Flat Chains: Problems

Robert Hardt

July 2011

The problems below are transcriptions from Professor Robert Hardt's first lecture at MSRI Graduate Summer Workshop on Geometric Measure Theory and Its Applications.

Exercise 1. Show that for the space of currents $\mathcal{D}_m(\mathbb{R}^n)$:

$$\text{mass convergence} \Longrightarrow \mathcal{F}\text{- convergence} \Longrightarrow \text{weak convergence}$$

where the mass and flat norms are defined by

$$\mathbb{M}(\varphi) = \sup_x \sup_{v_1,\dots,v_m \text{ o.n.}} \langle \varphi(x), v_1 \wedge \cdots \wedge v_m \rangle, \quad \varphi \in \mathcal{D}^m(\mathbb{R}^n),$$

$$\mathbb{M}(T) = \sup\{T(\varphi) : \varphi \in D^m(\mathbb{R}^n), \mathbb{M}(\varphi) \le 1\}, \quad T \in \mathcal{D}_m(\mathbb{R}^n)$$

and

$$\mathcal{F}(T) = \inf\{\mathbb{M}(T - \partial S) + \mathbb{M}(S) : S \in \mathcal{D}_{m+1}(\mathbb{R}^n)\}.$$

Exercise 2. Using atomic measures find sequences $P_i, Q_i \in \mathcal{D}_0(\mathbb{R})$ such that $\mathcal{F}(P_i) \to 0$ but $\mathbb{M}(P_i) \to \infty$ and $Q_i \to 0$ weakly as currents but $\mathcal{F}(Q_i) \to \infty$.

Exercise 3. Let $\mathbb{R}^4 = \mathbb{C}^2 = \{(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)\}$. For $v, w \in \mathbb{C}^2$ orthonormal show that

$$\langle dx_1 dy_1 + dx_2 dy_2, v \wedge w \rangle \le 1$$

with equality if and only if $w = iv$.

Exercise 4. Suppose $[0,1]^2$ is the unit square and for $1 \ge 2\delta$ let K_δ be the convex hull of the 4 balls with radius δ and centers (δ, δ), $(\delta, 1-\delta)$, $(1-\delta, \delta)$ and $(1-\delta, 1-\delta)$. Let

$$S_\delta = [[0,1]]^2 - [[K_\delta]].$$

Find δ_0 which minimizes $\mathbb{M}(\partial T - \partial S_\delta) + \mathbb{M}(S_\delta)$ where $T = \partial[[0,1]]^2$.

Exercise 5. Show that the following 1-dimensional currents in $\mathbb{R}^2$ have mass 1 but their boundaries have infinite mass.

$$\text{(Midget) } P(adx + bdy) = b(0,0)$$

$$\text{(Punk) } Q(adx + bdy) = \int_0^1 b(t,0)dt$$

Exercise 6. Recall a normal current is a current T such that T and ∂T have finite mass. Recall that

$$\mathbb{F}(\varphi) = \max\{\|\varphi\|, \|d\varphi\|\}, \quad \varphi \in \mathcal{D}^m(\mathbb{R}^n),$$

$$\mathbb{F}(T) = \sup\{T(\varphi) : \varphi \in D^m(\mathbb{R}^n), \mathbb{F}(\varphi) \leq 1\}, \quad T \in \mathcal{D}_m(\mathbb{R}^n)$$

and $\mathcal{F}$-convergence is equivalent to $\mathbb{F}$-convergence. Recall that the collection of (real) flat chains is the $\mathbb{F}$-closure of normal currents. We denote by $\mathbb{F}_m(\mathbb{R}^n)$ the collection of flat chains of dimension m in $\mathbb{R}^n$.

Show that Midget and Punk are not flat chains.

Exercise 7. Show that for any flat chain $T \in \mathbb{F}_m(\mathbb{R}^n)$,

$$\mathbb{M}(T) < \infty \iff \lim_{i\to\infty} \mathbb{M}(T - Q_i) = 0 \quad \text{for some normal currents } Q_i$$