Contents

1 Introduction

Introduction

In Lecture 1, we defined cluster algebras, based on a special kind of recursion called a *cluster pattern*. This recursion is very hard to solve in general (and even in particular!). In fact, it's so hard, that we have been willing to settle for "solutions" that merely write down the "vital statistics" of cluster variables (height, weight, Soc. Sec. #, etc.) so we can at least uniquely identify them. These kinds of solutions are not completely satisfactory, but they do allow us to prove some important properties of cluster algebras. Equally important (and possibly even more fun), the vital statistics display fascinating combinatorics, which connects them to other areas of math (e.g. root systems/reflection groups).

Introduction (continued)

Some of the vital statistics of cluster variables that we will talk about:

- denominator vectors,
- principal coefficients,
- **g**-vectors, and
- F-polynomials.

In fact, the previous slide maligns two of these statistics: Two of them (**g**-vectors and F-polynomials) together actually determine the cluster variable.

Besides vital statistics, we also explore the extent to which the recursion "collapses," in the sense that the underlying combinatorial structure is smaller than the n-regular tree.

Recall the example with $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbb{P} = \{1\}$ on page 4-5 of the notes for Lecture 1. In the example, the cluster algebra is supposed to "live" on the infinite path $\mathbb{T}_n$. But the real combinatorial backbone is a cycle with 6 vertices (i.e. 6 seeds) and 6 edges (i.e. 6 mutations). This is the *exchange graph* of the cluster pattern.

Another (dual) way to organize the combinatorics is to view the cluster variables as vertices of a simplicial complex, whose maximal simplices are the clusters. This is the *cluster complex*. In the example, there are 6 vertices (i.e. 6 cluster variables) and 6 maximal simplices (i.e. 6 clusters).

The fact that the exchange graph and the cluster complex are the same is just an artifact of low dimension. In general, the exchange graph is 1-dimensional, and the cluster complex is $(n-1)$-dimensional.

2 The exchange graph and the cluster complex

Review of the basic setup

The initial *exchange matrix* is a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$ with integer entries. The *coefficient semifield* is $\mathbb{P} = (\mathbb{P}, \oplus, \cdot)$ such that $(\mathbb{P}, \cdot)$ is an abelian group, and $\oplus$ is an auxiliary addition. ($\div$ yes, $-$ no.) The *ambient field* $\mathcal{F}$ is (isomorphic to) the field of rational functions in n variables, with coefficients in the group ring $\mathbb{Q}\mathbb{P}$. A *labeled seed* is a triple $(\mathbf{x}, \mathbf{y}, B)$, where

- B is an $n \times n$ *exchange matrix*,
- $\mathbf{y} = (y_1, \ldots, y_n)$ is a tuple of elements of $\mathbb{P}$ called *coefficients*,
- $\mathbf{x} = (x_1, \ldots, x_n)$ is a tuple (or "*cluster*") of algebraically independent elements of $\mathcal{F}$ called *cluster variables*.

By repeated *seed mutations*, we get a labeled seed for each vertex of the *n-ary tree* $\mathbb{T}_n$.

The *cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is the algebra generated by the set of all cluster variables in all seeds.

Unlabeled seeds

Two labeled seeds are *equivalent* if they can be made identical by simultaneously re-indexing the rows and columns of B and the entries of the tuples $\mathbf{x}$ and $\mathbf{y}$. That is, $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ is equivalent to $\Sigma' = (\mathbf{x}', \mathbf{y}', B')$ if there exists a permutation π of $[n]$ such that

$$x'_i = x_{\pi(i)}, \quad y'_i = y_{\pi(i)}, \quad b'_{ij} = b_{\pi(i)\pi(j)}$$

for all $i, j \in [n]$. An *unlabeled seed* (or usually just a *seed*) is an equivalence class of labeled seeds. The point is, we don't really care how the matrix is indexed. All we need to know, to do mutation, is the correspondence between entries in $\mathbf{x}$, entries in $\mathbf{y}$, and rows/columns of B.

The exchange graph

Given the initial (labeled) seed $(\mathbf{x}, \mathbf{y}, B)$, the *exchange graph* $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ is obtained from $\mathbb{T}_n$ by identifying vertices that map to the same (unlabeled) seed. We think of this as a graph whose vertices *are* seeds, and whose edges are mutations.

