LOWER BOUNDS FOR MIN'L Δ'ATIONS AND SIMPLICIAL COVERS OF CUBES

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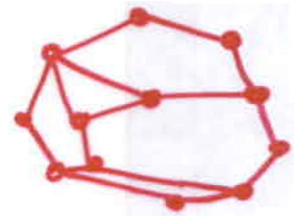
I. WHY COVERS?
II. PEBBLE SET BOUNDS
III. (NOT SO) SILLY VOLUME BOUNDS
IV. REAL ESTATE BOUNDS

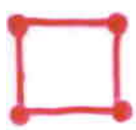
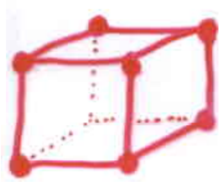
- d-polytope : region in $\mathbb{R}^d$ cut out by planes

  (n, d)-polytope

  $\uparrow$ # vertices   dimension



- d-cubes:

		# vertices	# facets
	0-cube $\square_0$	1	0
	1-cube $\square_1$	2	2
	2-cube $\square_2$	4	4
	3-cube $\square_3$	8	6
	4-cube $\square_4$	16	8
		$\vdots$	$\vdots$
	d-cube $\square_d$	2^d	$2d$

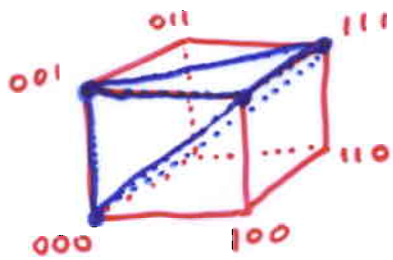
■ A standard Δ'ation of a cube: size $d!$

Every $\sigma \in S_d \longleftrightarrow$ simplex in this Δ'ation

EX. (d=3) $\sigma = (3,1,2) \longleftrightarrow$

$$\begin{array}{c} 000 \\ \downarrow_3 \\ 001 \\ \downarrow_1 \\ 101 \\ \downarrow_2 \\ 111 \end{array}$$

vertices of simplex



So 2-cube: Δ'ation size 2

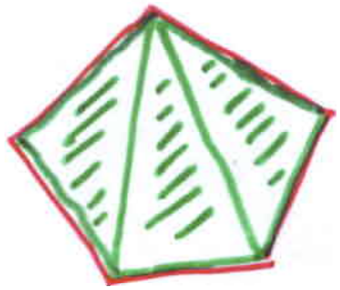
3-cube: Δ'ation size 6 $\longleftarrow$ is this smallest?

NO.

What is the size of the min'l Δ'ation of the d-cube?

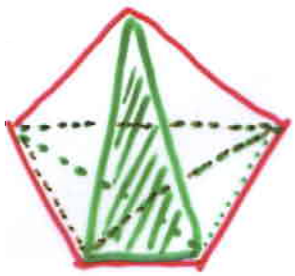
... Cottle '82, Sallee '82, Haiman '91, Hughes '94, Hughes-Anderson '96, Smith 2000, Orden-Santos '03, ...

(simplicial) cover of a polytope: P



ex. a triangulation

cover size: 3



cover size: 5

only allow simplices formed by vertices of P

Q. What's a minimal Δ'ation of P?
a minimal cover?
Let c(P) = size of min'l cover.

EX. not convex

min'l cover : size 2
min'l Δ'ation: " 10

min'l dissection: " 4

Why care?

.... about min'l Δ'ations?

- efficiency of simplicial algorithms
  e.g., fixed pt algorithms
  "fair division" procedures

- reflects (something about) the geometry of the polytope

.... about min'l covers?

- arise as images of simplicial Brouwer maps

- they give lower bounds for <u>ANY</u> kind of Δ'ation!

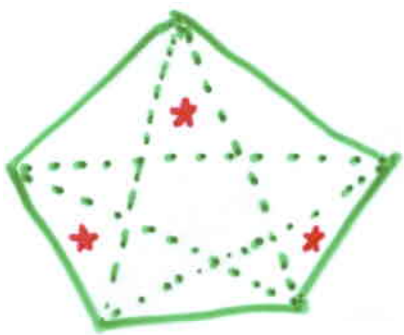
$$c(P) \leq T(P) \leq T_v(P) \leq T_{vf}(P)$$

$$c(P) \leq T_f(P)$$

LOWER BOUND TECHNIQUE: PEBBLE SETS

<u>Def'n</u>. A pebble set for a polytope P is a set of points (pebbles) such that every simplex of P contains at most one pebble.

