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1 Introduction

Introduction

In Lecture 1, we saw how a (finite) root system can be encoded by a Cartan matrix, which in turn can be encoded even more compactly by a Dynkin diagram. The Cartan matrix encodes* the relative lengths of simple roots the angles between them.

In this lecture, we will work both in the opposite direction and in greater generality. We will start with a *generalized* Cartan matrix A. This will be a matrix satisfying some mild rules that make it look like a Cartan matrix; these rules will essentially amount to requiring that A encodes* angles and relative lengths.

We can then pose the question of whether A is the Cartan matrix of a root system. Of course, we already know the answer: Yes if and only if the Dynkin diagram for A is in the Cartan-Killing classification, because Cartan-Killing is telling you whether a given specification of angles and relative lengths is realizable in Euclidean space.

There is also another direction we can go with a generalized Cartan matrix. We might insist on our specification of angles and relative lengths: Instead of asking if the specification is realizable in Euclidean space, we can use the specification to *define* what length and angle means, and then see what (possibly non-Euclidean) space we have defined.

This turns out to be not-too-hard to do, once we decide to try. The generalized Cartan matrix defines a (possibly infinite) collection of vectors called a *Kac-Moody root system*, or simply a *root system*. These turn out to be useful: An arbitrary exchange matrix B defines a generalized Cartan matrix A and thus a root system Φ. We will see that Φ has a lot to say about the cluster combinatorics of B.

2 (Kac-Moody) Root systems

Generalized Cartan matrices

A *generalized Cartan matrix* is an integer $n \times n$ matrix $A = (a_{ij})$ such that:

(i) $a_{ii} = 2$ for every $i \in [n]$;

(ii) $a_{ij} \leq 0$ for $i \neq j$

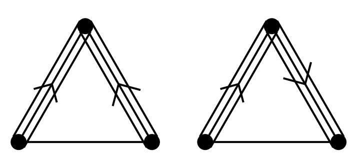
(iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

(iv) There exist positive, real $\delta_1, \ldots, \delta_n$ such that

$$\delta_i a_{ij} = a_{ji}\delta_j \text{ for all } i, j \in [n].$$

Condition (iv) says that A is *symmetrizable*.

Generalized Cartan matrices are encoded by Dynkin diagrams (with additional edge types allowed). Symmetrizability is a condition on cycles in the Dynkin diagram. Which of these is symmetrizable?



Simple roots and simple co-roots associated to A

Let V be a real vector space with a basis $\alpha_1, \ldots, \alpha_n$. The set $\Pi := \{\alpha_i : i \in [n]\}$ is called the set of *simple roots.* Define the *simple co-roots* to be $\alpha_i^\vee = \delta_i^{-1}\alpha_i$. Define a bilinear form K on V by $K(\alpha_i^\vee, \alpha_j) = a_{ij}$. K is symmetric:

$$K(\alpha_i, \alpha_j) = \delta_i K(\alpha_i^\vee, \alpha_j) = \delta_i a_{ij} = a_{ji}\delta_j = K(\alpha_j^\vee, \alpha_i)\delta_j = K(\alpha_j, \alpha_i).$$

Note also that $2\frac{\alpha_i}{K(\alpha_i,\alpha_i)} = \frac{K(\alpha_i^\vee,\alpha_i)\alpha_i}{K(\alpha_i,\alpha_i)} = \delta_i^{-1}\alpha_i = \alpha_i^\vee$. That is, roots and co-roots are related as before, with K replacing the standard Euclidean form. The matrix A is "the Cartan matrix" for these simple roots/co-roots: $a_{ij} = K(\alpha_i^\vee, \alpha_j)$.

The root system

We can define the reflection s_i orthogonal (in the sense of K) to the simple roots α_i:

$$s_i(x) = x - 2\frac{K(\alpha_i, x)}{K(\alpha_i, \alpha_i)}\alpha_i = x - K(\alpha_i^\vee, x)\alpha_i$$

The *root system* Φ associated to A is the set of all vectors obtained from simple roots by sequences of reflections orthogonal to simple roots. In some contexts, the roots in Φ and called "real" roots. There are also "imaginary" roots, but not for us.