Again, the example with $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbb{P} = \{1\}$.

$$\begin{array}{ccccc}
\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\
\begin{bmatrix} x_1 & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{x_2+1}{x_1} & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{x_2+1}{x_1} & \frac{x_1^2+(x_2+1)^2}{x_1^2 x_2} \end{bmatrix} \\
\updownarrow \mu_2 & & & & \updownarrow \mu_1 \\
\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\
\begin{bmatrix} x_1 & \frac{x_1^2+1}{x_2} \end{bmatrix} & & \begin{bmatrix} \frac{x_1^2+x_2+1}{x_1 x_2} & \frac{x_1^2+1}{x_2} \end{bmatrix} & & \begin{bmatrix} \frac{x_1^2+x_2+1}{x_1 x_2} & \frac{x_1^2+(x_2+1)^2}{x_1^2 x_2} \end{bmatrix}
\end{array}$$

The hexagon example works just as well with only labeled seeds. Here's an example that shows the need for unlabeled seeds. Here, the exchange graph is a cycle with 5 vertices.

$$\begin{array}{ccccc}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \longleftarrow \\
\begin{bmatrix} x_1 & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{x_2+1}{x_1} & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{x_2+1}{x_1} & \frac{x_1+x_2+1}{x_1 x_2} \end{bmatrix} & \\
\updownarrow \mu_2 & & & & & \textit{identify} \\
 & & & & & \textit{these} \\
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \longleftarrow \\
\begin{bmatrix} x_1 & \frac{x_1+1}{x_2} \end{bmatrix} & & \begin{bmatrix} \frac{x_1+x_2+1}{x_1 x_2} & \frac{x_1+1}{x_2} \end{bmatrix} & & \begin{bmatrix} \frac{x_1+x_2+1}{x_1 x_2} & \frac{x_2+1}{x_1} \end{bmatrix} &
\end{array}$$

The cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ depends on B. Up to *strong isomorphism*, $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ does not depend on $\mathbf{x}$. (Given two choices $\mathbf{x}$ and $\mathbf{x}'$, there is an algebra isomorphism of $\mathcal{F}$ mapping seeds to seeds, and thus inducing a isomorphism of cluster algebras.) In particular, $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ depends only on $(\mathbf{y}, B)$.

Conjecture 2.1 (Fomin and Zelevinsky, CDM Conj. 14(1))**.** *The exchange graph* $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ *depends only on* B.

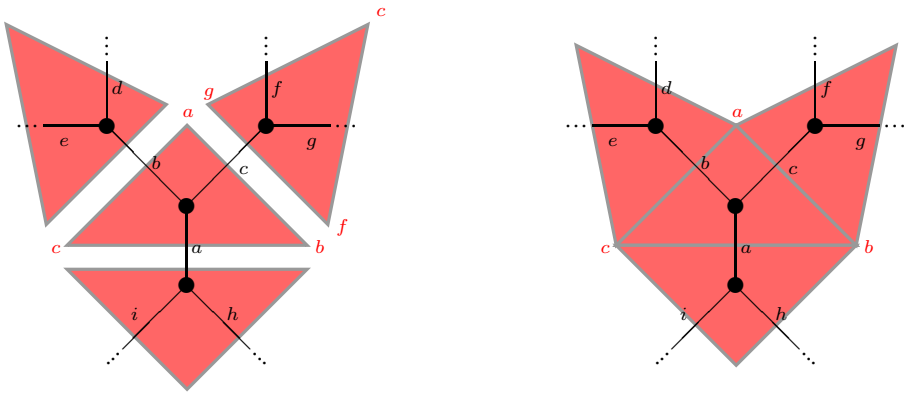
That is, if two labeled seeds are equivalent in the cluster pattern given by $(\mathbf{x}, \mathbf{y}, B)$, then the corresponding labeled seeds are equivalent in any cluster pattern $(\mathbf{x}', \mathbf{y}, B)$. You verified a case of this in an exercise.

When we form the exchange graph from $\mathbb{T}_n$, we lose the labeling of edges by $1, \ldots, n$. But we retain the information of which cluster variable in each seed is exchanged along each edge. This leads to two insights:

- Given a seed in the exchange graph, we can index the cluster, coefficient tuple and exchange matrix in that seed by the set of edges incident to that seed in the exchange graph.
- The exchange graph is equipped with a *connection*. This means that every edge μ connecting two vertices Σ and Σ' is equipped with a canonical bijection between the n edges incident to Σ and the n edges incident to Σ', fixing μ.

