The Cayley Trick

1. Mixed subdivisions.

Definition (fiber polytope perspective):

Let $A_1, \ldots, A_K$ be point sets in $\mathbb{R}^d$, let $n_i = |A_i|$.

Let $A = A_1 + \cdots + A_K$ be the Minkowski sum of the A_i's.

A (polyhedral) subdivision of A is called <u>mixed</u> if it is π-induced for the natural projection

$$\Delta^{n_1-1} \times \Delta^{n_2-1} \times \cdots \times \Delta^{n_K-1} \longrightarrow \operatorname{conv}(A)$$

$$(e_{i_1}, e_{i_2}, \ldots, e_{i_K}) \longmapsto a^{(1)}_{i_1} + a^{(2)}_{i_2} + \cdots + a^{(K)}_{i_K}$$

(where $e_1, \ldots, e_{n_i}$ are the vertices of Δ^{n_i-1} and $A_i = \{a^{(i)}_1, a^{(i)}_2, \ldots, a^{(i)}_{n_i}\}$).

... but this definition does not 'say much'...

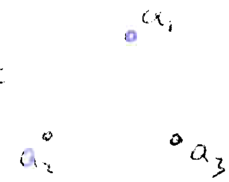
Definition ("down to earth")

- A subset of $A = A_1 + \dots + A_k$ is called a Minkowski cell if it can be written as $B_1 + \dots + B_k$, for certain subsets $B_1 \subseteq A_1, \dots, B_k \subseteq A_k$.
- A subdivision of $A = A_1 + \dots + A_k$ is mixed if all its cells are Minkowski cells.

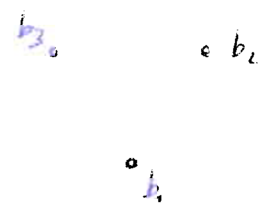
$$\boxed{\text{"A mixed subdivision is a subdivision into Minkowski cells"}}$$

