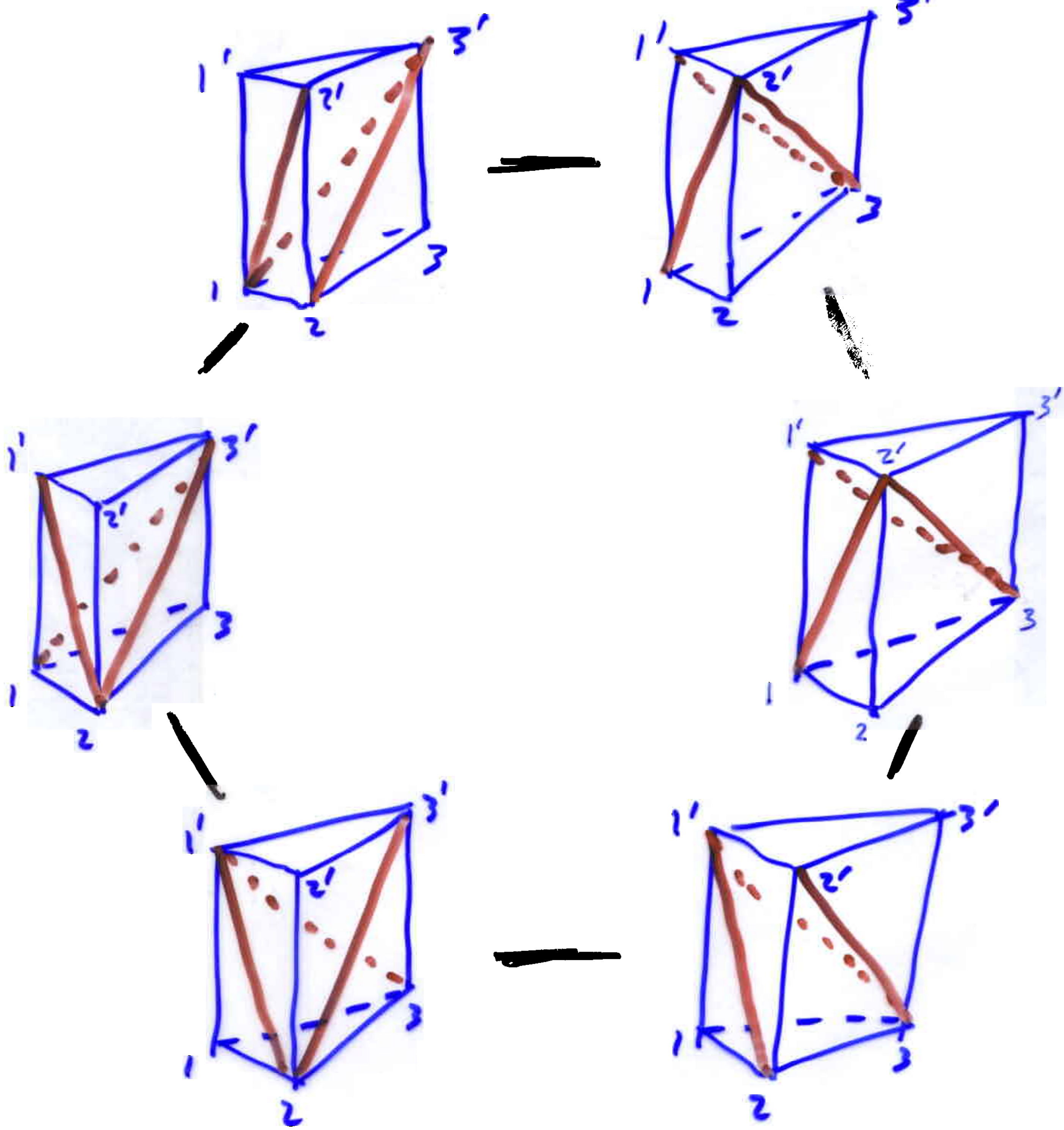
