Lecture 2: Exercises

Exercise 2a. *Assuming Conjecture 2.3, prove the following: If x is a cluster variable, then any two seeds containing x are related by a sequence of mutations that fix x. Conclude that Conjecture 2.3 implies Conjecture 2.2.*

You will want to use Exercises 1g and 1h.

Example with principal coefficients

Take $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbb{P} = \mathrm{Trop}(y_1, y_2)$, so $\widetilde{B} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{tikzcd}
\begin{array}{c} \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \end{array} \arrow[r, leftrightarrow, "\mu_1"] \arrow[d, leftrightarrow, "\mu_2"] & \begin{array}{c} \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} \frac{y_1+x_2}{x_1} & x_2 \end{bmatrix} \end{array} \arrow[r, leftrightarrow, "\mu_2"] & \begin{array}{c} \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \\ \begin{bmatrix} \frac{y_1+x_2}{x_1} & \frac{y_1^2 y_2 x_1^2+(y_1+x_2)^2}{x_1^2 x_2} \end{bmatrix} \end{array} \arrow[d, leftrightarrow, "\mu_1"] \\
\begin{array}{c} \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} x_1 & \frac{y_2 x_1^2+1}{x_2} \end{bmatrix} \end{array} \arrow[r, leftrightarrow, "\mu_1"] & \begin{array}{c} \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} \frac{y_1+y_1 y_2 x_1^2+x_2}{x_1 x_2} & \frac{y_2 x_1^2+1}{x_2} \end{bmatrix} \end{array} \arrow[r, leftrightarrow, "\mu_2"] & \begin{array}{c} \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \\ \begin{bmatrix} \frac{y_1+y_1 y_2 x_1^2+x_2}{x_1 x_2} & \frac{y_1^2 y_2 x_1^2+(y_1+x_2)^2}{x_1^2 x_2} \end{bmatrix} \end{array}
\end{tikzcd}$$

Exercise 2b. *Verify the principal coefficients example by hand (+ computer?).*

Exercise 2c. *In the principal coefficients example, compute $\mathbf{g}$-vectors and F-polynomials. Verify that Theorem 2.9 and Corollary 2.11 recover the general coefficients that you computed in Exercise 1f.*

Exercise 2d. *Use the (principal coefficients case of the) exchange relations to verify the following relations, which hold when $t \overset{k}{—\!—} t'$.*

$$F_{k;t} F_{k;t'} = \prod_{j=1}^{n} y_j^{[b^t_{n+j,\,k}]_+} \prod_{i=1}^{n} F_{i;t}^{[b^t_{i,\,k}]_+} + \prod_{j=1}^{n} y_j^{[-b^t_{n+j,\,k}]_+} \prod_{i=1}^{n} F_{i;t}^{[-b^t_{i,\,k}]_+}$$

$$\mathbf{g}_{k;t'} = -\mathbf{g}_{k;t} + \sum_{i=1}^{n} [b^t_{ik}]_+ \, \mathbf{g}_{i;t} - \sum_{j=1}^{n} [b^t_{n+j,\,k}]_+ \, \mathbf{b}_j$$

How do $F_{i;t}$ & $F_{i;t'}$ relate if $i \neq k$? Same question for $\mathbf{g}_{i;t}$ & $\mathbf{g}_{i;t'}$.

Exercises, in order of priority

There are more exercises than you can be expected to complete in a day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

2b, 2c, 2d,

2a.