OPTIMIZATION ON THE SPACE OF TRIANGULATIONS MY NOTES

Suppose we pay a price for each triangle (or edge, etc) used in the triangulation. Then the total cost of the triangulation is the sum of prices for those "simplices" we have chosen.

GOAL: Find the optimal (minimal, maximal) triangulation!!

Examples: Suppose we take the following situations

1) Given a point set in $\mathbb{R}^2$, with price \$1 for any of its triangles. What is the the cheapest triangulation?

ANSWER: Use a triangulation with points in the boundary (no interior) because # of triangles $= 2n-2-n_b$. TRIVIAL.

SAME PROBLEM in $\mathbb{R}^d$ is NP-HARD!!!

2) Given a point set in $\mathbb{R}^2$, now suppose we have to pay ZERO or \$1 for the edges (an evil person sets the prices). Can we find a triangulation of weight ZERO? NP-complete problem

3) Find a triangulation of $A \subseteq \mathbb{R}^d$ that minimizes the largest diameter of its simplices.

4) Find a triangulation of $A \subseteq \mathbb{R}^d$ that minimizes the sum of the "surface area" of its d-simplices (volumes of $(d-1)$-simplices)

5) Find a triangulation of $A \subseteq \mathbb{R}^2$ and touches all points of A and minimizes the largest triangle area.

We will occupy time to discuss

1) MLT problem: Minimum Length Triangulation problem
Find a triangulation $A \subseteq \mathbb{R}^2$, touching all points and minimizing the sum of Euclidean lengths of edges present in triangulation.

2) MST problem: Minimum size triangulation problem:
Find a triangulation of $A \subseteq \mathbb{R}^d$, $d \geq 3$ that uses the smallest number of d-dimensional simplices.

MLT problem: Famous open problem to decide the complexity! stated in 1974 by Garey-Johnson in their book.

Common mistake: MLT is the same as GREEDY triangulation (greedy means we try to fit the shortest edge first, if that fails try second shortest, etc) False even for vertices of n-gons.

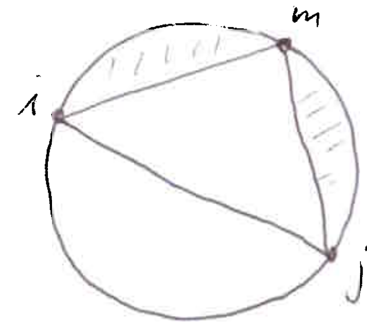
another mistake: MLT contains a minimum weight Hamiltonian cycle (a shortest length cycle passing through ALL vertices) FALSE

Thm: 1980 Klincsek Finding the minimum weight triangulation of the vertices of convex n-gon can be done in polynomial time.

ALGORITHM: MLT for n-gons

INPUT Vertices $M_1, \dots, M_n$ in clockwise direction

Denote $C(i,j)$ the MLT subgraph involving nodes $M_i, M_{i+1}, \dots, M_j$



Step 1 For $k=1$, $i=1,2,\dots n$, $j=i+k$
set $C(i,j) = \text{distance}(M_i, M_j)\ (= d(M_i,M_j))$

Step 2 Let $k := k+1$, for $i=1,2,\dots n$, $j=i+k$ let

$$C(i,j) := d(M_i,M_j) + \min_{i<m<j} [C(i,m)+C(m,j)] \quad (*)$$

For each pair (i,j) let $L(i,j)$ be the index where $C(i,j)$ in $(*)$ is achieved

Step 3 If $k<n$ goto step 2, otherwise the MLT has weight $C(1,n)$

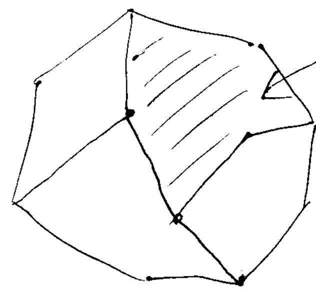
Step 4 To find the edges involved in MLT, backtrace along pointers $L(i,j)$. Edge $\overline{M_1M_n}$ is in the MLT obviously

Step 5 For each $\overline{M_iM_j}$, edge belonging to MLT, with $j > i+1$, let $l = L(i,j)$ then $\overline{M_iM_l}$ and $\overline{M_lM_j}$ belong to MLT.

EXACTLY same algorithm works for non-convex polygons.

Klinseck's algorithm gives a general "approximation technique" that produces a triangulation VERY close to MLT:

1) Find enough edges that partition convex hull of A into polygons



2) Fill Holes by using Klincsek's algorithm.

Work by Levcoupolus, Lingas, et al best approximations.

EXACT computation:

IDEA: Find as many edges as you can that MUST belong to the MLT. Hope that the resulting graph has only polygonal gaps. If this is true then use Klincsek.

Lemma (B.T Yang) If given an edge, the region inside the circles of radius e has no other point $\Rightarrow$ $e \in MLT$

In particular, shortest edge belongs to MLT

We say an edge is LIGHT if it is NOT crossed by a shorter edge

(1996) The study of light edges is closely related to MLT (Aichholzer, Aurenhammer, Taschwer, Rote, Cheng, Katoh and Xu)

Theorem (Aichholzer et. al.) If light edges form a triangulation then this is precisely the MLT.

Lemma: (Pairs of triangulations intersect nicely! When they touch all points.)
Consider T_1, T_2 two triangulations of point set $S \subseteq \mathbb{R}^2$ with n points. Let $E(T_1), E(T_2)$ be the collection of edges of T_1, T_2 respectively. Then there exist a Perfect matching (a bijection) $\varphi : E(T_1) \to E(T_2)$ such that $e, \varphi(e)$ are either identical or they cross!!

Proof of Theorem: Call L the triangulation of light edges
$W(L) \leq W(T)$ for any other T because by matching $\sum_{e \in L} \text{length}(\varphi(e)) \geq \sum_{e \in L} \text{length}(e)$

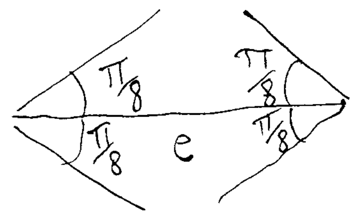
Done

1995 Dickerson and Keil: A triangulation is LM (locally minimal) if one cannot flip to get a smaller triangulation.

NOTE MLTs are LMT's

DEF: LMT-skeleton (set of edges belonging to ALL LMT's)

Lemma: The edge e belongs to MLT then it is necessary that at least one of the regions contains no other vertex.



Lemma: Each candidate edge e to belong to an LMT must be crossed by at least one longer edge. If e is a candidate, and it does not intersect any other candidate edge, then e is in MWT

Using LMT-skeleton people have been to compute MLT for point sets with 1000 points !!! (Snoeyink et al. 1997) BUT it fails sometimes

MINIMAL SIZE TRIANGULATION

Theorem: It is an NP-hard to find the smallest size triangulation (Below, Richter-Gebert, J) for 3-d polytopes.

One of the main "switches" is built using Schönhardt's polytope. You need this to construct a polytope that BIG triangulations only if it does not use a point, but triangulations can be SMALL if one uses the extra point: