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1 Introduction

Introduction

A (Kac-Moody) root system Φ defines a group W of transformations, generated by the reflections orthogonal (in the sense of the symmetric bilinear form K) to the simple roots. This naturally gives W the structure of a *Coxeter group*. Coxeter groups are defined abstractly within the framework of *combinatorial group theory*. That is, we are given a *presentation* of a group by *generators and relations*. The abstract algebra encodes the geometry surprisingly well: Not only does each root system define a Coxeter group, but also each Coxeter group can be represented geometrically by specifying a root system. But we need a root system given by a "generalized" generalized Cartan matrix for a "non-crystallographic" Coxeter group.

In this lecture, we'll provide some basic background on Coxeter groups that will be useful for understanding Cambrian lattices and sortable elements. Standard references include (BB), (B), and (H). A summary, written specifically for use with sortable elements and Cambrian lattices, can be found in Section 2 of (INF).

2 Combinatorics

Coxeter groups

A Coxeter group is a group with a certain *presentation*. Choose a finite generating set $S = \{s_1, \ldots, s_n\}$ and for every $i < j$, choose an integer $m(i,j) \geq 2$, or $m(i,j) = \infty$. Define: $W = \langle S \mid s_i^2 = 1,\ \forall i \text{ and } (s_i s_j)^{m(i,j)} = 1,\ \forall i < j \rangle$. Why would anyone write this down?

Exercise 4Ba. *Let Φ be a Kac-Moody root system with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and define $S = \{s_1, \ldots, s_n\}$ for s_i as in Lecture 3B. Define $m(i,j)$ to be $\frac{2\pi}{\pi - \text{angle}(\alpha_i, \alpha_j)}$. Show that the group W' generated by S satisfies the relations given above.*

The exercise shows that W' is a homomorphic image of the abstract Coxeter group W. In fact, the two are isomorphic. Thus all of our root systems examples yield Coxeter group examples.

Coxeter group examples

We'll focus on two examples:

- The dihedral group of order 8:
$$B_2 = \langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 \rangle.$$
This is the Coxeter group associated to the root system B_2. Its elements are
$$1,\ s_1,\ s_2,\ s_1s_2,\ s_2s_1,\ s_1s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1s_2 = s_2s_1s_2s_1.$$

- The symmetric group S_{n+1} (AKA A_n): This is the group of permutations of $[n+1]$. Writing $s_i = (i\ \ i{+}1)$, the symmetric group is a Coxeter group with
$$m(i,j) = \begin{cases} 3 & \text{if } j = i+1, \text{ or} \\ 2 & \text{if } j > i+1. \end{cases}$$
This is the Coxeter group associated to the root system A_n, constructed explicitly as $\{e_j - e_i : i, j \in [n+1],\ i \neq j\}$ in Exercise 1k.1. This construction leads to a representation of S_{n+1} as permutations of the coordinates.

Coxeter diagrams

The *Coxeter diagram* of a *Coxeter system* (W, S) is a graph with

- Vertex set: $\{1, \ldots, n\}$.
- Edges: $i — j$ if $m(i, j) \geq 3$.
- Edge labels: $m(i, j)$. By convention, we omit edge labels "3."

The dihedral group of order 8 has a diagram with two vertices connected by an edge labeled 4.

The diagram for A_n is

Obs: Non-edges $\leftrightarrow$ $m(i, j) = 2$ $\leftrightarrow$ s_i and s_j commute.

Reflections

The set S is called the *simple reflections*. The set $T = \{wsw^{-1} : w \in W, s \in S\}$ is called the set of *reflections* in W. Why?

Exercise 4Bb. *Suppose that* W *is the (Coxeter) group defined (under the name* W'*) in Exercise 4Ba. Show that*

1. *For every reflection* $t \in T$*, there is a unique positive root* $\beta \in \Phi_+$ *such that* t *is the reflection orthogonal to* t *(in the sense of* K*).*
2. *For every root* β*, the reflection orthogonal to* t *(in the sense of* K*) is an element of* T*.*

Thus, reflections are in bijection with positive roots! We'll write β_t for the positive root associated with $t \in T$. Furthermore, T is the complete set of elements of W that act as reflections.

Reflections in B_2 and A_n

B_2: $S = \{s_1, s_2\}$, $T = \{s_1, s_2, s_1s_2s_1, s_2s_1s_2\}$.

There are 4 positive roots.