Minkowski sums with summands in all the A_i's

Example. $A_1 =$



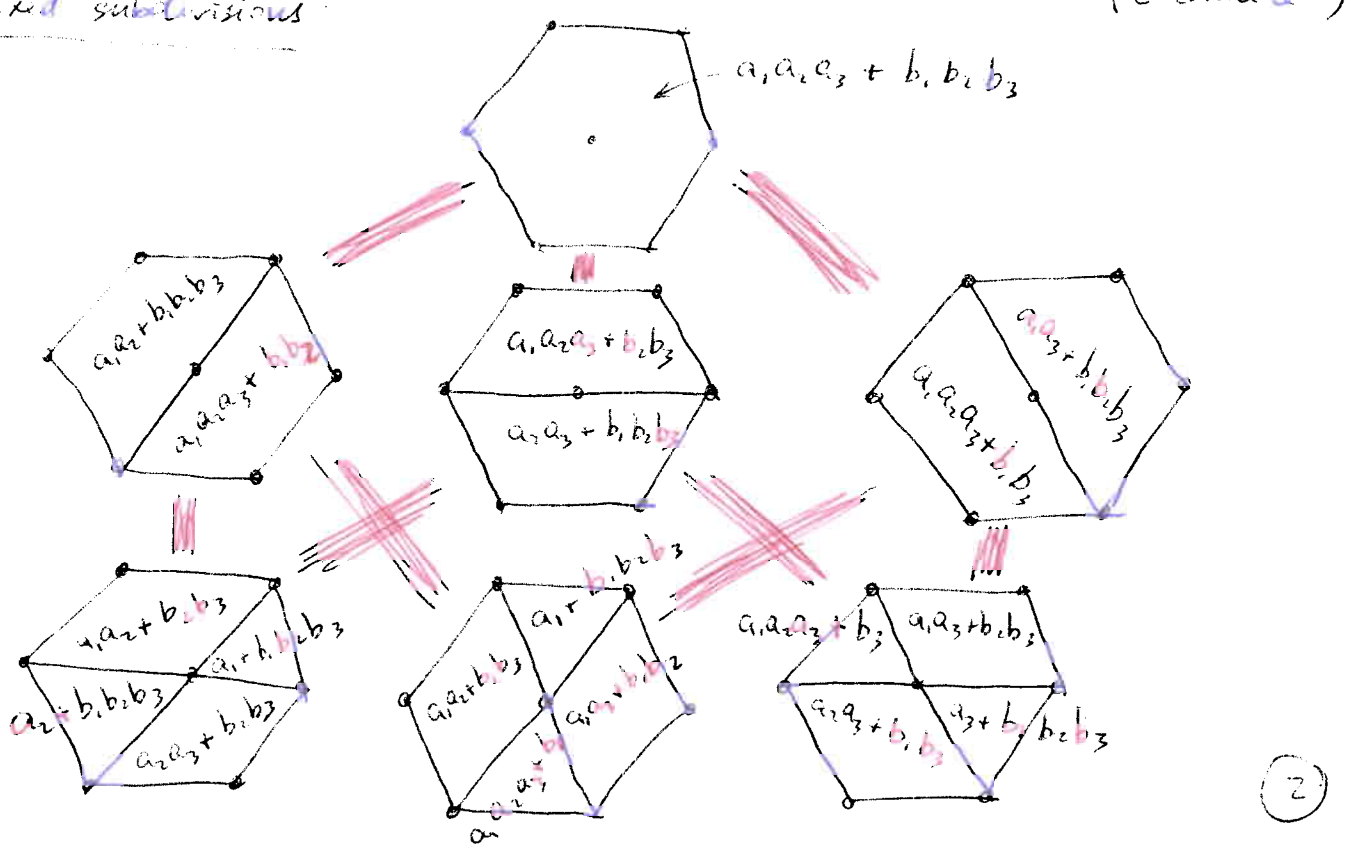
$A_2 =$



$A_1 + A_2 =$

5 pts (2 "coincide")

Mixed subdivisions



<u>Fine mixed subdivisions</u>: a (full-dimensional) Minkowski cell $B_1 + B_2 + \cdots + B_k$ is called <u>fine</u> if

- all the B_i's are affinely independent
- they are "transversal" to one another. That is to say, their affine spans are complementary affine subspaces (in particular, $\boxed{\Sigma(|B_i|-1) = d}$)

<u>Example</u>: if <u>$d=2$</u>, a fine Minkowski cell may consist of

- 3 independent points in one A_i, plus a single point in every other A_i.
- 2 independent points in two of the A_i's (with the two segments they produce non-parallel) plus a single point in every other A_i

<u>Defn</u>: A mixed subdivision is called <u>fine</u> if it consists of only fine Minkowski cells.

<u>Remark</u>: . fine Minkowski cells are the minimal full-dimensional Minkowski cells. Hence, fine mixed subdivisions are the minimal (w.r.t. refinement) mixed subdivisions. They are the analogue of triangulations in the mixed world.

2. $\boxed{\text{The Cayley Trick}}$

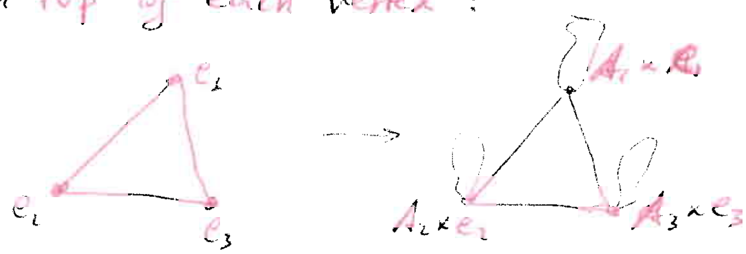
Let $A_1, \dots, A_k$ be point sets in $\mathbb{R}^d$. We call Cayley embedding of $A_1, \dots, A_k$ the following point set $C(A_1, \dots, A_k) \subseteq \mathbb{R}^{d+k-1} = \mathbb{R}^d \times \mathbb{R}^{k-1}$

$$C(A_1, \dots, A_k) = A_1 \times \{e_1\} \cup A_2 \times \{e_2\} \cup \dots \cup A_k \times \{e_k\},$$