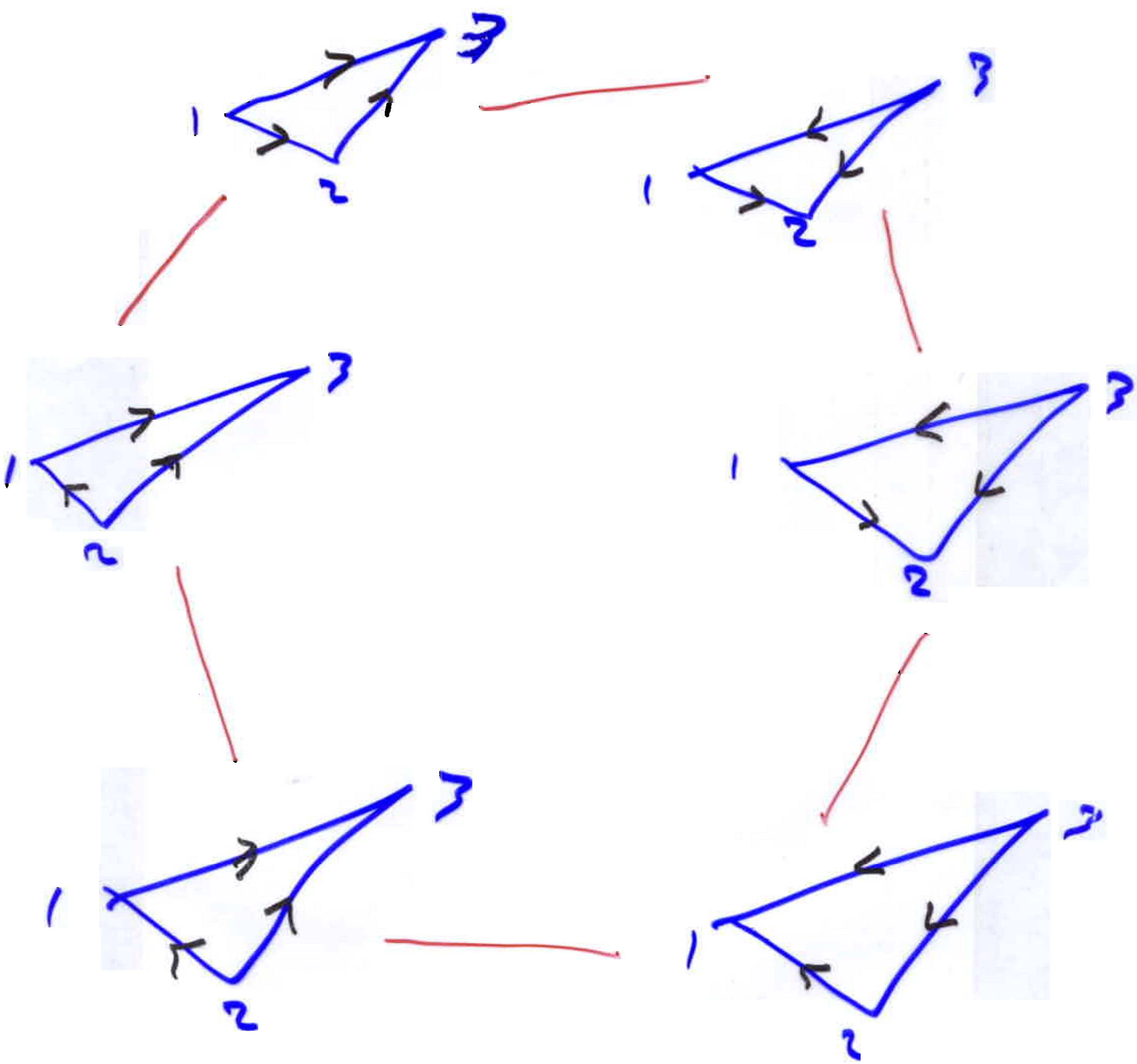
IV. The construction.

