Lecture 1: Exercises

Exercise 1a. *Show that $\mathbb{P}$ is torsion-free as a multiplicative group. Why doesn't your argument prove a similar result about fields?*

Exercise 1b. *Show that $\mu_k(B)$ is an exchange matrix.*

Exercise 1c. *Show that matrix mutation can be equivalently defined by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\}; \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{otherwise.} \end{cases}$$

Exercise 1d. *Show that each mutation μ_k is an involution on labeled seeds.*

Example

Let $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and take $\mathbb{P} = \{1\}$, so that $\mathbf{y} = (1,1)$.

$$\begin{tikzcd}
\begin{matrix}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\ \left[x_1 \ \ x_2\right]\end{matrix} \arrow[r, leftrightarrow, "\mu_1"] \arrow[d, leftrightarrow, "\mu_2"] & \begin{matrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ \left[\frac{x_2+1}{x_1} \ \ x_2\right]\end{matrix} \arrow[r, leftrightarrow, "\mu_2"] & \begin{matrix}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\ \left[\frac{x_2+1}{x_1} \ \ \frac{x_1^2+(x_2+1)^2}{x_1^2 x_2}\right]\end{matrix} \arrow[d, leftrightarrow, "\mu_1"] \\
\begin{matrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ \left[x_1 \ \ \frac{x_1^2+1}{x_2}\right]\end{matrix} \arrow[r, leftrightarrow, "\mu_1"] & \begin{matrix}\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\ \left[\frac{x_1^2+x_2+1}{x_1 x_2} \ \ \frac{x_1^2+1}{x_2}\right]\end{matrix} \arrow[r, leftrightarrow, "\mu_2"] & \begin{matrix}\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ \left[\frac{x_1^2+x_2+1}{x_1 x_2} \ \ \frac{x_1^2+(x_2+1)^2}{x_1^2 x_2}\right]\end{matrix}
\end{tikzcd}$$

Cluster variables: $x_1, \quad x_2, \quad \frac{x_2+1}{x_1}, \quad \frac{x_1^2+(x_2+1)^2}{x_1^2 x_2}, \quad \frac{x_1^2+x_2+1}{x_1 x_2}, \quad \frac{x_1^2+1}{x_2}$

Exercise 1e. *Verify the previous example by hand.*

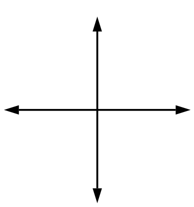
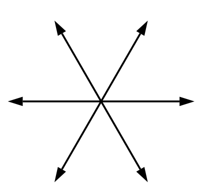
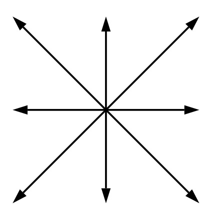
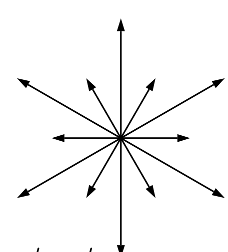
Exercise 1f. *Redo the example with $\mathbb{P}$ a general semifield, and initial coefficients $\mathbf{y} = (y_1, y_2)$. Find all labeled seeds $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ in the exchange pattern. Are there still finitely many distinct labeled seeds?*

Exercise 1g. *Show that, for any cluster in a cluster pattern, no cluster variable occurs twice. (Hint: prove something much, much stronger. In fact, if you read the directions very strictly in a previous exercise, you should have proved the stronger thing already.)*

Exercise 1h. *Suppose $t \overset{k}{\text{---}} t'$. Let x_k and x'_k be the cluster variables in $\mathbf{x}_t$ and $\mathbf{x}_{t'}$ that are related by the exchange relation. Show that $x_k \neq x'_k$ (Hint: if you did the previous exercise the way I have in mind, this becomes easy.)*

Root systems of rank 2

The *rank* of a root system is the dimension of its linear span. These are the root systems of rank 2 (up to scaling and rotation):

$A_1 \times A_1$:  A_2:  B_2:  G_2: 

Exercise 1i. *Verify that the collections of vectors shown on the previous slide are indeed root systems.*

It is acceptable to do Exercise 1i visually, without writing anything. Condition (ii) is easy. To get (i) and (iii), you can just check that, for any $\alpha, \beta \in \Phi$, the vector $\sigma_\alpha(\beta)$ is in Φ and differs from β by an integer multiple of α.

Exercise 1j. *Show that the four root systems shown on the previous page are the only root systems of rank 2, up to scaling and rotation.*

To get you started on Exercise 1j, note that up to scaling and rotation, we may as well have the root $\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in Φ. What vectors β have the property that $\langle \alpha^\vee, \beta \rangle$ and $\langle \beta^\vee, \alpha \rangle$ are both integers?

Exercise 1k. *1. Show that* $\{e_j - e_i : i, j \in [n+1], i \neq j\}$ *is a rank-n root system. (It is called* A_n*.)*
2. Show that $\{\pm e_i : i \in [n]\} \cup \{\pm e_j \pm e_i : i, j \in [n], i < j\}$ *is a rank-n root system. (It is called* B_n*.)*
3. Show that $\{\pm 2e_i : i \in [n]\} \cup \{\pm e_j \pm e_i : i, j \in [n], i < j\}$ *is a rank-n root system. (It is called* C_n*.)*
4. Show that $\{\pm e_j \pm e_i : i, j \in [n], i < j\}$ *is a rank-n root system. (It is called* D_n*.)*

These will be repetitive, so re-use your work. A different one:

Exercise 1l. *Show that the following is a rank-4 root system. (It is called* F_4*.)*

$$\{\pm e_j \pm e_i : i, j \in [4], i < j\} \cup \{\pm e_i : i \in [4]\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}$$

Exercise 1m. *For each part of Exercise 1k, we propose a set of positive roots below. Verify that this is a valid choice. Also, find the corresponding set of simple roots. 1.* $\{e_j - e_i : i, j \in [n+1], i < j\}$ *2.* $\{e_i : i \in [n]\} \cup \{e_j \pm e_i : i, j \in [n], i < j\}$ *3.* $\{2e_i : i \in [n]\} \cup \{e_j \pm e_i : i, j \in [n], i < j\}$ *4.* $\{e_j \pm e_i : i, j \in [n], i < j\}$

Exercises, in order of priority

There are more exercises than you can be expected to complete in a day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

1b, 1d, 1e, 1i, 1k.1,

1k.2–4, 1m, 1f, 1g, 1h, 1j, 1l, 1c, 1a.