where $\{e_1, \dots, e_k\}$ is an affine basis ("simplex") in $\mathbb{R}^{k-1}$

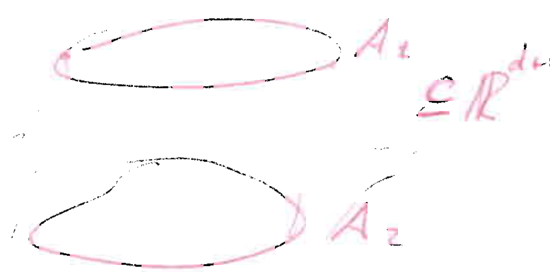
Idea: we take a simplex in $\mathbb{R}^{k-1}$ and place one of the A_i's on top of each vertex:

$k=3$:



Examples:

if $k=2$: we are placing A_1 and A_2 on two parallel hyperplanes in $\mathbb{R}^{d+1}$:



if $\overbrace{A_1 = A_2 = \dots = A_k}^{A}$, then

$$C(A_1, \dots, A_k) = A \times \Delta^{k-1} \quad \left(\text{product with a simplex}\right)$$

Theorem (the Cayley Trick): Let $A_1, \dots, A_k$ be as before.

Then, there is a bijection between

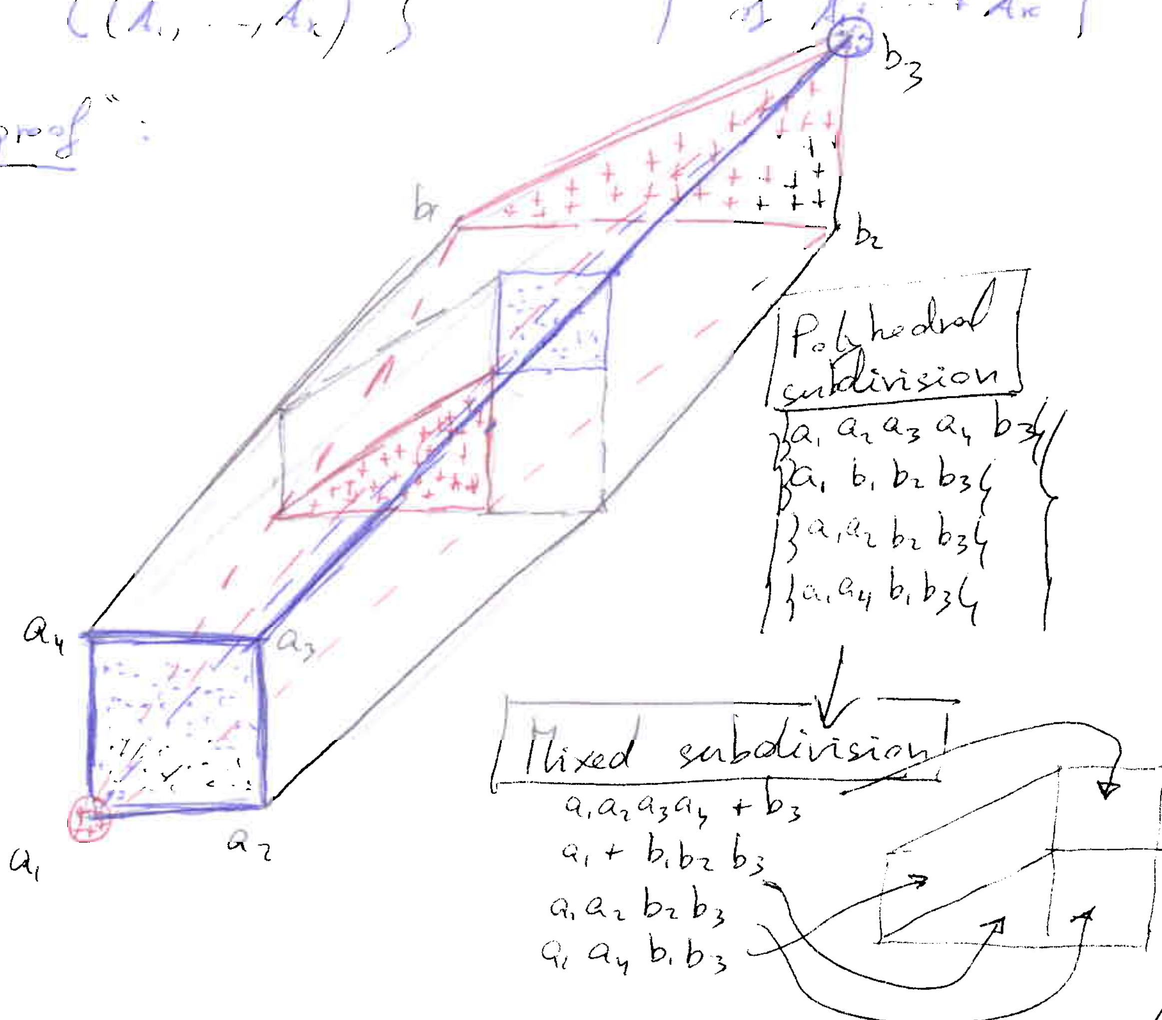
$$\left\{ \begin{array}{c} \text{polyhedral subdivisions} \\ \text{of} \\ C(A_1, \dots, A_k) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{mixed subdivisions} \\ \text{of} \\ A_1 + \dots + A_k \end{array} \right\}$$

