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MSRI Graduate Summer School Lecture 1: Intro to Cluster Algebras

Reference: Cluster Algebras I, II, III, IV by Fomin + Zelevinsky

Cluster algebras: class of commutative rings, introduced in 2000 by Fomin & Zelevinsky

Original motivation: Lusztig's dual canonical basis & total positivity

Since 2000, cluster algebra structures have been connected to:

- coordinate rings of Grassmannians, flag varieties, other homog. spaces...
- quiver reps
- Teichmüller theory
- invariant theory
- tropical calculus
- Poisson geometry
- Lie theory
- combinatorics (associahedra, perfect matchings, cluster complex...)
- integrable systems
- total positivity

Cluster Algebra Portal:

www.math.lsa.umich.edu/~fomin/cluster.html

Reference for conferences, software, etc.

Fall 2012: Semester-long program on cluster algebras

Def: (F & Z) A clust. alg. $\mathcal{A}$ is a certain subalgebra of $k(x_1, \dots, x_n)$, the field of rational functions over $\{x_1, \dots, x_n\}$. Generators are constructed by a series of exchange relations which in turn induce all relations satisfied by the generators.

Def: An $n \times n$ integral matrix B is skew-symmetrizable if $\exists\ d_1, d_2, \dots, d_n \in \mathbb{Z}^+$ s.t. $d_i b_{ij} = -d_j b_{ji}\ \forall\ i,j$

Any skew-symmetric matrix is skew-symmetrizable. More generally, can start from skew-symmetric matrix & scale the columns by pos. integers $d_1 \dots d_n$.

We can associate a (coefficient-free) cluster algebra $\mathcal{A}(B)$ to any such matrix B.

Start w/ a seed $(\{x_1, \dots, x_n\}, B)$.

From this seed we can mutate in each of n directions, obtaining n more seeds.

Columns of B encode the exchange relations:

For $k \in \{1, \dots, n\}$, $$X_k X_k' = \prod_{b_{ik} > 0} X_i^{|b_{ik}|} + \prod_{b_{ik} < 0} X_i^{|b_{ik}|}$$

This defines a new cluster variable X_k'.

For $k \in \{1, \dots, n\}$, $\exists$ another seed for $\mathcal{A}$ consists of the clusters $\{X_1, \dots, \hat{X_k}, \dots, X_n\} \cup \{X_k'\}$ and matrix $\mu_k(B)$, where

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k=i \text{ or } k=j \\ b_{ij} & \text{if } b_{ik}b_{kj} \le 0 \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

($\mu_k(B)$ is again skew-symmetrizable)

Start from the initial seed & apply all possible sequences of mutations: this produces the set of all cluster variables (possibly infinite).

<u>Def</u>: The cluster algebra $\mathcal{A}(B)$ is the subalgebra of $k(x_1, \dots, x_n)$ generated by all cluster variables.

<u>Example 1: Rank 2 cluster algebras</u>:

Let $\mathcal{F} = \mathbb{Q}(y_1, y_2)$ — rational functions in y_1 & y_2.

Fix $b, c \in \mathbb{Z}^+$ and define $B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$

Then $y_1 y_1' = y_2^c + 1$. We refer to y_1' as y_0.

Also, $y_2 y_2' = y_1^b + 1$. Refer to y_2' as y_3.

$\mu_1(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ and $\mu_2(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$

$\mu_1^2 = \mu_2^2 = id$, $\mu_1 \circ \mu_2 (B) = \mu_2 \circ \mu_1 (B) = B$

Our cluster variables are $\{y_m\}_{m \in \mathbb{Z}}$ — they satisfy

$$y_{m-1}y_{m+1} = \begin{cases} y_m^b + 1 & m \text{ odd} \\ y_m^c + 1 & m \text{ even} \end{cases} \qquad (\star)$$

The cl. alg. $A(b,c)$ is the subring of $\mathcal{F}$ gen. by the y_m $\forall m$

Clust var's: y_m

Exchange relations: $(\star)$ (note: 3-term)

Clusters: $\{y_m, y_{m+1}\}$ $\forall m$

Exchange graph: $\cdots \overset{\{y_0,y_1\}}{\bullet} - \overset{\{y_1,y_2\}}{\bullet} - \overset{\{y_2,y_3\}}{\bullet} - \overset{\{y_3,y_4\}}{\bullet} - \bullet \cdots$

Note: one can express every cluster variable as a rational expression in terms of the variables of a single cluster.

Laurent phenomenon (F-Z): Given any seed $(\{x_1,\dots,x_n\}, B)$ for any cluster algebra and any cluster variable x, one can express x as Laurent poly in variables $\{x_1,\dots,x_n\}$.