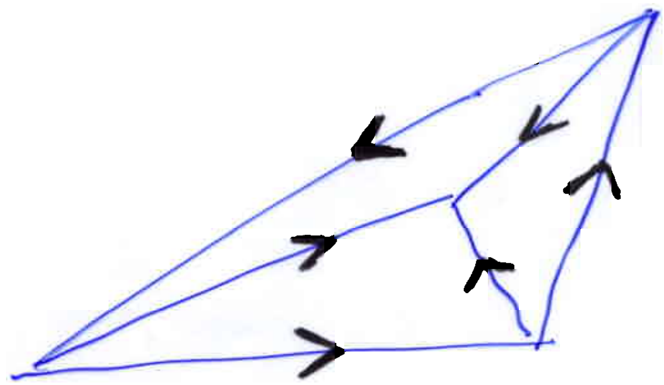
$\boxed{\text{Basic building block:}}$ triangulations of a triangular prism:

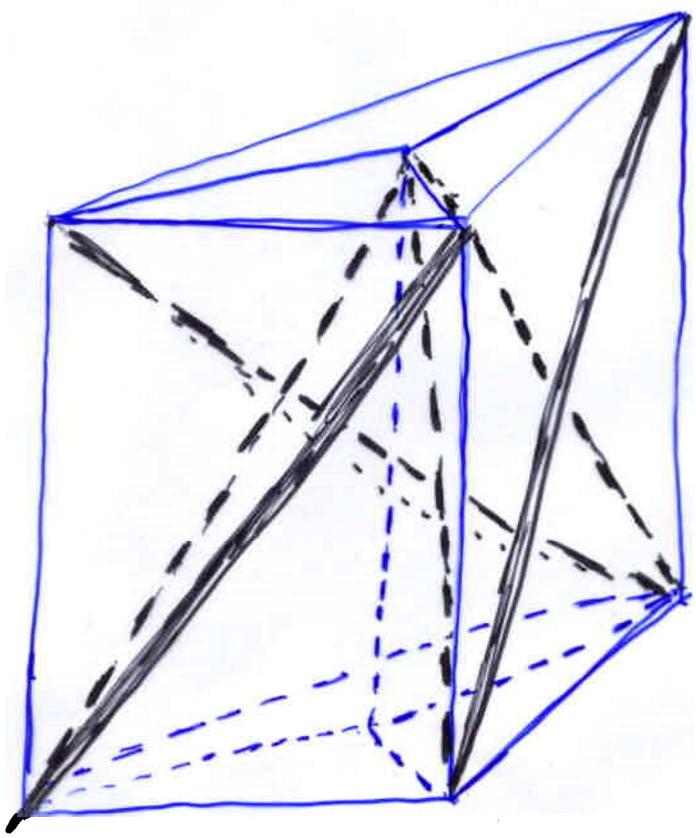


A more compact version of the same picture:



... and we can glue the acyclic orientations of different triangles of a 2-dimensional simplicial complex :





... as long as the orientation of edges is locally acyclic (acyclic on every triangle of the complex).

$\boxed{\text{Key idea:}}$ Let P be a convex polytope. Assume all its 2-faces are triangles. Then:

- Every triangulation of $P \times I$ induces a triangulation of $\tau \times I$, where τ is a 2-face of P.
- Triangulations of $\tau \times I$ correspond to acyclic orientations of the edges of τ.
- Triangulations of $\bigcup_{\tau \text{ a 2-face}} (\tau \times I)$ correspond to <u>locally acyclic orientations</u> of the 1-skeleton of P.
- Flips in triangulations of $P \times I$ either do not change the triangulation of $\bigcup_{\tau} (\tau \times I)$ or correspond to the reversal of a single edge in the locally acyclic orientation.

Key picture:

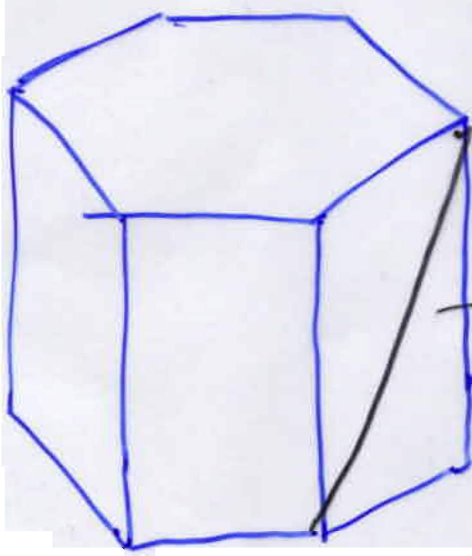
$P \subseteq \mathbb{R}^d$

τ, a triangular 2-face of P

acyclic orientations of τ

$\updownarrow$

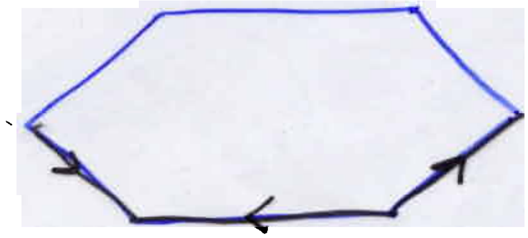
triangulations of $\tau \times I$



$\tau \times I$, a "triangular prism" 3-face of $P \times I$

$P \times I \subseteq \mathbb{R}^{d+1}$

Key picture:



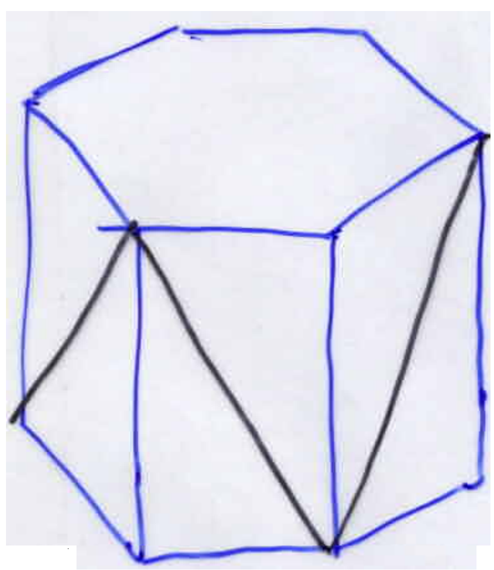
$P \subseteq \mathbb{R}^d$

K, a set of triangular 2-faces of P

locally acyclic orientations of the 1-skeleton of K

$\updownarrow$

triangulations of $K \times I$



$P \times I \subseteq \mathbb{R}^{d+1}$

Beware: not all triangulations of $K \times I$ need to be extendable to the whole polytope $P \times I$.

But clearly: every triangulation of $P \times I$ induces a l.a.o. of K. A flip either leaves the l.a.o. unchanged or reverses a single edge.

Our goal Find a polytope P with triangular 2-faces, and a locally acyclic orientation of its 1-skeleton <u>without reversible edges</u> (i.e: in which the reversal of any edge produces a triangle cycle : )