Exchange graphs and simplicial complexes

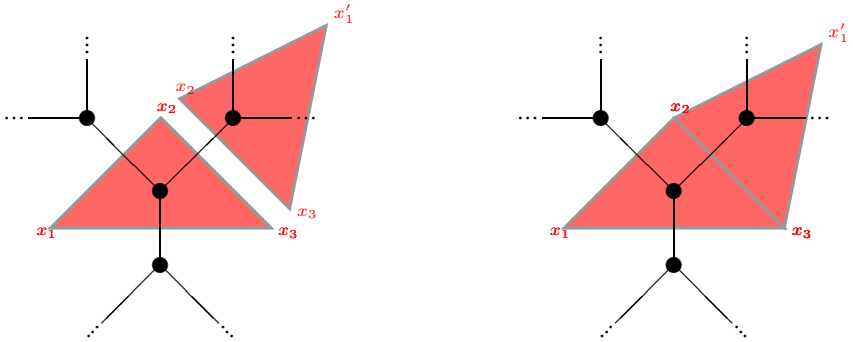
An n-regular graph with connection defines a simplicial complex: For each vertex Σ of the graph, think of the set of n edges incident to Σ as an abstract $(n-1)$-simplex Δ_Σ. (Get it straight: Vertices of the simplex are edges of the graph.) For each edge μ connecting two vertices Σ and Σ', identify the vertices of Δ_Σ and $\Delta_{\Sigma'}$ (except μ) according to the connection.



The cluster complex

Conjecture 2.2. *The simplicial complex defined by the exchange graph has vertices specified by the cluster variables.*

It is easy to label the vertices of the complex by cluster variables: The conjecture is that every vertex of the complex is labeled by a distinct cluster variable. When Conjecture 2.2 holds, call this complex the *cluster complex*.



Failure of the conjecture might look like this: Suppose a rank-2 cluster algebra has distinct cluster variables a, b, c, d, and x. Suppose also that the exchange graph is a 6-cycle, and that the clusters in the 6 seeds are

$$\{a,b\},\ \{b,x\},\ \{x,c\},\ \{c,d\},\ \{d,x\},\ \{x,a\}.$$

Conjecture 2.3 (Fomin and Zelevinsky, CDM Conj. 14(3))**.** *For every cluster variable x, the seeds whose clusters contain x induce a connected subgraph of* $\mathrm{Ex}(\mathbf{x},\mathbf{y},B)$.

Exercise 2a. *Assuming Conjecture 2.3, prove the following: If x is a cluster variable, then any two seeds containing x are related by a sequence of mutations that fix x. Conclude that Conjecture 2.3 implies Conjecture 2.2.*

You will want to use Exercises 1g and 1h.

3 Denominator vectors

$\mathbb{Z}^n$-gradings

The ring $\mathbb{R}[x]$ of real polynomials in x is a $\mathbb{N}$-*graded algebra*. First of all, this means that $\mathbb{R}[x]$ is an $\mathbb{N}$-graded vector space $\bigoplus_{n\in\mathbb{N}} V_n$, where each V_n is a vector space. (Take $V_n = \mathbb{R}x^n$.) Also, this means that, if $x \in V_p$ and $y \in V_q$, then $xy \in V_{p+q}$. The Laurent polynomial ring $\mathbb{R}[x,x^{-1}]$ is an $\mathbb{Z}$-graded algebra in the same way.

A $\mathbb{Z}^n$-graded algebra is the same thing, except that the graded pieces are indexed by integer vectors, and "$p+q$" is interpreted as vector addition. For example, the Laurent polynomial ring $\mathbb{R}[x_1,x_1^{-1},\ldots,x_n,x_n^{-1}]$ is $\mathbb{Z}^n$-graded with $V_{i_1\cdots i_n} = \mathbb{R}x_1^{i_1}\cdots x_n^{i_n}$. We will discuss two important $\mathbb{Z}^n$-gradings of a cluster algebra, given by *denominator vectors* and $\mathbf{g}$-*vectors*.