$A_n = S_{n+1}$: $S = \{\text{adjacent transpositions } (i\ \ i{+}1)\}$, $T = \{\text{all transpositions } (i\ \ j)\}$.

The positive roots are $\{e_j - e_i : i, j \in [n+1], i < j\}$ by Exercise 1m.1.

Reduced words and the word problem

Since W is generated by S, each element w of W can be written (in many ways!) as a *word* in the "alphabet" S. A word of minimal length, among words for w, is called a *reduced word* for w. The *length* $\ell(w)$ of w is the length of a reduced word for w.

Solution to the *word problem* for W (J. Tits): Any word for w can be converted to a reduced word for w by a sequence of

- braid moves: $s_is_js_i \cdots \leftrightarrow s_js_is_j \cdots$ ($m(i, j)$ letters)
- nil moves: delete s_is_i.

Any two reduced words for w are related by a sequence of braid moves.

Exercise 4Bc. *Find all reduced words for* $4321 \in S_4$.

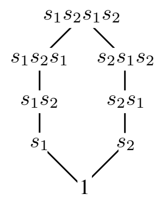
Inversions

An *inversion* of $w \in W$ is a reflection $t \in T$ such that $\ell(tw) < \ell(w)$. The notation $\mathrm{inv}(w)$ means $\{$inversions of $w\}$. If $a_1 \cdots a_k$ is a reduced word for w, then write $t_i = a_1 \cdots a_i \cdots a_1$. Then $\mathrm{inv}(w) = \{t_i : 1 \leq i \leq k\}$. The sequence $t_1, \ldots, t_k$ is the *reflection sequence* for the reduced word $a_1 \cdots a_k$.

Weak order (*Right* weak order)

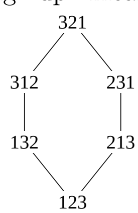
The *weak order* on a Coxeter group W sets $u \leq w$ if and only if a reduced word for u occurs as a prefix of some reduced word for w. The covers are $w \lessdot ws$ for $w \in W$ and $s \in S$ with $\ell(w) < \ell(ws)$. Equivalently, $u \leq w$ if and only if $\mathrm{inv}(u) \subseteq \mathrm{inv}(w)$. The weak order is ranked by the length function ℓ. It is a *meet semilattice* in general, and a *lattice* when W is finite. *Alert:* This is "right" weak order. There is also a "left" weak order.

Example: $B_2 = \left\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 \right\rangle$



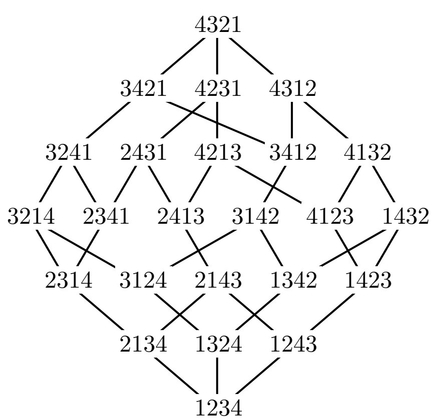
Inversions and weak order in S_{n+1}

We will write a permutation π in one-line notation $\pi_1 \cdots \pi_{n+1}$. Then the cover relations in the weak order are transpositions of adjacent entries. Going "up" means putting the entries out of numerical order. The weak order on S_3:



Inversions are $\mathrm{inv}(\pi) = \{$transpositions $(i\ \ j) : i$ comes before j in $\pi\}$, and this is the origin of the term "inversion."

The weak order on S_4



Cover reflections

A *cover reflection* of $w \in W$ is an inversion t of w such that $tw = ws$ for some $s \in S$. The name "cover reflection" refers to the fact that w covers tw in the weak order. Indeed, for each cover $ws \lessdot w$, there is a cover reflection wsw^{-1} of w. The set of cover reflections of w is written $\mathrm{cov}(w)$. In S_{n+1}:

$$\mathrm{cov}(\pi) = \{\text{transpositions } (i\ \ j) : i \text{ immediately before } j \text{ in } \pi\}.$$

3 Geometry

Bringing geometry into the picture

Exercise 4Bd. *Show that the diagram of a Coxeter system associated to a Kac-Moody root system has the following properties.*

1. *Each edge is unlabeled or has label* 4*,* 6 *or* ∞*.*
2. *Any cycle has an even number of* 4*'s and an even number of* 6*'s.*

Exercise 4Be. *Given a Coxeter group* W *whose diagrams satisfy the conditions of Exercise 4Bd, show that there is a Kac-Moody root system associated to* W*.*

In fact, there are many! In general, we can make a "generalized" generalized Cartan matrix and root system for any Coxeter group if we allow non-integer entries and add an additional technical condition.