One can check that every root is either *positive* (in the nonnegative span of Π) or *negative* (in the nonpositive span of Π). To each root is associated a coroot: (Apply the same sequence of reflections to a simple root and to its corresponding coroot. Or, see Exercise 3Bb.)

Exercise 3Ba. *Show that the action of each s_i is an isometry (in the sense of K). That is, $K(s_i(x), s_i(y)) = K(x, y)$ for any $x, y \in V$.*

Exercise 3Bb. *Prove: If β is a root, then its associated co-root $\beta^\vee$ is $2\frac{\beta}{K(\beta,\beta)}$.*

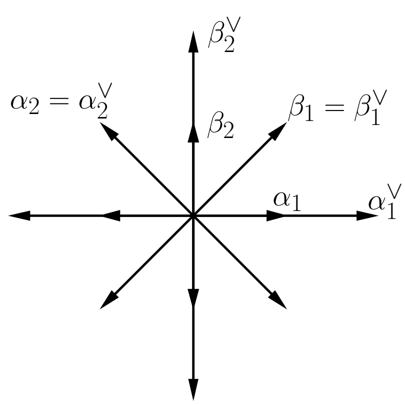
Note that we *can't* define $\gamma^\vee$ for a general vector $\gamma \in V$, because we don't know that $K(\gamma, \gamma)$ is positive.

The set $\Phi^\vee$ of all co-roots is a root system in its own right. (The *dual root system.*) The simple roots of $\Phi^\vee$ are the simple co-roots of Φ. The Cartan matrix of $\Phi^\vee$ is A^T.

Given a root β, there is a corresponding reflection t with $t(x) = x - K(\beta^\vee, x)\beta_i$. Each of these reflections permutes Φ.

Dual root system example

Let $A = \left[\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right]$. This is symmetrizable with $\delta_1 = \frac{1}{2}$ and $\delta_2 = 1$. (Other choices of δ_1, δ_2 work.) The roots and co-roots are shown here. The non-simple positive roots are $\beta_1 = 2\alpha_1 + \alpha_2$ and $\beta_2 = \alpha_1 + \alpha_2$. The non-simple positive co-roots are $\beta_1^\vee = \alpha_1^\vee + \alpha_2^\vee$ and $\beta_2^\vee = \alpha_1^\vee + 2\alpha_2^\vee$.

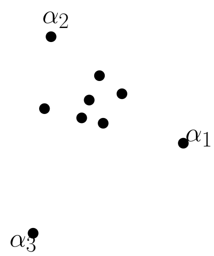


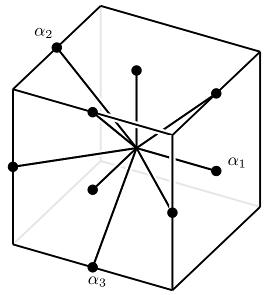
Finite type classification

The previous example is not new. It is a root system in the sense of Lecture 1. So how do we know whether A defines a root system in the sense of Lecture 1?

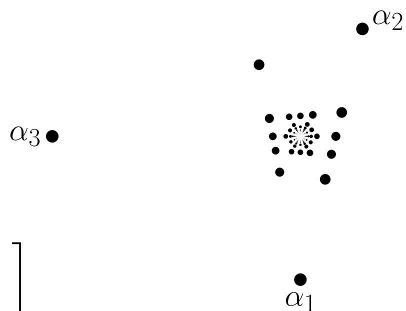
One answer: See if A is in the Cartan-Killing classification. Another answer: Check if K is positive definite. When K is positive definite, it gives V the structure of a Euclidean vector space, so all of the constructions in this lecture coincide with the constructions in Lecture 1. The second of these two answers is the key to actually determining the Cartan-Killing classification.