The Laurent phenomenon

Once again, take $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbb{P} = \{1\}$.

$$\begin{matrix} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\ [x_1\ \ x_2] \end{matrix} \overset{\mu_1}{\longleftrightarrow} \begin{matrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ \left[\frac{x_2+1}{x_1}\ \ x_2\right] \end{matrix} \overset{\mu_2}{\longleftrightarrow} \begin{matrix} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\ \left[\frac{x_2+1}{x_1}\ \ \frac{x_1^2+(x_2+1)^2}{x_1^2x_2}\right] \end{matrix} \overset{\mu_1}{\longleftrightarrow} \begin{matrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ \left[v\ \ \frac{x_1^2+(x_2+1)^2}{x_1^2x_2}\right] \end{matrix}$$

We calculated $v = \frac{x_1^2+x_2+1}{x_1x_2}$. This is a Laurent polynomial in x_1 and x_2. *A priori, it need only have been a rational function.*

$$\begin{aligned} v = \frac{1+\frac{x_1^2+(x_2+1)^2}{x_1^2x_2}}{\frac{x_2+1}{x_1}} &= \frac{x_1}{x_2+1}\cdot\frac{x_1^2x_2+x_1^2+(x_2+1)^2}{x_1^2x_2} \\ &= \frac{x_1}{x_2+1}\cdot\frac{(x_2+1)(x_1^2+x_2+1)}{x_1^2x_2} \end{aligned}$$

The Laurent phenomenon is the assertion that this kind of cancellation *always* happens.

Theorem 2.4 (Fomin and Zelevinsky, CA I, CA II)**.** *Every cluster variable in $\mathcal{A}(\mathbf{x},\mathbf{y},B)$ is a Laurent polynomial in $\mathbf{x}$, whose coefficients are integer polynomials $\mathbf{y}$.*

Conjecture 2.5 (Fomin and Zelevinsky, CA I)**.** *Every cluster variable in $\mathcal{A}(\mathbf{x},\mathbf{y},B)$ is a Laurent polynomial in $\mathbf{x}$, whose coefficients are polynomials $\mathbf{y}$ with nonnegative integer coefficients.*

Why is this hard? After all, the coefficients in the exchange relations have nonnegative coefficients.

$$x_k' = \frac{y_k\prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k\oplus 1)x_k}.$$

A priori, something like the following could happen: $\frac{x^2+1}{x+1} = x-1$.

Denominator vectors

Writing terms in a Laurent polynomial over a common denominator, we get $\frac{\text{polynomial}}{\text{monomial}}$. The *denominator vector* of a cluster variable is the degree sequence of its denominator. In our favorite example, $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbb{P} = \{1\}$:

x_1,	x_2,	$\frac{x_2+1}{x_1}$,	$\frac{x_1^2+(x_2+1)^2}{x_1^2x_2}$,	$\frac{x_1^2+x_2+1}{x_1x_2}$,	$\frac{x_1^2+1}{x_2}$
$[-1,0]$,	$[0,-1]$,	$[1,0]$,	$[2,1]$,	$[1,1]$,	$[0,1]$

Denominator vectors are a $\mathbb{Z}^n$-grading. Indeed, $\mathcal{A}(\mathbf{x},\mathbf{y},B)$ is an $\mathbb{Z}^n$-graded subalgebra of the Laurent polynomial ring $\mathbb{ZP}[x_1,x_1^{-1},\ldots,x_n,x_n^{-1}]$. Conjecturally, denominator vectors distinguish cluster variables.

Conjecture 2.6. *Different cluster variables have different denominator vectors.*

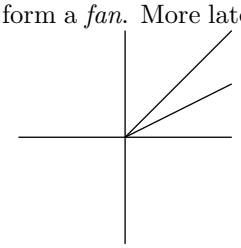
If this conjecture is true, then we can model the cluster complex by knowing the list of possible denominator vectors and knowing which denominator vectors are in the same cluster. In fact, much more than Conjecture 2.6 is probably true...

Cluster monomials

A *cluster monomial* is a monomial in the cluster variables in some single cluster (an ordinary monomial, *not* a Laurent monomial).

Conjecture 2.7. *Different cluster monomials have different denominator vectors.*

This would imply that denominator vectors form a *fan*. More later... For now a picture, for our favorite example.



4 Principal coefficients

Tropical semifields

There is a particularly nice choice of coefficient semifield. Let $u_1, \ldots, u_m$ be formal symbols called *tropical variables*. $\mathrm{Trop}(u_1, \ldots, u_m)$ is the free abelian group generated by $u_1, \ldots, u_m$. Its elements are formal products of the form $\prod_{i=1}^m u_i^{a_i}$ with $a_i \in \mathbb{Z}$ and multiplication given by

$$\prod_{i=1}^m u_i^{a_i} \cdot \prod_{i=1}^m u_i^{b_i} = \prod_{i=1}^m u_i^{a_i+b_i}.$$

We define an auxiliary addition $\oplus$ in $\mathrm{Trop}(u_1, \ldots, u_m)$ by

$$\prod_{i=1}^m u_i^{a_i} \oplus \prod_{i=1}^m u_i^{b_i} = \prod_{i=1}^m u_i^{\min(a_i,b_i)}.$$

The triple $(\mathrm{Trop}(u_1, \ldots, u_m), \oplus, \cdot)$ is a semifield.