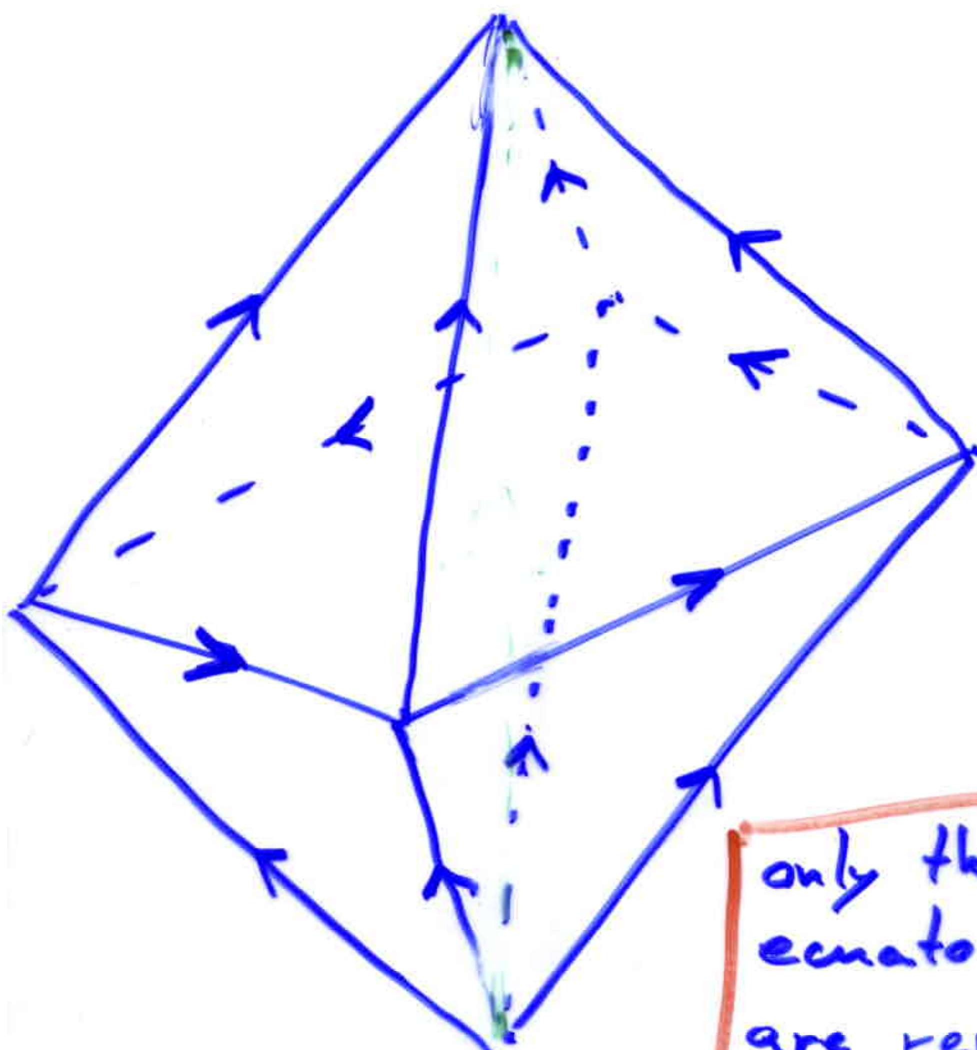
<u>Remark</u>: each 2-face prevents the reversal of at most one edge. In particular, in P we need:

$$\# \text{2-faces} \geq \# \text{edges}$$

which is impossible in 3D :

$$(e - f = v - 2)$$

Example: if P = octahedron, the best we can hope for is getting $v - 2 = 4$ reversible edges. And we can actually achieve that:



only the ecuatorial edges are reversible.

Our polytope: the 24-cell.

It is a 4-dimensional polytope with face vector $(24, 96, 96, 24)$

24 vertices

96 edges

96 triangles

24 octahedra.

Vertices: $(\pm1, \pm1, \pm1, \pm1) \rightarrow$ 16 vertices

$(\pm2, 0, 0, 0) \rightarrow$ 2 vertices

$(0, \pm2, 0, 0) \rightarrow$ " "

$(0, 0, \pm2, 0) \rightarrow$ " "

$(0, 0, 0, \pm2) \rightarrow$ " "

or: $(\pm1, \pm1, 0, 0) \rightarrow$ 4 vertices

$(\pm1, 0, \pm1, 0) \rightarrow$ " "

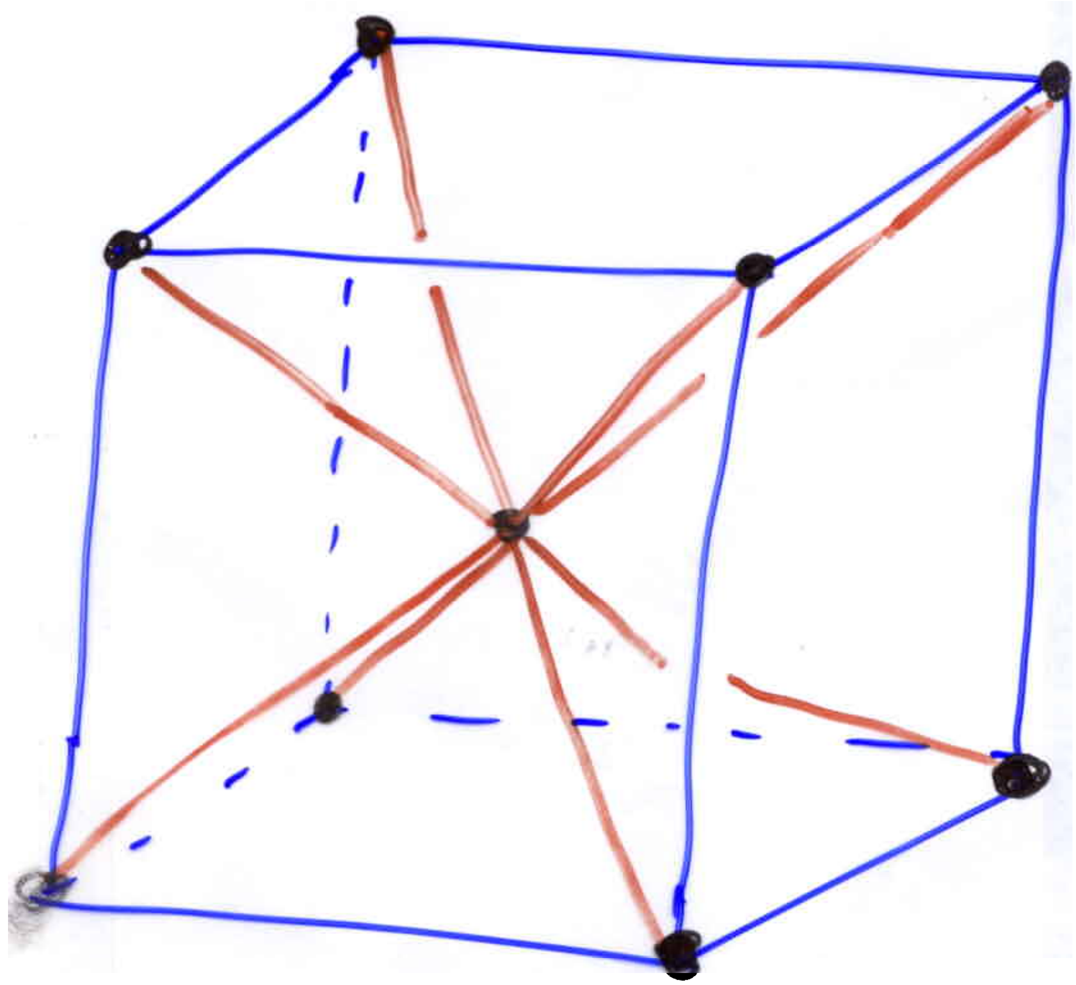
$(\pm1, 0, 0, \pm1) \rightarrow$ " "

$(0, \pm1, \pm1, 0) \rightarrow$ " "

$(0, \pm1, 0, \pm1) \rightarrow$ " "

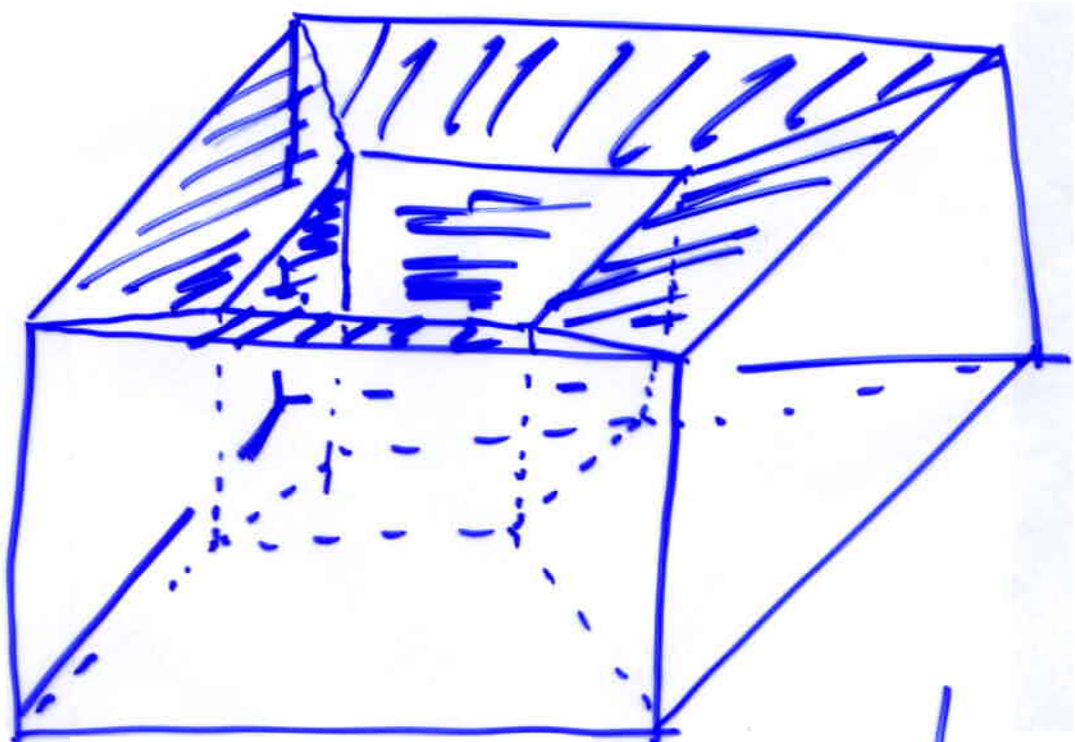
$(0, 0, \pm1, \pm1) \rightarrow$ " "