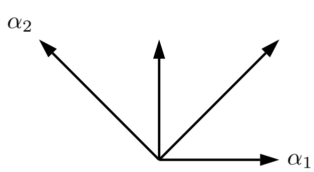
The Tits cone

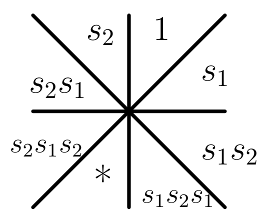
Define $D = \bigcap_{\alpha_i \in \Pi} \{x \in V^* : \langle x, \alpha_i \rangle \geq 0\}$. This is an n-dimensional simplicial cone in the dual space V^*. The set $\mathcal{F}(A)$ of all cones wD and their faces is a fan in V^* which we call the *Coxeter fan.* Its maximal cones are in bijection with elements of W (i.e. the map $w \mapsto wD$ is injective). The union of the cones of $\mathcal{F}(A)$ is a convex subset of V^* known as the *Tits cone* and denoted $\text{Tits}(A)$. The cones wD are the *regions* in $\text{Tits}(A)$ defined by the reflecting hyperplanes $\left\{\beta^\perp : \beta \in \Phi\right\}$.

Tits cone example: B_2

$$D = \bigcap_{\alpha_i \in \Pi} \{x \in V^* : \langle x, \alpha_i \rangle \geq 0\}$$

In this case, $\text{Tits}(A)$ is all of V^*. We'll label each region wD by w.

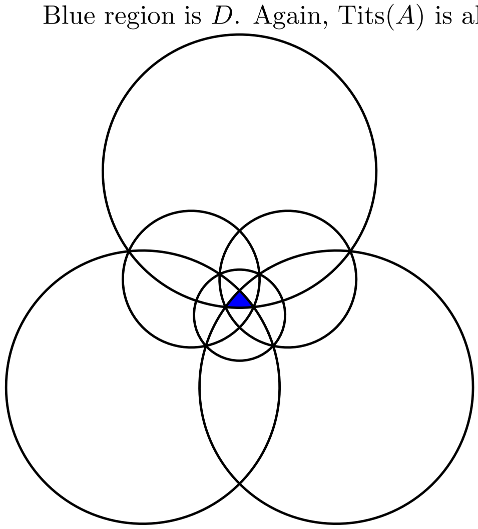




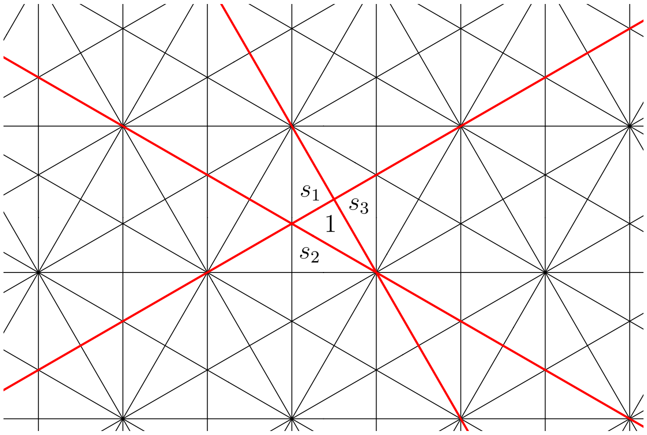
$$* = s_1s_2s_1s_2 = s_2s_1s_2s_1$$

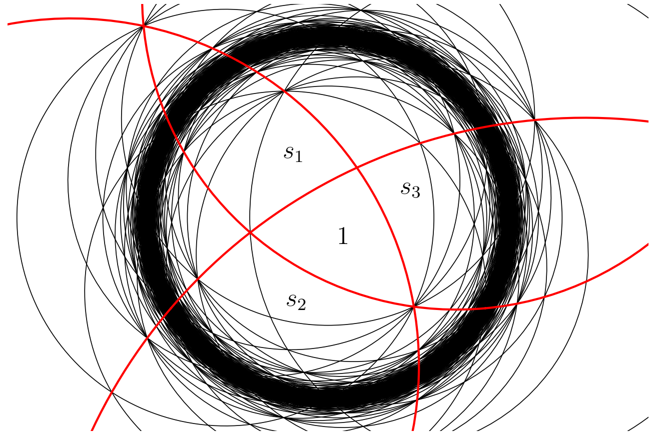
Tits cone example: S_4

Blue region is D. Again, $\text{Tits}(A)$ is all of V^* Largest circles: hyperplanes for s_1, s_2, and s_3. (s_2 on top.)

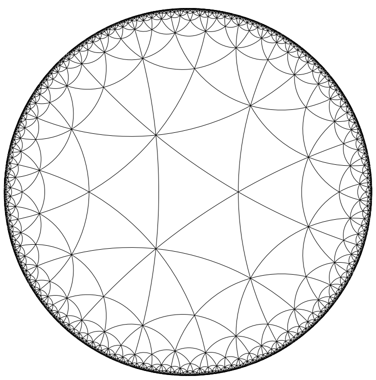


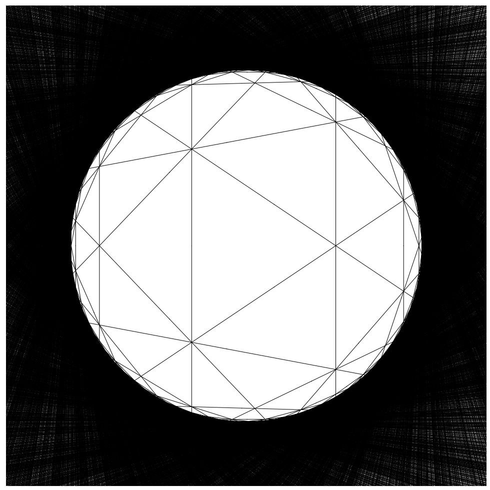
Tits cone example: an affine root system

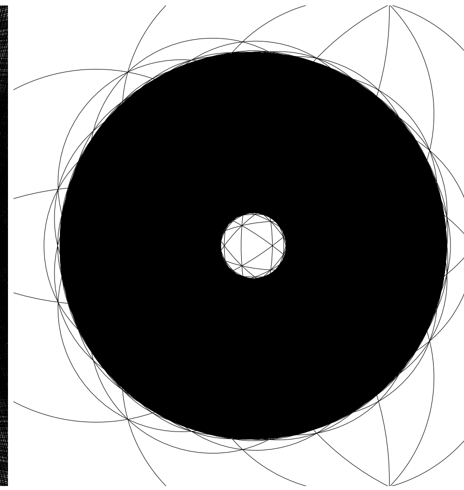




Tits cone example: a hyperbolic root system







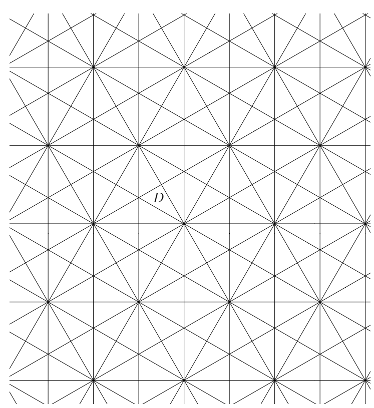
How the combinatorics shows in the geometry

Words are paths from D.

Reduced words: paths that don't cross any hyperplane twice. "Walls" are labeled by S. Crossing a wall $\leftrightarrow$ tacking a letter on right.

inv(w)*: reflections whose hyperplanes separate wD from D.*

cov(w)*: inversions whose hyperplanes define facets of wD*



The weak order, geometrically (S_4)

[The picture can't be reduced to notes. It illustrates that you move up in the weak order from 1 by crossing hyperplanes away from D.]

References

(BB) A. Björner and F. Brenti, "Combinatorics of Coxeter groups." Graduate Texts in Mathematics, **231**.

(B) N. Bourbaki, "Lie groups and Lie algebras. Chapters 4–6." Elements of Mathematics.

(H) J. E. Humphreys, "Reflection groups and Coxeter groups." Cambridge studies in advanced mathematics **29**.

(INF) N. Reading and D. Speyer, "Sortable elements in infinite Coxeter groups." Transactions AMS **363**.

Exercises, in order of priority

There are more exercises than you can be expected to complete in a *half* day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

4Ba, 4Bc, 4Bd,

4Bb, 4Be.