Lecture 4A: Exercises

Exercise 4Aa. *Show that ω is skew-symmetric.*

The *Euler form* E associated to B is $E(\alpha_i^\vee, \alpha_j) = \begin{cases} \min(b_{ij}, 0) & \text{if } i \neq j, \text{ or} \\ 1 & \text{if } i = j. \end{cases}$

Exercise 4Ab. *Show that $\omega(\beta, \gamma) = E(\beta, \gamma) - E(\gamma, \beta)$ and $K(\beta, \gamma) = E(\beta, \gamma) + E(\gamma, \beta)$.*

Exercise 4Ac. *The Reflection condition implies that $-\beta \in C(v')$. If we reverse the roles of v and v' above and consider the root $\gamma' \in C(v')$, then the Reflection condition is the assertion that $\gamma \in C(v)$.*

Exercise 4Ad. *Prove linear independence of $C(v)$ in the case of a reflection framework.*

Exercises, in order of priority

There are more exercises than you can be expected to complete in a *half* day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

4Aa, 4Ab,

4Ac, 4Ad.

Lecture 4B: Exercises

Exercise 4Ba. *Let Φ be a Kac-Moody root system with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and define $S = \{s_1, \ldots, s_n\}$ for s_i as in Lecture 3B. Define $m(i, j)$ to be $\frac{2\pi}{\pi - \text{angle}(\alpha_i, \alpha_j)}$. Show that the group W' generated by S satisfies the relations given above.*

The exercise shows that W' is a homomorphic image of the abstract Coxeter group W. In fact, the two are isomorphic. Thus all of our root systems examples yield Coxeter group examples.

Exercise 4Bb. *Suppose that W is the (Coxeter) group defined (under the name W') in Exercise 4Ba. Show that*

1. *For every reflection $t \in T$, there is a unique positive root $\beta \in \Phi_+$ such that t is the reflection orthogonal to t (in the sense of K).*
2. *For every root β, the reflection orthogonal to t (in the sense of K) is an element of T.*

Exercise 4Bc. *Find all reduced words for $4321 \in S_4$.*

Exercise 4Bd. *Show that the diagram of a Coxeter system associated to a Kac-Moody root system has the following properties.*

1. *Each edge is unlabeled or has label 4, 6 or ∞.*
2. *Any cycle has an even number of 4's and an even number of 6's.*

Exercise 4Be. *Given a Coxeter group W whose diagrams satisfy the conditions of Exercise* **??**, *show that there is a Kac-Moody root system associated to W.*

Exercises, in order of priority

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4Ba, 4Bc, 4Bd,

4Bb, 4Be.