and $(j+1)$-th points $(0, *, 1)$ with all the edges between the j-th and $(j+1)$-th horizontal strips:



(or



Question: How many flips does this triangulation have?

Answer:

- $2k(4k-1)$ in circuits on the XY-plane.
- "Only" $4k-2$ in circuits involving the points $(0,j,1)$

Crucial observation:

If we call $n = |A| = (2k+1)^2 + 2k$, the planar triangulated grid had "about $3n$" triangulations and with the 3rd dimension we have killed about n of them.

Actually, we can kill about another n of them using another line of points behind the planar triangulated grid:

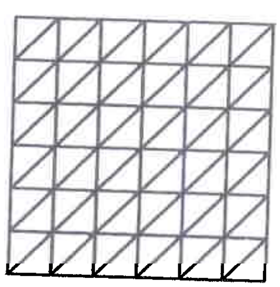


Figure 7.2: A triangular grid in the plane has about $3n$ flips, but...

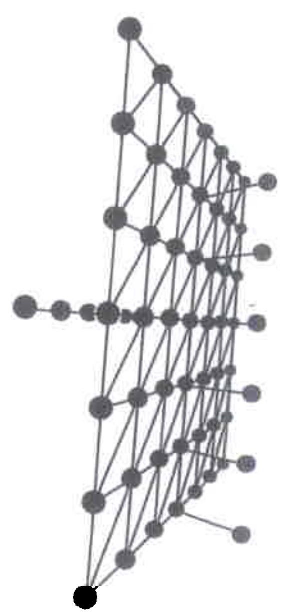
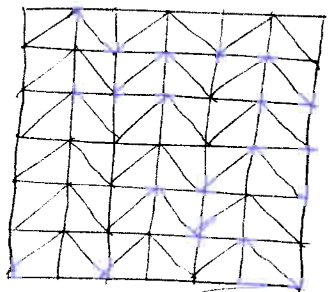
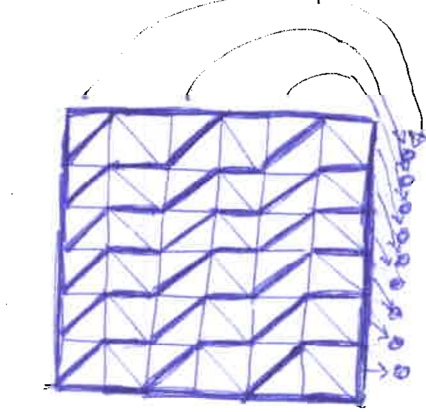


Figure 7.3: ...about $2n$ of them can be killed by the third dimension.

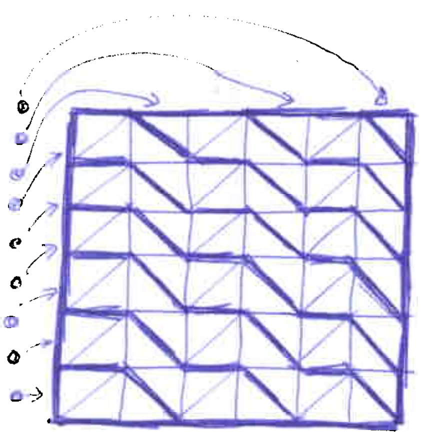
What if we start with something with less flips in 2D? For example:



This has about $2n$ flips.



We can kill about n of them joining zig-zag strips to 9 pts above the plane.



... and about another n joining zig-zag strips in the other direction to 9 pts below the plane.

After this is done, all the flips on circuits contained in the XY-plane are killed. But, have we introduced new flips?

How many?

The count of flips (1)

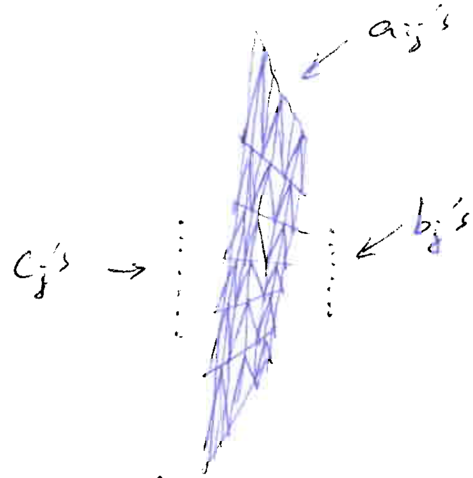
Preliminary remarks:

(1) notation: - Let a_{ij}, $i,j = 0, \dots, 2k+1$ be the points in the planar grid.

- Let b_j, $j = 1, \dots, \frac{3k}{2}$ be the pts in front of the plane. They lie on a line parallel to the vector $(0,1,0)$.

(In other words $b_j = (0, t_j, 1)$ with $t_1 < t_2 < \dots < t_{\frac{3k}{2}}$)

- Let c_j, $j = 1, \dots, \frac{3k}{2}$ be the pts behind the xy-plane. They lie on a line parallel to the previous one. For example, $c_j = (0, t_j, -1)$ with $t_1 < t_2 < \dots < t_{\frac{3k}{2}}$.