Positivity conjecture (F-Z): All coefficients in the above Laurent expansion are positive.

Ex: Let $b=1$, $c=1$. Start w/ $\{y_1, y_2\}$, then

$$y_3 = \frac{y_2+1}{y_1}, \quad y_4 = \frac{y_3+1}{y_2} = \frac{\frac{y_2+1}{y_1}+1}{y_2} = \frac{y_1+y_2+1}{y_1 y_2}$$

Cancellation here!!

$$y_5 = \frac{y_4+1}{y_3} = \frac{\frac{y_1+y_2+1}{y_1y_2}+1}{(y_2+1)/y_1} = \frac{y_1y_2+y_1+y_2+1}{y_2(y_2+1)} = \frac{y_1+1}{y_2}$$

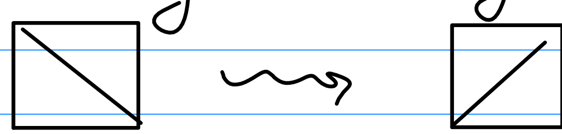
$$y_6 = \frac{y_5+1}{y_4} = \frac{\frac{y_1+1}{y_2}+1}{(y_1+y_2+1)/y_1y_2} = y_1 \;!$$ Periodic.

Note: all Laurent polynomials, all coeff's pos.

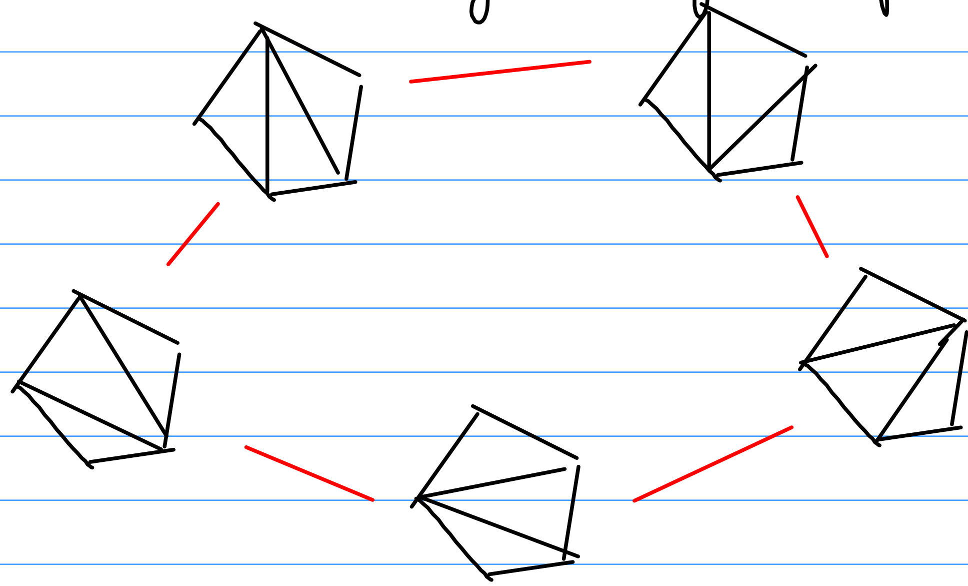
Only finitely many cluster algebras so finite type.
This is of type A_2.

Example 2: Consider a polygon w/ $n+3$ sides, choose any triangulation T.
(Will have $n+3$ boundary segments, &
n diagonals.)

Note: The set of all triangulations of an $(n+3)$-gon are connected by elementary moves called flips:

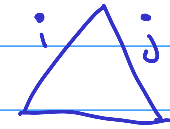


Eg there are 5 triangulations of a pentagon

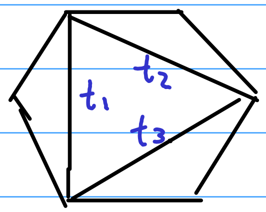


Can associate an $n \times n$ matrix $B(T)$ to T.
First label the n diagonals of T from 1 to n.
$B(T) = (b_{ij})$ where

$$b_{ij} = \#\left\{\begin{array}{l}\text{triangles w/ sides } i \,\&\, j \text{, w/} \\ j \text{ following } i \text{ in clockwise order}\end{array}\right\}$$



$$- \#\left\{\begin{array}{l}\text{triangles w/ sides } i \,\&\, j \text{, w/} \\ j \text{ following } i \text{ in counterclockwise order}\end{array}\right\}$$

eg. $T =$ 

$$\rightsquigarrow \quad B(T) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

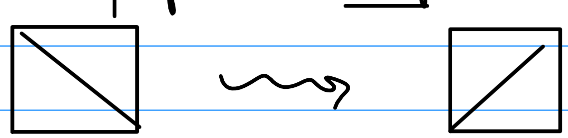
This gives a construction of a cluster alg $\mathcal{A}(B(T))$ assoc. to each triangulation T of a polygon.