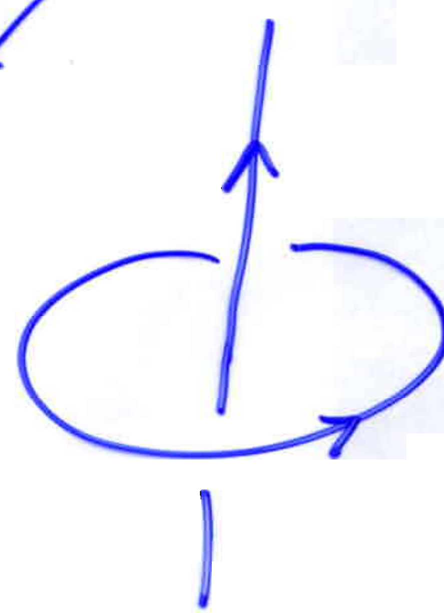
Observation: the boundary of the 24-cell can be (combinatorially) obtained from the boundary of a 4-cube, dividing each of the 8 3-cubes into 6 half-octahedra:



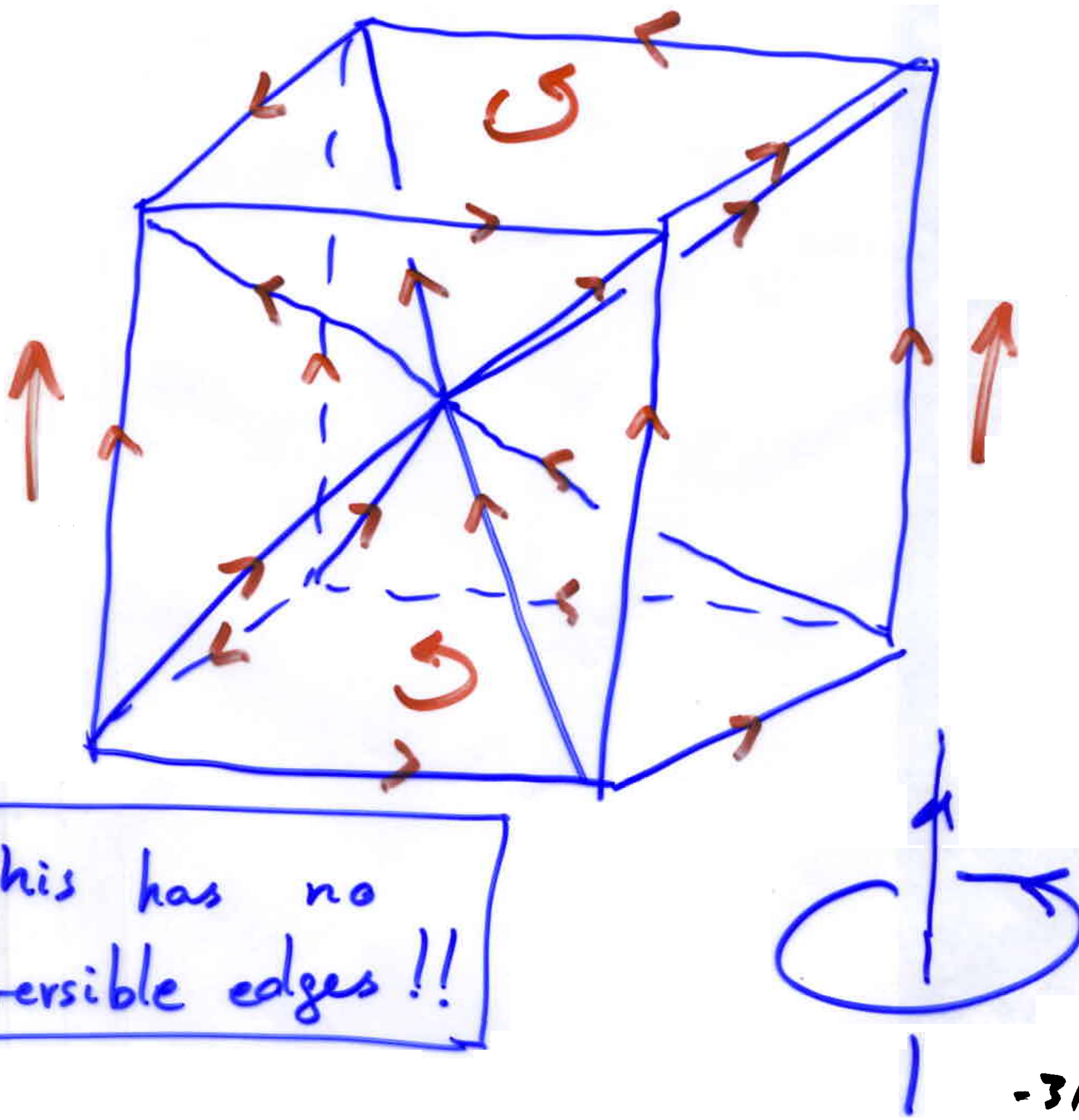
Our locally acyclic orientation:

Consider the 8 3-cubes in the 4-cube as forming 2 cycles of 4 3-cubes each (this is the well-known decomposition of a 3-sphere into 2 solid tori):





Using this (directed) cycles we build our locally acyclic orientation of the 1-skeleton of the 24-cell:



This has no reversible edges !!

What have we proved, exactly?

"The graph of single-edge reversals in locally acyclic orientations of the 1-skeleton of the 24-cell has an isolated element" (Hence, it is disconnected)

$$\Big\downarrow \text{translation}$$

"There is a way to triangulate the 96 3-faces "triangle $\times I$" of $P \times I$ which is invariant under flips".

Hence, if we can extend this to a triangulation of $P \times I$, the graph of triangulations of $P \times I$ will be disconnected. If this can be done unimodularly, the toric Hilbert scheme will be disconnected, too.

... but we can add points to P if we wish (as long as the 96 triangles remain faces)

<u>Thm</u> (Santos, 2001). Adding the origin to P, the 50 point configuration $(\mathrm{vert}(P) \cup \{0\}) \times \{0,1\}$ has a triangulation which extends the one in the previous slides. In particular, the graph of triangulations of those points is disconnected.

Moreover, <u>the triangulation in question is unimodular.</u> In particular, the toric Hilbert scheme of those points is disconnected.

More precise quantitative results

Let $A = \left(\left\{ \begin{array}{c} \text{vertices of} \\ \text{24-cell} \end{array} \right\} \cup O \right) \times \{0, 1\}$

$(d = 5, \ n = 50)$. The toric Hilbert scheme of A and the graph of triangulations of A have:

- at least 13 connected components.
- with dimension $\geqslant 96$ each.
- with at least 3^{48} monomial ideals / unimodular triangulations each.

Asymptotics of the number of connected components:

By "stacking" $k+1$ copies of the 24-cell one on top of another we can get a big number of connected components. We take the same triangulation of $\{\text{vertices 24-cell}\} \cup \{0\}$ in each copy, and we have a choice of (at least) 3 ways of giving l.a.o. to each copy:
- the one we have described...
- its opposite one...
- and any globally acyclic orientation.

Corollary: For any $k = 1, 2, \dots$ we can construct a point set with $25(k+1)$ points in dimension 5 and whose graph of triangulations has at least 3^k connected components.

That is to say,

the graph of triangulations can have exponentially many connected components.