(2) For reasons that will become apparent later, we place the b_j's very close to one another, and the same for the c_j's. Our goal is that

$\boxed{\text{ALL THE SEGMENTS } \overline{b_i c_j} \text{ intersect the planar grid in one and the same triangle}}$

The count of flips (2)

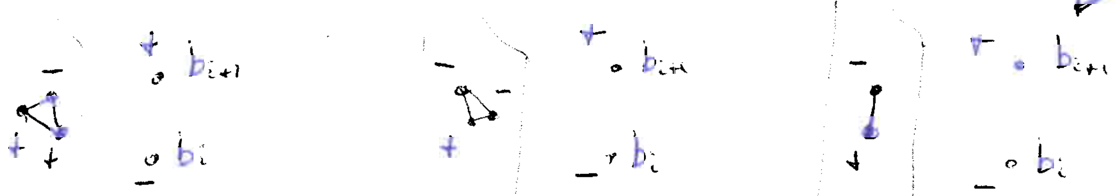
We count flips by looking at pairs of adjacent tetrahedra. Equivalently, by looking at interior triangles:

(a) If our interior triangle τ is in the planar grid, its incident tetrahedra are $\tau \cup \{b_i\}$ and $\tau \cup \{c_j\}$ for some b_i and c_j. Since the edge $\overline{b_i c_j}$ is not in our triangulation, we need that edge to intersect the triangle τ in order to have our circuit triangulated. By remark ② this happens for only one τ. Hence, we have (at most) <u>one flip of this type</u>.

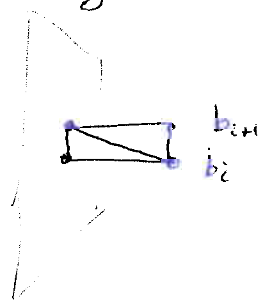
(b) If our interior triangle τ uses two points in the grid and one outside (say b_i), there are two possibilities:

(b.1) the other two points of τ form an edge interior to the "zone" joined to b_i. Then the circuit contained in our pair of tetrahedra is contained in the planar grid and "by construction" there is <u>no flip</u> in it.

(b2) the other two points of τ form an edge in the common boundary of the zone joined to b_i and the zone joined to (say) b_{i+1} (the other case, b_{i-1}, is treated equally). Then the circuit looks as follows:



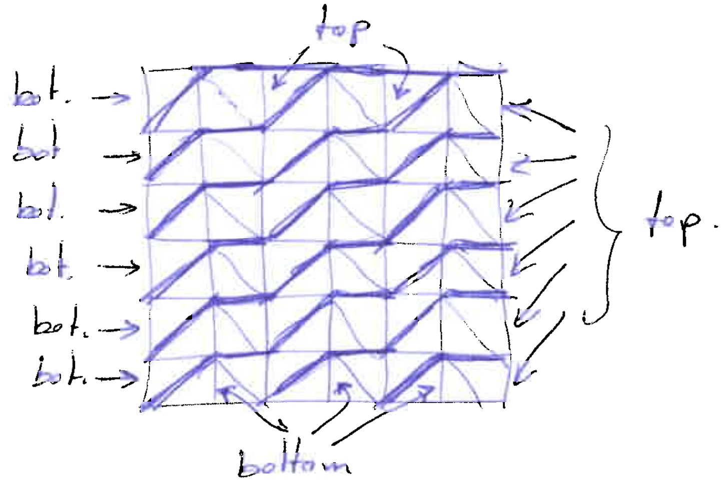
The triangulation of our circuit is



This flip exists if the link condition is satisfied.

Equivalently: we have one such flip for each vertical segment in the grid for which "all the" grid triangles incident to it have the top edge in the boundary between the b_i and the b_{i+1} zone.

Changing "top edge" to "bottom edge" gives the case of b_i and b_{i-1}. The edges in question are marked below:



This gives $6k-1$ flips "on the b side" and we have to count another $6k-1$ "on the c side".

(c) If our interior triangle τ uses one point in the grid and two outside (say b_i and b_{i+1}), then the incident tetrahedra are two consecutive edges in the common boundary between the b_i and the b_{i+1} zones, both joined to $\overline{b_i b_{i+1}}$. Cases:

(c.1) the edges are in the zig-zag

No flip is produced, because we do not have a triangulation of the circuit ("long edge" missing )

(c.2) the edges are collinear

No flip is produced, because they do not have the same link.

<u>Conclusion:</u> we have $\simeq 4k^2$ points and $\simeq 12k$ flips.

That is to say, if we call $n = (2k+1)^2 + 4k$ = # of points

$$\boxed{\text{We have } \simeq 6\sqrt{n} \text{ flips}}$$

(The construction can be modified to give $\simeq 4\sqrt{n}$)