Cluster algebras of geometric type

Take $\mathbb{P} = \mathrm{Trop}(u_1, \ldots, u_m)$. Elements of $\mathbb{P}$ are Laurent monomials in $u_1, \ldots, u_m$. Thus the group rings $\mathbb{Z}\mathbb{P}$ and $\mathbb{Q}\mathbb{P}$ are just the rings of Laurent polynomials in $u_1, \ldots, u_m$. A *cluster algebra of geometric type* is a cluster algebra having $\mathbb{P} = \mathrm{Trop}(u_1, \ldots, u_m)$ for its coefficient semifield.

Geometric type simplifies the story considerably: Each $\mathbf{y}_t$ in the cluster pattern is a collection of Laurent monomials in $u_1, \ldots, u_m$. Recall that mutation for coefficients looks like

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k. \end{cases}$$

But y_k is $\prod_{i=1}^m u_i^{a_i}$ for some integers a_i, so we can rewrite:

$$y_k = \frac{\prod u_i^{[a_i]_+}}{\prod u_i^{[-a_i]_+}} \qquad \text{and} \qquad (y_k \oplus 1)^{-1} = \prod u_i^{[-a_i]_+}.$$

Thus

$$\begin{aligned} y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} &= \begin{cases} \prod u_i^{b_{kj}[a_i]_+} & \text{if } b_{kj} \geq 0 \\ \prod u_i^{-b_{kj}[-a_i]_+} & \text{if } b_{kj} \leq 0 \end{cases} \\ &= \prod u_i^{\mathrm{sgn}(b_{kj})[a_i b_{kj}]_+} \end{aligned}$$

Extended exchange matrices

In geometric type, if y_k is $\prod_{i=1}^m u_i^{a_i}$, then coefficient mutation replaces y_j by $y_j \prod u_i^{\mathrm{sgn}(b_{kj})[a_i b_{kj}]_+}$. This looks like *matrix mutation*. A Y-seed $(\mathbf{y}, B)$ of geometric type (i.e. $\mathbb{P} = \mathrm{Trop}(u_1, \ldots, u_m)$ can be encoded by an *extended exchange matrix* $\widetilde{B} = (b_{ij})_{i \in [n+m],\, j \in [n]}$. The top square matrix $(b_{ij})_{i,j \in [n]}$ is B. The bottom $m \times n$ matrix is given by $y_j = \prod_{i=1}^m u_i^{b_{(n+i)\,j}}$.

The punchline: If we encode every Y-seed in a Y-pattern of geometric type by an extended exchange matrix, then mutation of Y-seeds is given by matrix mutation. That is, in direction k:

$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}; \\ b_{ij} + \mathrm{sgn}(b_{kj})[b_{ik} b_{kj}]_+ & \text{otherwise.} \end{cases}$$

Exchange relations in geometric type

Exchange relations are also much simpler in geometric type: Again, write a coefficient as $y_k = \prod_{i=1}^m u_i^{a_i}$, and define $x_{n+i} = u_i$ for $i = 1, \ldots, m$.

$$\frac{1}{y_k \oplus 1} = \frac{1}{\prod u_i^{-[-a_i]_+}} = \prod u_i^{[-a_i]_+} \quad \text{and} \quad \frac{y_k}{y_k \oplus 1} = \frac{\prod u_i^{a_i}}{\prod u_i^{-[-a_i]_+}} = \prod_{i=1}^m u_i^{[a_i]_+}.$$

Thus the exchange relation

$$x_k' = \frac{y_k \prod x_j^{[b_{jk}]_+} + \prod x_j^{[-b_{jk}]_+}}{(y_k \oplus 1) x_k} \qquad \text{becomes} \qquad x_k' = \frac{\prod u_i^{[a_i]_+} \prod x_j^{[b_{jk}]_+} + \prod u_i^{[-a_i]_+} \prod x_j^{[-b_{jk}]_+}}{x_k}.$$

Again encoding the Y-seed $(\mathbf{y}, B)$ by an extended exchange matrix $\widetilde{B}$, we have $a_i = b_{ik}$. So the exchange relation is

$$x_k' = \frac{\prod_{i=1}^{n+m} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n+m} x_i^{[-b_{ik}]_+}}{x_k}.$$

Principal coefficients

The cluster algebra with *principal coefficients* associated to an exchange matrix B is the cluster algebra of geometric type with:

- Coefficient semifield: $\mathbb{P} = \mathrm{Trop}(y_1, \ldots, y_n)$; and
- Initial coordinate tuple: $\mathbf{y} = (y_1, \ldots, y_n)$.

Up to isomorphism, this depends only on B, so we write $\mathcal{A}_\bullet(B)$. The initial extended exchange matrix is $\widetilde{B} = \begin{bmatrix} B \\ I \end{bmatrix}$. This is important for at least the following reasons:

- The cluster pattern with principal coefficients has the "largest" exchange graph.
- Cluster variables with principal coefficients determine cluster variables with any other choice of initial coordinates.

We'll explain in the next few slides. But first, an example.

Example with principal coefficients

Take $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbb{P} = \mathrm{Trop}(y_1, y_2)$, so $\widetilde{B} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{array}{ccccc}
\begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \\
\begin{bmatrix} x_1 & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{y_1+x_2}{x_1} & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{y_1+x_2}{x_1} & \frac{y_1^2 y_2 x_1^2 + (y_1+x_2)^2}{x_1^2 x_2} \end{bmatrix} \\
\updownarrow \mu_2 & & & & \updownarrow \mu_1 \\
\begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \\
\begin{bmatrix} x_1 & \frac{y_2 x_1^2 + 1}{x_2} \end{bmatrix} & & \begin{bmatrix} \frac{y_1 + y_1 y_2 x_1^2 + x_2}{x_1 x_2} & \frac{y_2 x_1^2 + 1}{x_2} \end{bmatrix} & & \begin{bmatrix} \frac{y_1 + y_1 y_2 x_1^2 + x_2}{x_1 x_2} & \frac{y_1^2 y_2 x_1^2 + (y_1+x_2)^2}{x_1^2 x_2} \end{bmatrix}
\end{array}$$

Coverings of exchange graphs

Let $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ and $\mathcal{A}(\mathbf{x}', \mathbf{y}', B)$ be cluster algebras. Notice: Same B in both. Coefficient semifields may differ. We say $\mathrm{Ex}(\mathbf{x}', \mathbf{y}', B)$ *covers* $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ if, for every pair t_1, t_2 of vertices of $\mathbb{T}_n$,

$$\Sigma'_{t_1} \sim \Sigma'_{t_2} \quad \Longrightarrow \quad \Sigma_{t_1} \sim \Sigma_{t_2}.$$

Theorem 2.8 (Fomin and Zelevinsky, CA IV)**.** *The exchange graph* $\mathrm{Ex}_\bullet(B)$ *covers the exchange graph of any other cluster pattern with initial exchange matrix* B*.*

That is, the cluster pattern with principal coefficients has the "largest" exchange graph among cluster patterns with initial exchange matrix B. Incidentally, the "smallest" exchange graph for B is obtained from taking $\mathbb{P} = \{1\}$. But recall that, conjecturally, these exchange graphs all coincide (Conjecture 2.1).

Some notation with a bit more detail

Recall that we wrote $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ for the labeled seed associated to a vertex t of $\mathbb{T}_n$. To specify individual cluster variables, coeficients, and matrix entries, while still keeping track of t, we will write

$$\mathbf{x}_t = (x_{i;t}, \ldots, x_{n;t}), \quad \mathbf{y}_t = (y_{1;t}, \ldots, y_{n;t}), \quad \text{and} \quad B_t = (b_{ij}^t).$$

We will also write

$$\widetilde{B}_t = B_t = (b_{ij}^t)$$

for extended exchange matrices in a cluster pattern of geometric type.

5 F-polynomials and g-vectors

X's and F's

In the principal coefficients case, each $x_{i;t}$ is a rational function in $x_1, \ldots, x_n, y_1, \ldots, y_n$. We write

$$X_{i;t}(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

for this rational function. We also define $F_{i;t}$ to be the rational function

$$F_{i;t}(y_1, \ldots, y_n) = X_{i;t}(1, \ldots, 1, y_1, \ldots, y_n)$$

The Laurent Phenomenon (Theorem 2.4) says that $X_{i;t}$ is a Laurent polynomial in $(x_1, \ldots, x_n)$ whose coefficients are integer polynomials in $(y_1, \ldots, y_n)$. This implies that $F_{i;t}$ is an integer polynomial called an *F-polynomial*.

Theorem 2.9 (Fomin and Zelevinsky, CA IV)**.** *Consider a cluster pattern over coefficient semifield $\mathbb{P}$ with initial seed $(\mathbf{x}, \mathbf{y}, B)$. Then the cluster variables are*

$$x_{i;t} = \frac{X_{i;t}|_{\mathcal{F}}(x_1, \ldots, x_n, y_1, \ldots, y_n)}{F_{i;t}|_{\mathbb{P}}(y_1, \ldots, y_n)}.$$

This formula exhibits a *separation of additions* phenomenon:

- In the numerator, we evaluate $X_{i;t}$ as a rational function in $\mathcal{F}$ (the field of rational functions in $\mathbf{x}$ with coefficients in $\mathbb{QP}$).
- In the denominator, we evaluate $F_{i;t}$ as a polynomial in $\mathbb{P}$, using the auxiliary addition $\oplus$.
- Actually, to evaluate $F_{i;t}$ as a polynomial in $\mathbb{P}$, we need Conjecture 2.5, which says that $X_{i;t}$ is a Laurent polynomial in $(x_1, \ldots, x_n)$ whose coefficients are *nonnegative* integer polynomials in $(y_1, \ldots, y_n)$. This implies that $F_{i;t}$ is a polynomial with nonnegative integer coefficients.
- However, with Conjecture 2.5 unproven, the formula for $x_{i;t}$ still makes sense: The function $X_{i;t}$ is defined by iterating exchange relations, each of which has positive coefficients. If we never "cancel" common factors from numerator and denominator, we obtain a *subtraction-free* expression for $X_{i;t}$. This leads to a subtraction-free (rational!) expression for F, which we can evaluate as a rational function in $\mathbb{P}$.

g-vectors

The Laurent phenomenon implies that $X_{i;t}$ lives in the ring of Laurent polynomials in $\mathbf{x}$, with coefficients integer polynomials in $\mathbf{y}$. We define a new $\mathbb{Z}^n$-grading of this ring:

$$\deg(x_i) = \mathbf{e}_i\,, \qquad \deg(y_j) = -\mathbf{b}_j$$

where $\mathbf{e}_i$ is the standard basis vector and $\mathbf{b}_j$ is the j^{th} column of B.

Proposition 2.10 (Fomin and Zelevinsky, CA IV)**.** *Each $X_{i;t}$ is homogenous with respect to the new grading.*

This is easy for t adjacent to t_0 in $\mathbb{T}_n$. E.g.: $\left[\begin{array}{rrrr} 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 1 \\ -3 & 0 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

The degree of $X_{i;t}$ is the integer vector $\mathbf{g}_{i;t}$, called the $\mathbf{g}$-*vector*. Together, the $\mathbf{g}$-vector and the F-polynomial determine the cluster variable (in arbitrary coefficients). The following is a corollary of Theorem 2.9 (separation of additions) and Proposition 2.10 (homogeneity of cluster variables).

Corollary 2.11 (Fomin and Zelevinsky, CA IV)**.** *Consider a cluster pattern over coefficient semifield $\mathbb{P}$ with initial seed $(\mathbf{x}, \mathbf{y}, B)$. Then the cluster variables are*

$$x_{i;t} = \frac{F_{i;t}|_{\mathcal{F}}(\hat{y}_1, \ldots, \hat{y}_n)}{F_{i;t}|_{\mathbb{P}}(y_1, \ldots, y_n)} \, \mathbf{x}^{\mathbf{g}_{i;t}}.$$

Each $\hat{y}_j$ is the (degree-0 homogeneous) element $y_j \prod_{i=1}^n x_i^{b_{ij}}$. (These entries are *not* b_{ij}^t, but rather b_{ij}.) The monomial $\mathbf{x}^{\mathbf{g}_{i;t}}$ is $x_1^{g_1} \cdots x_n^{g_n}$, where $\mathbf{g}_{i;t} = (g_1, \ldots, g_n)$.

Exercise 2b. *Verify the principal coefficients example by hand (+ computer?).*

Exercise 2c. *In the principal coefficients example, compute* **g***-vectors and F-polynomials. Verify that Theorem 2.9 and Corollary 2.11 recover the general coefficients that you computed in Exercise 1f.*

Exercise 2d. *Use the (principal coefficients case of the) exchange relations to verify the following relations, which hold when $t \overset{k}{\text{---}} t'$.*

$$F_{k;t} F_{k;t'} = \prod_{j=1}^n y_j^{[b_{n+j,\,k}^t]_+} \prod_{i=1}^n F_{i;t}^{[b_{i,\,k}^t]_+} + \prod_{j=1}^n y_j^{[-b_{n+j,\,k}^t]_+} \prod_{i=1}^n F_{i;t}^{[-b_{i,\,k}^t]_+}$$

$$\mathbf{g}_{k;t'} = -\mathbf{g}_{k;t} + \sum_{i=1}^n [b_{ik}^t]_+ \, \mathbf{g}_{i;t} - \sum_{j=1}^n [b_{n+j,\,k}^t]_+ \, \mathbf{b}_j$$

How do $F_{i;t}$ & $F_{i;t'}$ relate if $i \neq k$? Same question for $\mathbf{g}_{i;t}$ & $\mathbf{g}_{i;t'}$.

Some conjectures

Conjecture 2.12. *Each F-polynomial has constant term* 1*.*

Conjecture 2.13. *Each F-polynomial has a unique monomial of maximal degree. It has coefficient* 1 *and is divisible by all the other monomials.*

Conjecture 2.14. *For each $t \in \mathbb{T}_n$, the vectors $\mathbf{g}_{i;t} : i \in [n]$ are a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

Conjecture 2.15. *Different cluster monomials have different* **g***-vectors.*

We can interpret this as the statement that **g**-vectors define a fan.

Conjecture 2.16. *In a principal-coefficients cluster pattern, if seeds have equivalent extended exchange matrices, then the seeds are equivalent.*

A collection of integer vectors is called *sign coherent* if, for each $i \in [n]$, the i^{th} components of the vectors all have weakly the same sign.

Conjecture 2.17. *Given a cluster pattern with principal coefficients, for each $t \in \mathbb{T}_n$, the rows of the bottom half of $\widetilde{B}_t$ are sign-coherent.*

Conjecture 2.18. *For each $t \in \mathbb{T}_n$, the* **g***-vectors $\mathbf{g}_{i;t} : i \in [n]$ are sign-coherent.*

Various of these conjectures are equivalent to each other.

Many of them are proved in special cases. Most, for example, are known for skew-symmetric B by results of the paper (QP2)

Cluster algebras of finite type

A cluster algebra is of *finite type* if it has only finitely many distinct cluster variables. We have seen two examples. (The exchange graph was a 5-cycle and a 6-cycle respectively.) The obvious questions are: Which initial data lead to cluster algebras of finite type? Is the answer dependent on coefficients, or only on B? The next lecture will be devoted to answering this question by explaining results of (CA II).

As part of the answer, we will describe Fomin and Zelevinsky's combinatorial model for exchange graphs of finite type, in terms of root systems. This model and another type of model, in terms of (finite/infinite) reflection groups and (Kac-Moody) root systems, are the subject of the remaining lectures.

Combinatorial models for cluster complexes

What could/should we expect from a combinatorial model? Ideally, we want a combinatorial model of the exchange graph and/or cluster complex, with a combinatorial recipe for explicitly writing down the seeds: exchange matrices, coefficients, and cluster variables. (Even if we accomplish this, we're not really necessarily modeling the cluster *algebra* by doing this. We're only modeling the cluster pattern.)

Less ideally, we might settle for a model that got the exchange graph right and explicitly gave us denominator vectors and/or **g**-vectors and/or principal coefficients. One might do the same thing replacing "combinatorial" by "algebraic" or "geometric" throughout. This is an active area of research. In each case, the word "explicitly" is key.

References

(QP2) H. Derksen, J. Weyman, and A. Zelevinsky, "Quivers with potentials and their representations II: applications to cluster algebras." Journal AMS **23**.

(CA I) S. Fomin and A. Zelevinsky, "Cluster algebras I: Foundations." Journal AMS **15**.

(CA II) S. Fomin and A. Zelevinsky, "Cluster algebras II: Finite type classification." Invent. Math. **154**.

(CDM) S. Fomin and A. Zelevinsky, "Cluster algebras: Notes for the CDM-03 conference." CDM 2003: Current Developments in Mathematics, International Press, 2004.

(CA IV) S. Fomin and A. Zelevinsky, "Cluster algebras IV: Coefficients." Compositio Mathematica **143**.

Exercises, in order of priority

There are more exercises than you can be expected to complete in a day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

2b, 2c, 2d,

2a.