Let's consider $\mu_1(B(T))$.

From $$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} & \text{if } b_{ik} b_{kj} \le 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

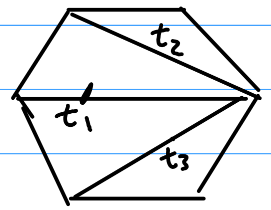
we get $$\mu_1(B(T)) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Suppose we perform a <u>flip</u> in the triangulation, i.e.



local picture

eg. let's flip t_1.

$T' =$ 

. $B(T') = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

Claim: Let T be triangulation, and T' be new triangulation obtained by flipping t_i. Then

$$B(T') = \mu_i(T).$$

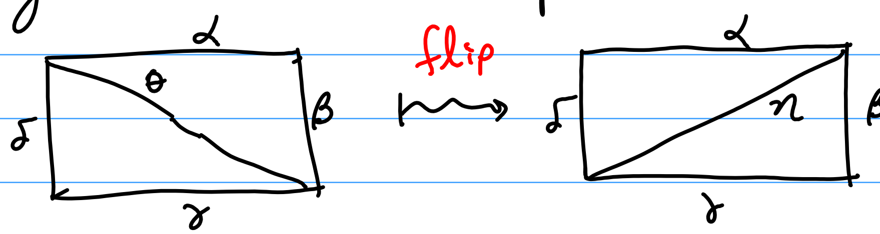
Cor: Given a $(n+3)$-gon and a triangulation T, the cluster alg $\mathcal{A}(B(T))$ does not depend on T, only on the number $n+3$.

Thm: We have bijections

cluster variables x_γ $\longleftrightarrow$ diagonals γ of $(n+3)$ gon

clusters $\longleftrightarrow$ triangulations.

exchange relation $\longleftrightarrow$ flips



Exchange relation: $x_\theta x_\eta = x_\alpha x_\gamma + x_\beta x_\delta$.

Next week: Gregg will explain more about how to generalize this from polygons to arbitrary surfaces.

(Sometimes one would like to assign a variable to the boundary segments in the polygon. Such a variable will never get flipped, & will be present in every triangulation (cluster).)

To do so, use cluster algebras w/ frozen variables (coefficients).

To define such a cl. alg use rectangular $m \times n$ matrix B, $m > n$, whose top $n \times n$ part is skew-symmetrizable.
Start w/ seed $(\{\underbrace{x_1, x_2, .., x_n}_{\text{clust.}}, \underbrace{x_{n+1}, .., x_m}_{\text{coeff}}\}, B)$.

We only mutate in directions $1, 2, .., n$.

Columns of B encode the <u>exchange relations</u>:

For $k \in \{1, \dots, n\}$, $$x_k x_k' = \prod_{b_{ik} > 0} x_i^{|b_{ik}|} + \prod_{b_{ik} < 0} x_i^{|b_{ik}|}$$

now coeff var's may appear

Mutating a matrix works same:

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} & \text{if } b_{ik} b_{kj} \leq 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

Def: Let $\Sigma = (\{ \underbrace{x_1, x_2, \dots, x_n}_{\text{clust.}}, \underbrace{x_{n+1}, \dots, x_m}_{\text{coeff}} \}, \overset{n \times n}{B})$ be a seed. Then the cluster algebra $\mathcal{A}(\Sigma)$ is the $\mathbb{Z}[x_{n+1}, \dots, x_m]$-subalgebra of $\mathbb{Z}[x_{n+1}, \dots, x_m](x_1, \dots, x_n)$ generated by all cluster variables.

(If more time, mention quiver mutation)

Exercises

1. Check that if $B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ & we denote the initial cluster variables as (y_1, y_2), then the set of all cluster variables is in bijection with $\mathbb{Z}$ (denote them by y_m, $m \in \mathbb{Z}$), & they satisfy:

$$y_{m-1} y_{m+1} = \begin{cases} y_m^b + 1 & m \text{ odd} \\ y_m^c + 1 & m \text{ even} \end{cases}$$

2. For which b and $c \in \mathbb{Z}$ is the set of cluster variables finite?

3. Prove the following claim.

<u>Claim:</u> Let T be triangulation, and T' be new triangulation obtained by flipping t_i. Then

$$B(T') = \mu_i(T).$$

4. Explicitly compute all cluster variables & seeds associated to the cluster algebra coming from a pentagon.