... which restricts to bijections between

$$\left\{ \begin{array}{c} \text{coherent (regular)} \\ \text{subdivisions of} \\ C(A_1, \dots, A_k) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{coherent } (\pi_H\text{-coherent}) \text{ mixed} \\ \text{subdivisions of} \\ A_1 + \dots + A_k \end{array} \right\}$$

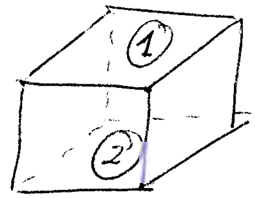
$$\left\{ \begin{array}{c} \text{triangulations} \\ \text{of} \\ C(A_1, \dots, A_k) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{fine mixed} \\ \text{subdivisions} \\ \text{of } A_1 + \dots + A_k \end{array} \right\}$$

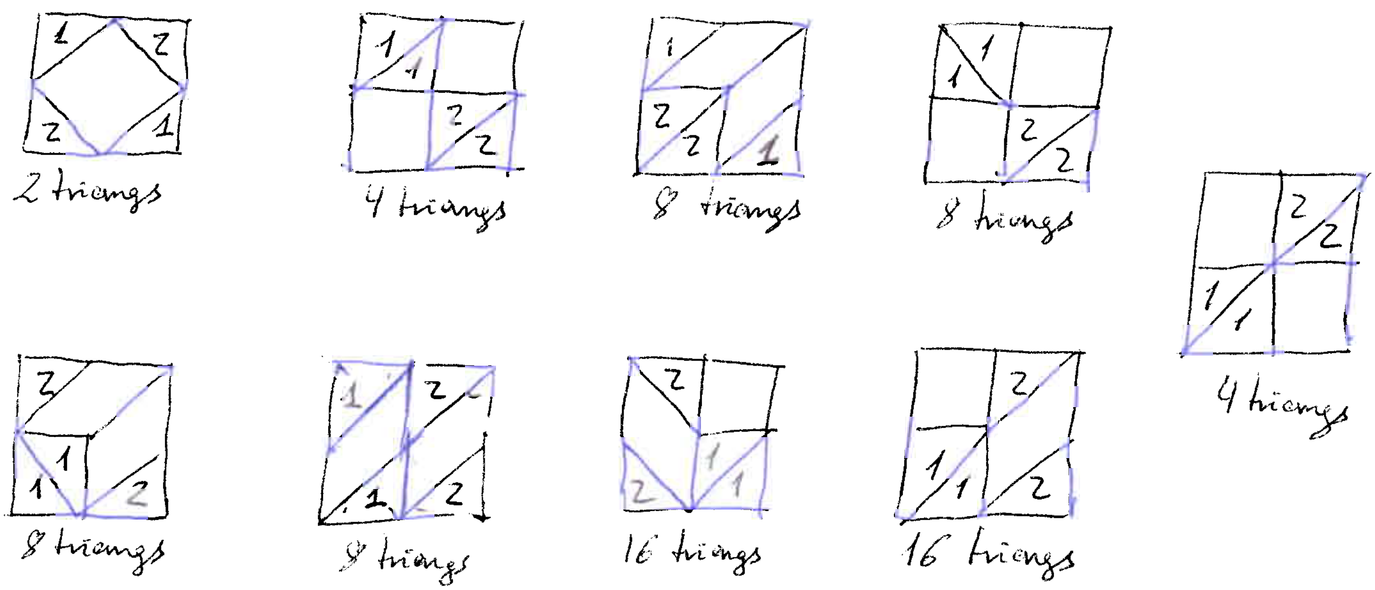
"1-picture proof":



