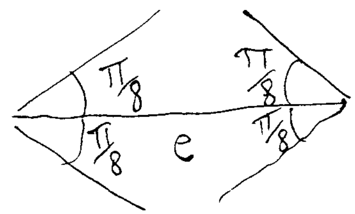
DEF: LMT-skeleton (set of edges belonging to ALL LMT's)

Lemma: The edge e belongs to MLT then it is necessary that at least one of the regions contains no other vertex.



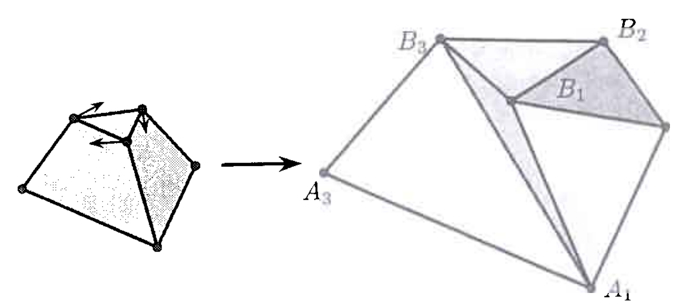
Lemma: Each candidate edge e to belong to an LMT must be crossed by at least one longer edge. If e is a candidate, and it does not intersect any other candidate edge, then e is in MWT

Using LMT-skeleton people have been to compute MLT for point sets with 1000 points!!! (Snoeyink et al. 1997) BUT it fails sometimes

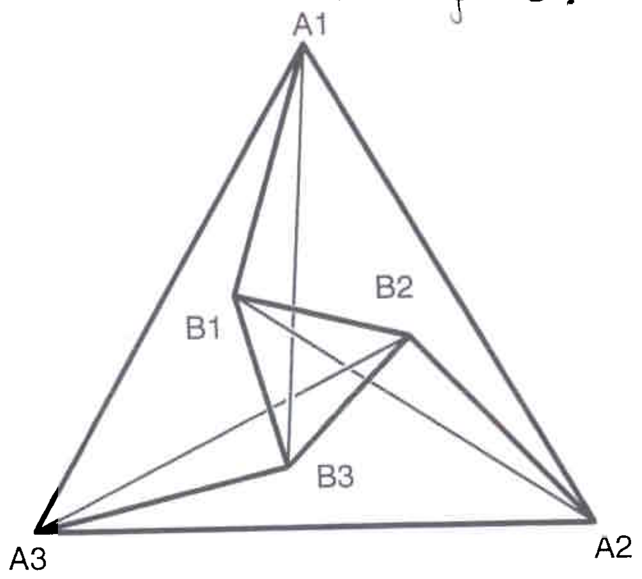
MINIMAL SIZE TRIANGULATION

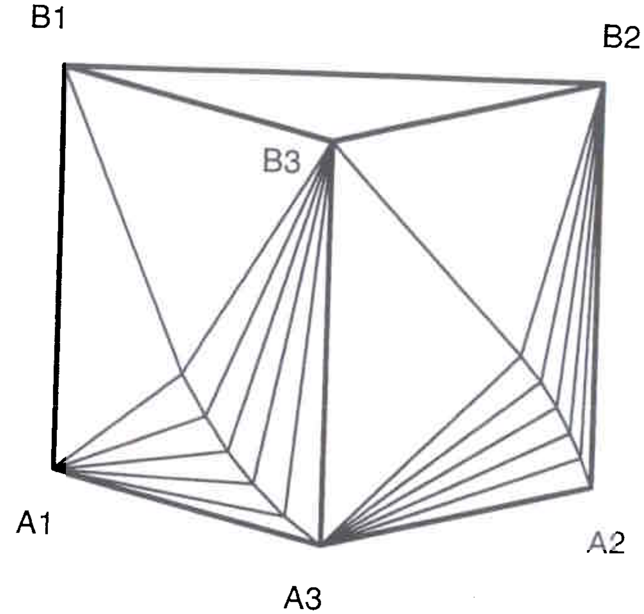
Theorem: It is an NP-hard to find the smallest size triangulation (Below, Richter-Gebert, J) for 3-d polytopes.

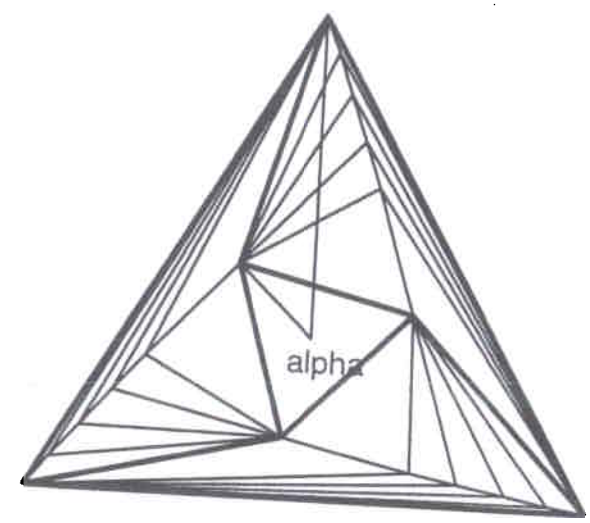
One of the main "switches" is ~~build~~ using Schönhardt's polytope. You need this to construct a polytope that BIG triangulations only if it does not use a point, but triangulations can be SMALL if one uses the extra point:



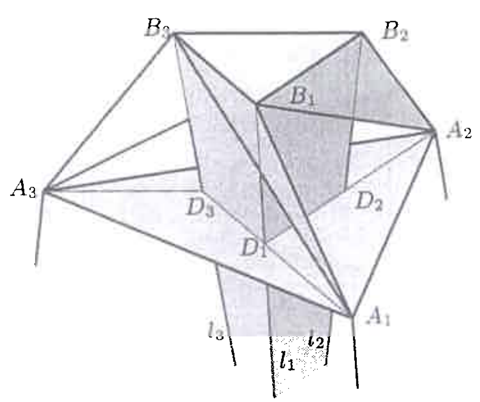
To construct the "switch polytope" we glue to the sides of Schönhardt a chain of triangles. 6







If an extra point is added to the configuration $\mathcal{A}$, such that it lies in the cone defined by planes $l_1 = A_1 B_1 B_3$, $l_2 = A_2 B_1 B_2$, $l_3 = A_3 B_2 B_3$ then we can do <u>SMALL</u> triangulations, smaller than without the point.



Region is bounded by planes l_1, l_2, l_3

__Corollary__:

For any fixed dimension, it is NP-hard to compute ~~the~~ a smallest triangulation of a point set A.

How about concrete families of polytopes?

What can be done?

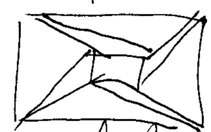
Suppose we are given a concrete point set, weights for each simplex how can we find an optimal triangulation

a) minimizing sum of weights of d-simplices present

b) minimizing maximal weight among d-simplices present

EXAMPLES: Consider an m-prism. The smallest number of tetrahedra in a triangulation is $2m + \lceil \frac{m}{2} \rceil - 5$.

WARNING: Size of maximal triangulation depends on the coordinates (more precisely on the order type).

Do picture of cube vs 

EXAMPLE: Regular d-cube. Maximal triangulation $d!$ simplices but we do not know minimal triangulation for $d \leq 7$.

For $d=8$ optimal is between 5522 and 13136 (more this afternoon)

How to compute, calculate concrete examples? We model the problem as LINEAR & INTEGER PROGRAMMING PROBLEMS !!

$$\min cx \quad \text{subject to} \quad Ax \leq b$$

Simplex method offers practical solution

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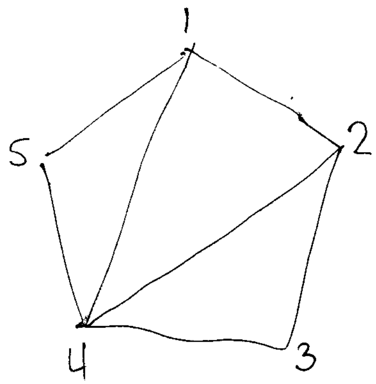
when asking about integral solutions, we have to use Exhaustive Enumeration techniques.

THINK of triangulations as LONG vectors of ZEROS/ONES.

Let $A \subseteq \mathbb{R}^d$ be a point set with n points. Each triangulation is a vector with $\binom{n}{d+1}$ entries (one per possible d-simplex)

$$(V_T)_\sigma = \begin{cases} 0 & \text{if } \sigma \text{ not in } T \\ 1 & \text{if } \sigma \text{ in } T \end{cases}$$

EXAMPLE: Pentagon, each triangulation is represented by a vector in $\mathbb{R}^{10}$



$(0,1,0,0,0,1,1,0,0,0)$ ← There will be ONE VARIABLE PER d-simplex

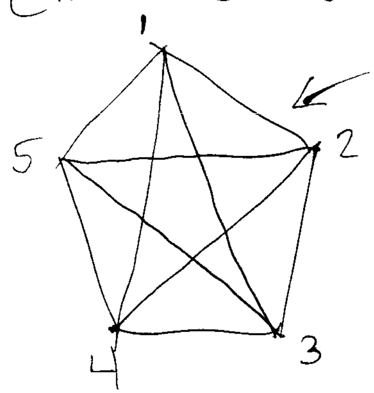
Definition: Let U_A = convex hull (0-1 vectors of triangulations)

NOTE: $U_A \subseteq \mathbb{R}^{\binom{n}{d+1}}$ (it lives in high dimension)

GOAL: PROVIDE, equations and inequalities describing U_A

A) Equalities

Chambers are regions that result from "overlapping" ALL triangulations



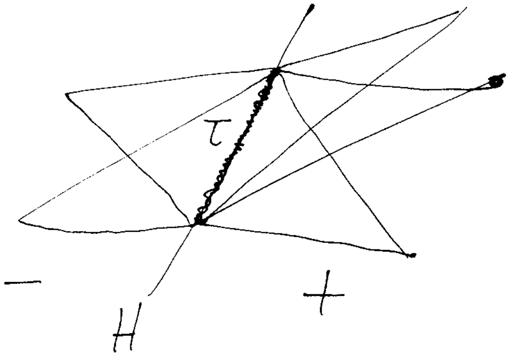
← 11 chambers

$$X_{245} + X_{134} + X_{133} + X_{124} + X_{235} = 1$$

in general $\boxed{\displaystyle\sum_{\substack{\sigma \text{ contains} \\ \text{chamber } \gamma}} X_\sigma = 1}$

· Second kind of equalities

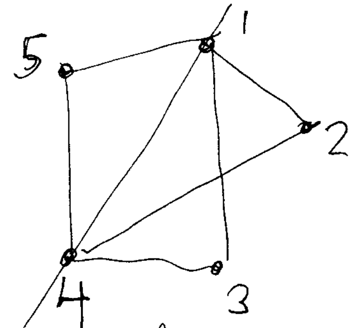
· Consider a hyperplane H spanned by a $(d-1)$-simplex τ



$$\sum_{\substack{\sigma \subseteq H^+ \\ \tau \subset \sigma}} x_\sigma - \sum_{\substack{\sigma' \subseteq H^- \\ \tau \subset \sigma'}} x_{\sigma'} = 0$$

(Both sides of τ should be covered or NONE)

Example:

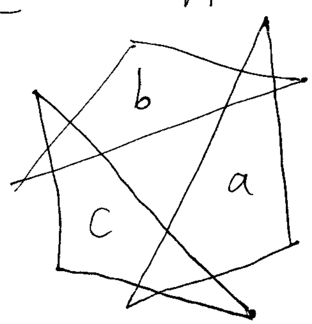


$Z_{145} - Z_{124} - Z_{134} = 0$

Theorems: The 0/1 solutions of the above system of equations are EXACTLY the vectors of triangulations.

Observation: If you wish to optimize over triangulations that use ALL points of A, how would you do it? $\sum_{\substack{\sigma \text{ contains} \\ \text{vertex } p}} x_\sigma \geq 1$

INEQUALITIES: Suppose you have a bunch of triangle intersecting pairwise



$x_a + x_b + x_c \leq 1$ is an inequality satisfied

What are the edges of the universal polytope?

Theorem 9.1.10. *Let T_1 and T_2 be two distinct triangulations of $\mathcal{A}$. The following statements are equivalent:*

1. *v_{T_1} and v_{T_2} are not neighbors in $Q_{\mathcal{A}}$.*
2. *v_{T_1} and v_{T_2} are not neighbors in $U_{\mathcal{A}}$.*
3. *There exist two triangulations T_3 and T_4 of $\mathcal{A}$ (different from T_1 and T_2) such that $v_{T_1} + v_{T_2} = v_{T_3} + v_{T_4}$.*
4. *There exist partitions $T_1 = R_1 \cup L_1$ and $T_2 = R_2 \cup L_2$ such that $L_1 \cup R_2$ and $R_1 \cup L_2$ are two other triangulations of $\mathcal{A}$, different from T_1 and T_2.*

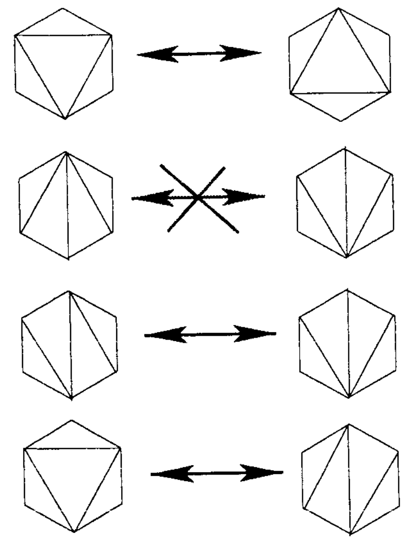


Figure 9.6: adjacencies in the universal of a hexagon

Flips are particular cases of "edges of the universal."

SECONDARY POLYTOPE is a projection of the universal polytope.

There is an interesting connection between MINIMAL SIZE TRIANGULATIONS and SMALLEST NUMBER OF FLIPS NEEDED to transform a triangulation T_1 into T_2:

Each flip corresponds to stacking up another simplex inside a triangulation of a ball or a polytope.

Theorem minimal size triangulations of convex 3-polytopes are between $n-3$ and $2n-10$ for large values of n.

implies diameter of the graph of triangulations of n-gon is bounded by $2n-10$.