Another finite example





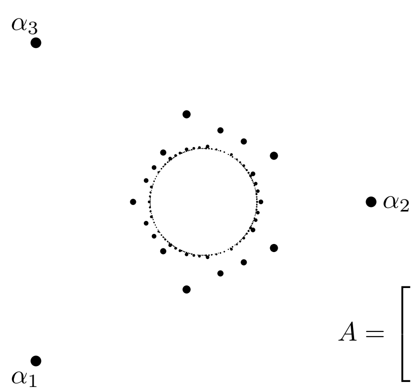
An infinite example



$$A = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Or, swap the -1 and -3.

Another infinite example



$$A = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

3 Root systems and hyperplane arrangements

Dual vector spaces

The *dual space* to V is $V^* = \{\text{linear maps } \varphi : v \to \mathbb{R}\}$. This is a n-dimensional vector space where $c \cdot \varphi$ is the map sending $x \in V$ to $c\varphi(x)$, and $\varphi + \psi$ is the map sending x to $\varphi(x) + \varphi(y)$. We often want to think of the maps in V^* as "elements of a vector space," not as "maps," so we often write $\langle \varphi, x \rangle$ to denote $\varphi(x)$. Then we'll usually use non-Greek letters for elements of V^* and just think of V^* and V as two vector spaces with a pairing $\langle \cdot, \cdot \rangle : (V^* \times V) \to \mathbb{R}$.

Given a basis $x_1, \ldots, x_n$ of V, the *dual basis* is the unique basis $x_1^*, \ldots, x_n^*$ of V^* such that $\langle x_i^*, x_j \rangle = \delta_{ij}$.

Given a linear map $f : V \to W$, there is a dual linear map $f^* : W^* \to V^*$ given by $f^*(\varphi) = \varphi \circ f$. If f is written in matrix form in terms of bases for V and W, then f^* is given by the transpose of the matrix, in terms of the dual bases for W^* and V^*.

Since V and V^* are both n-dimensional, they are isomorphic, so it is useful sometimes to identify them. But there is no canonical identification. Instead, we identify a basis of $x_1, \ldots, x_n$ of V with the *dual basis* $x_1^*, \ldots, x_n^*$ of V^*. This has the effect of making $\langle \cdot\,, \cdot \rangle$ into a Euclidean form ("inner product"), such that the identified basis is orthonormal. We won't do this, except implicitly, to draw pictures.

Danger: Infinite-dimensional vector spaces are different!

Fundamental weights

Write $\rho_1, \ldots, \rho_n$ for the dual basis to the simple *co*-roots $\alpha_1^\vee, \ldots, \alpha_n^\vee$. These are the *fundamental weights.* V is often called the *root space* and V^* is called the *weight space.* Each reflection t associated to a root β is linear map $t : V \to V$, so there is a dual map $t^* : V^* \to V^*$.

Exercise 3Bc. *Show that the action of s_i^* on the basis of fundamental weights is*

$$s_i^*(\rho_j) = \begin{cases} \rho_j - \sum_{k=1}^n K(\alpha_k^\vee, \alpha_j)\rho_k & \text{if } i = j, \text{ or} \\ \rho_j & \text{if } i \neq j. \end{cases}$$

Exercise 3Bd. *Show that $t^* : V^* \to V^*$ is a reflection (in the sense that it fixes a hyperplane and has an eigenvector -1). Indeed, show that t^* fixes $\beta^\perp = \{x \in V^* : \langle x, \beta \rangle = 0\}$.*

Root systems and hyperplane arrangements

The geometry of roots in V is manifested in the geometry of the associated reflecting hyperplanes $\left\{\beta^\perp : \beta \in \Phi\right\}$ in V^*. (A *hyperplane arrangement.*)

[Many pictures omitted from notes.]

References

(K) V. Kac, "Infinite-dimensional Lie algebras." Cambridge University Press, 1990.

(INF) N. Reading and D. Speyer, "Sortable elements in infinite Coxeter groups." Transactions AMS **363**.

Exercises, in order of priority

There are more exercises than you can be expected to complete in a *half* day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

3Ba, 3Bb,

3Bd, 3Bc.