3. Triangulations of cubes

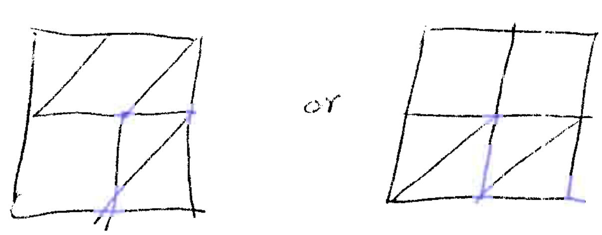
Application 1: Since 3-Cube = C (square, square) we can picture all the triangulations of the 3-cube as 2-dimensional objects:





Total = $2+8+4+8+8+16+8+16+4 = \boxed{74 \text{ triangulations}}$

Beware: some good-looking decompositions are not mixed, hence they do not represent triangulations of the cube:

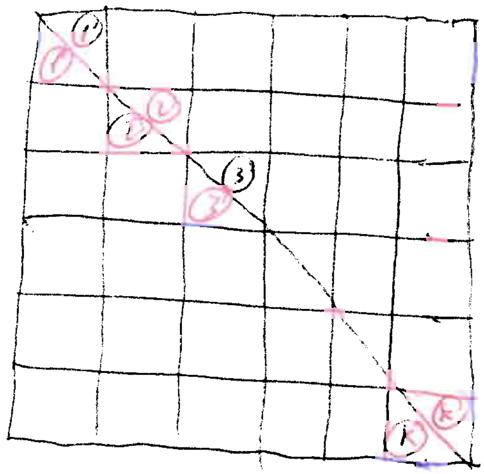


(no "good" way of putting the labels (1) and (2) into the triangles)

Application 2: Efficient triangulations of high-dimensional cubes

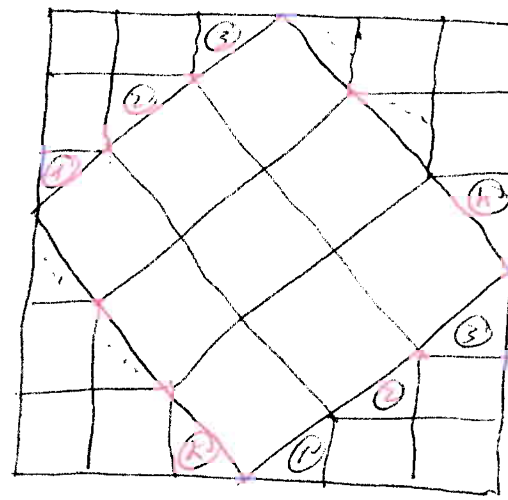
Since $(\text{square} \times \Delta^{k-1}) = C(\text{square}, \text{square}, \ldots, \text{square})$ we can picture its triangulations as 2-dimensional objects.

Example:



k^2+k simplices

$\Rightarrow$ unimodular triangulation (maximal number of simplices)



$\simeq \frac{3k^2}{4} + k$ simplices

This is the minimal triangulation of $\text{square} \times \Delta^{k-1}$

Now: suppose you have a triangulation of the $(k-1)$-cube into t simplices. This gives you a subdivision of the $(k+1)$-cube into t "$\text{square} \times \Delta^{k+1}$"'s and you can refine each of those with about $\frac{3k^2}{4}$ simplices

$\Longrightarrow$ asymptotically, you can triangulate the n-cube with about $\left(\sqrt{\frac{3}{4}}\right)^n \cdot n!$ simplices