<u>EX</u>.



P= pentagon

$c(P) \geq 3$

↑

because:

THE SIZE OF A PEBBLE SET OF P $\leq$ SIZE OF ANY COVER of P.

Why? Every simp. in cover contains at most one pebble.

<u>Thm</u> (DELOERA - PETERSON - S., 2002)
Every n-vertex, d-dim'l, polytope P has a pebble set of size (n-d).

<u>COR</u>. $c(P) \geq n-d$.

- These observations true for <u>any</u> polytope.

For d-cubes $\square_d$:

d	$n-d = 2^d - d$ lower bound	known: $T_{vf}(\square_d)$	
1	1	1	
2	2	2	
3	5	5	
4	12	16	Cottle '82
5	27	67	Hughes-Anderson '96
6	58	308	"
7	121	1493	"

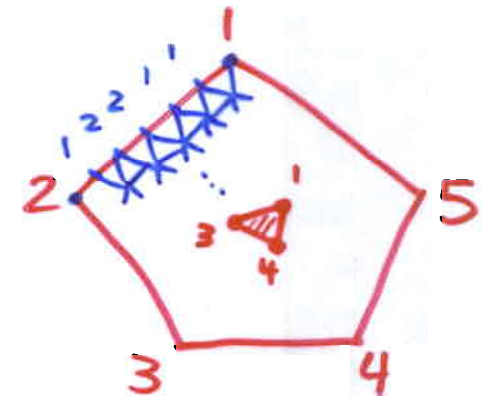
higher T_{vf} not exactly known!

- ... haven't used geometry of the cube yet.

<u>CHALLENGE:</u> Find better pebble set constructions specific to cubes.

consequence: POLYTOPAL SPERNER LEMMA

- Δ'ate a (n,d)-polytope P
- Label vertices of Δ'ation in "Sperner way".
- A full cell is a d-simplex in Δ'ation whose labels are distinct.

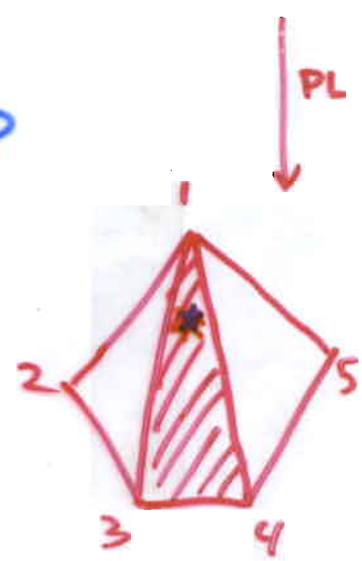


- Atanassov (1996) showed that a Δ'ated, Sperner-labelled n-gon has at least $(n-2)$ full cells.
- Atanassov conjectured, and we proved:

Thm (deLoera-Peterson-S.)
A Δ'ated, Sperner-labelled (n,d)-polytope has at least $(n-d)$ full cells.

- Cor. Usual sperner's lemma: $n=d+1$.

proof idea: Construct PL map $P \to P$ mapping labels to vertices:



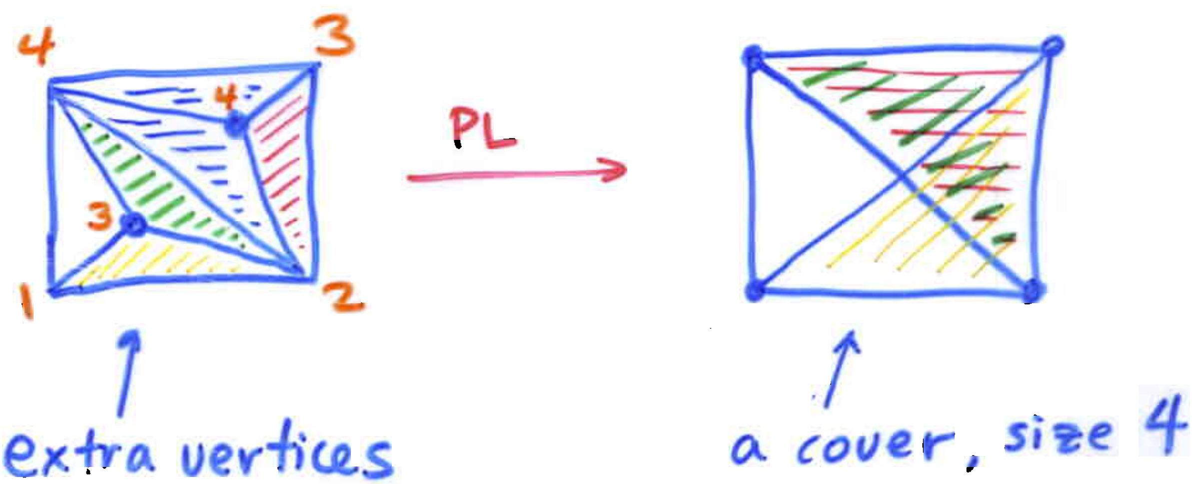
then inverse images of pebbles live in full cells!

Note also: images of full cells form a cover!

THESE IDEAS SHOW!

Prop'n. $c(P) \leq T(P)$.

Pf. To any Δ'ation of P, give a Sperner labelling. Under the assoc. PL map, images of full cells form a simplicial cover, of size no larger than the Δ'ation.



MORAL: Lower bounds for covers give lower bounds for any kind of Δ'ation!

(e.g. extra vertices, non face-to-face, ...)

... not just for $T_v(P)$, $T_f(P)$, $T_{vf}(P)$

most often in literature

LOWER BOUND TECHNIQUE: VOLUME CALCULATIONS

FACT. In unit d-cube every simplex has volume a multiple of $\frac{1}{d!}$

Def'n: a simplex of volume $\frac{\mathring{z}}{d!}$ is a "class $\mathring{z}$" simplex.

A (NOT SO) SILLY BOUND:

$$\frac{\text{The vol. of a polytope } P}{\text{vol. of largest possible simplex of } P} \le c(P)$$

- Even for cubes, exact values of this not known for large d (but there are bounds)
  the Hadamard det. problem
- For a 3-cube, the largest simp. has class 2.
  So $\boxed{c(\square_3) \geq 3} = \frac{3!}{2}$
- For a 4-cube, largest simp. has class 3.
  So $\boxed{c(\square_4) \geq 8} = \frac{4!}{3}$
- Smith (2000) used hyperbolic volume!
  got $c(\square_3) \geq 5$, $c(\square_4) \geq 15$, and great bounds for other d.

LOWER BOUND TECHNIQUE:

EXTERIOR FACE REAL ESTATE

Can we do better than Smith (2000)?

YES.

Notice:

- cube faces are again cubes.
- simplices with exterior faces must cover those cubes

Let

$f(d, c, d', c')$ = max'l # of exterior d'-faces of class c' that a d-simplex of class c in the cube can have.

How to count this?

- Solve the LP that measures exterior face real estate in each dim d':

let $\boxed{X_c = \#\,d\text{-simps class-}c \text{ used in the cover.}}$

$$\min \sum X_c$$

subject to:

$$\sum_{c=1}^{\mathrm{maxvol}(d)} \sum_{c'=1}^{c} \frac{c'}{d!} F(d,c,d',c') X_c \geq 2^{d-d'} \binom{d}{d'}$$

for each $d' = 1, \ldots, d$.

For d:	0	1	2	3	4	5	6	7	8	...
maxvol(d) known to be:	1	1	1	2	3	5	9	32	56	...

...	9	10	11	12
	144	320	1458	3645

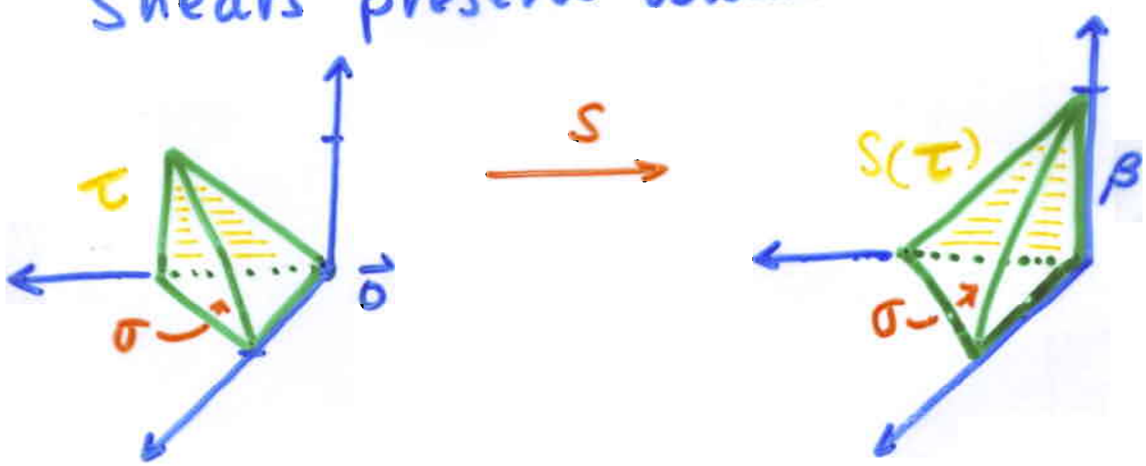
Let $M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & & & & \\ 0 & \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_d \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$ be a d-simp. in the cube.

A k-face of M picks $(k+1)$ cols. These are corners of that face.

A k-face of cube picks k rows & row 1. Outside these rows, all other rows are constant on that face

(1) $\text{class}(M) = |\det(M)|$.

(2) Shears preserve volume:



$$\begin{array}{c} \left[\begin{array}{c|c} 1\;1\;1\;1 & 1 \cdots 1 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\;\sigma & \star \\ \hline 0 & \star \end{array}\right] \\ \tau \end{array} \xrightarrow{S} \begin{array}{c} \left[\begin{array}{c|c} 1\;1\;1\;1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\;\sigma & 0 \\ \hline 0 & \beta_{ext} \end{array}\right] \\ S(\tau) \end{array}$$

③ τ has a "footprint" $(\tau \cap \sigma)$ in σ

τ has a "shadow" $(S(\tau) \cap \beta)$ in β

④ intersections of ext. faces are exterior,

so the footprint and shadow of τ

are exterior faces of

simplices: σ and β, resp.

⑤ Hence

if $f(d,c,d',c') \neq 0$ then

$$f(d,c,d',c') \leq \sum_{\delta=1}^{d'} \sum_{\gamma=0}^{c'} f(d',c',\delta,\gamma) \cdot f\left(d-d', \tfrac{c}{c'}, d'-\delta, \tfrac{c'}{\gamma}\right)$$

where set $f(d,c,0,1) := 1$

and use $f(d,c,d',c') = 0$ if

$d' > d$

$c' \nmid c$

$c < \mathrm{maxvol}(d)$

$c' < \mathrm{maxvol}(d')$

Use this recursion to get upper bd.

$$F(d,c,d',c') \geq f(d,c,d',c').$$

Contrast with Hughes '94, Hughes-Anderson '96

- also used an LP, but
- needed to compute every possible simplex config., one for each variable
- followed exterior facet flags
- huge program
- bounded $T_v(\square_d)$ not $T(\square_d)$

d	LOWER BOUNDS $c(\square_d)$, $T(\square_d)$: Smith (2000)	Bliss-S. (2003)	$T_v(\square_d)$ Hughes-Anderson (1996)	UPPER BOUNDS $T_{vf}(\square_d)$
1	1	1	= 1	= 1
2	2	2	= 2	= 2
3	5	5	= 5	= 5
4	15	16	= 16	= 16
5	48	60	= 67	= 67
6	174	250	270	= 308
7	681	1,118	1,175	= 1,493
8	2,863	4,680	5,522	13,136
9	12,811	21,384	26,593	105,341
10	60,574	95,708	131,269	928,780
11	300,956	516,465	665,272	
12	1,564,340	2,906,455	?	
gen'l	$\frac{1}{2} \cdot \frac{6^{d/2}\, d!}{(d+1)^{\frac{d+1}{2}}}$	not yet...	not better than Smith	$(.8159)^d\, d!$ (Orden-Santos, 2003)

FURTHER QUESTIONS:

Q. Can this LP be solved to give an asymptotic lower bound? (to compare with Smith (2000)).

(any feasible sol'n to dual LP will yield a lower bound)

Q. Find a cover of the 5-cube of size < 67.

(may have more symm. than min'l Δ'ation)

Q. Find a "good" pebble set construction for any specific class of polytopes. (exceeding n-d)

Q. Is there some d-cube for which adding interior vertices gives a smaller Δ'ation?

($d \geq 5$)

References:

(w/ A. Bliss)
"Lower bounds for minimal triangulations of cubes..."
in preparation.

(w/ J.A. DeLoera, E. Peterson)
"A polytopal generalization of Sperner's lemma"
J. Combin. Theory. Ser. A 100 (2002), 1-26.

For these, and papers applying generalizations of Sperner, Tucker lemmas to "fair division" problems, see my home page:

www.math.hmc.edu /~su