$\Downarrow$ $0.866^n \, n!$

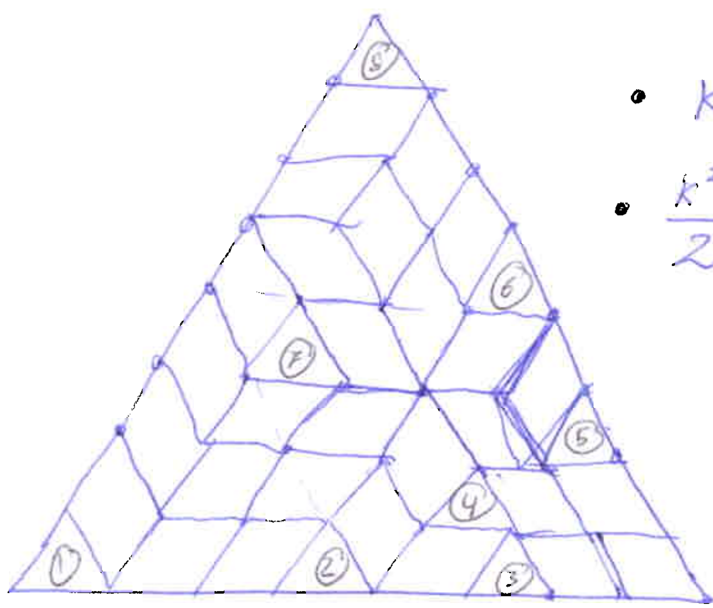
Remember: (Haiman '91: $0{,}840^n \, n!$) (Orden-Santos '03: $0.816^n \, n!$)

triangle (k-1)-simplex

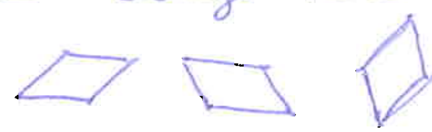
8. Triangulations of $\Delta^2 \times \Delta^{k-1}$.

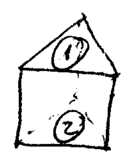
Since $\Delta^2 \times \Delta^k = \mathcal{C}(\underbrace{\Delta^2, \Delta^2, \dots, \Delta^2}_{k})$, we can picture its triangulations as mixed subdivisions of $\Delta^2 + \dots + \Delta^2$.

That is to say: a triangle of side k:

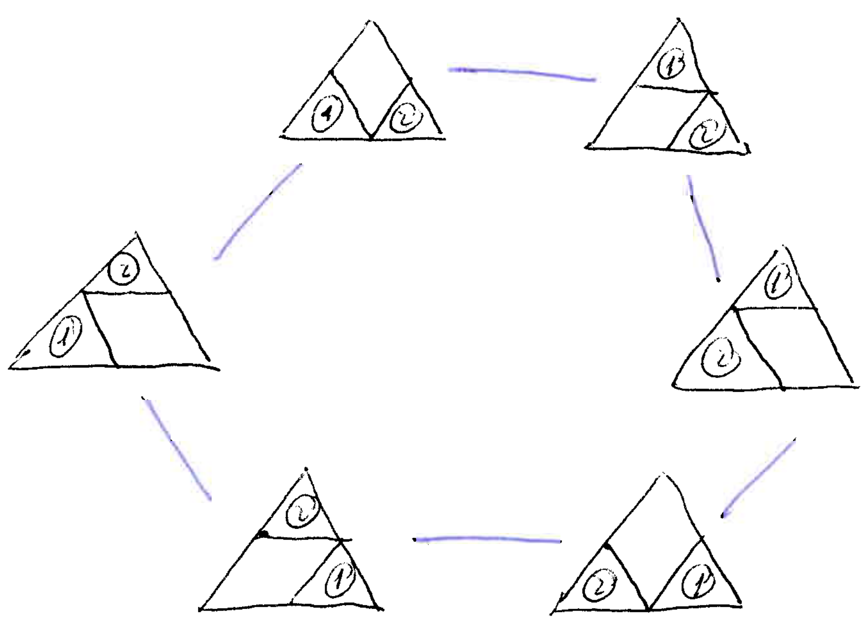


- k upward triangles △
- $\frac{k^2-k}{2}$ "lozenge" tiles:

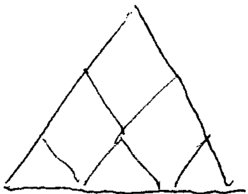


Example: the six triangulations of 

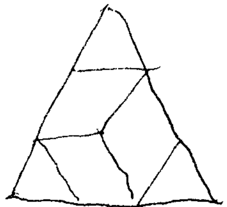
(again!)



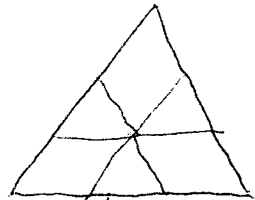
Example: the 108 triangulations of $\Delta^2 \times \Delta^2$:



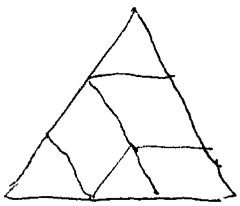
3 lozenge tilings
18 triangulations



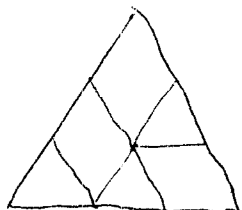
2 tilings
12 triangulations



1 tiling
6 triangulations



6 tilings
36 triangulations

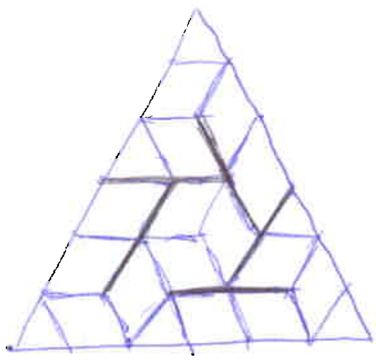


6 tilings
36 triangulations

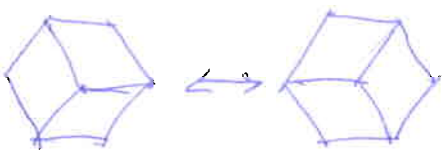
The number of triangulations of $\Delta^2 \times \Delta^k$:

k	lozenge tilings of $k \cdot \Delta^2$	triangulations of $\Delta^2 \times \Delta^{k-1}$
1	1	$1 \times 1! = 1$
2	3	$3 \times 2! = 6$
3	18	$18 \times 3! = 108$
4	187	$187 \times 4! = 2244$
5	3135	$3135 \times 5! =$ 18[illegible]100
6	81462	⋮
7	3198404	
8	186498819	
9	15952438877	
10	1983341709785	
11		
12		
13		
14	17665249123640876125464	

A non-regular triangulation of $\Delta^2 \times \Delta^5$:



Flips between triangulations:

 and 

With this approach:

∘) The number of triangulations of $\Delta^2 \times \Delta^k$ is between α^{k^2} and β^{k^2}, with

$$\alpha = \sqrt[4]{\frac{27}{16}} = 1.13975\ldots \quad \beta = \sqrt[6]{6} = 1.34800\ldots$$

∘) All triangulations of $\Delta^2 \times \Delta^k$ are connected by flips.

Research problem: generalize this to $\Delta^3 \times \Delta^k$.

5 The permutahedron, again

Zonotope = Minkowski sum of segments

Lawrence polytope = Cayley embedding of segments.

Hence

Theorem: There is a bijection between zonotopal tilings of the zonotope $A_1 + \cdots + A_k$ and polyhedral subdivisions of the Lawrence polytope $C(A_1, \to A_k)$.

Simplest example: $A_1 = A_2 = A_3 = \cdots = A_k$

So: $A_1 + \cdots + A_k$ = segment

Fine Mixed subdivisions of it = monotone paths in the k-cube

$$C(A_1, \ldots, A_k) = \text{prism over } \Delta^{k-1}$$

Corollary: there is a bijection between monotone paths in the k-cube and triangulations of the prism over Δ^k (both give $k!$ elements).

And:

$$\text{secondary polytope}\left(\Delta^{k-1} \times I\right) = \text{permutahedron} = \text{monotone path polytope}\left(I^k\right)$$