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1 Introduction

Introduction

We have seen, in Lecture 3A, that the combinatorics of finite root systems is intricately related to the combinatorics of cluster algebras of finite type. This insight led to Fomin and Zelevinsky's combinatorial model, organized around denominator vectors. It is not immediately apparent how to extend this almost-positive roots model to cluster algebras of infinite type.

Instead, we describe a different approach to combinatorial models. This approach uses the combinatorics of the Coxeter group W in an essential way, along with the geometry of the associated root system and arrangement of reflecting hyperplanes. Specifically, the combinatorics of reduced words is at play through the *sortable elements* in W, and the geometry of the root system enters the picture through the *Cambrian fans*. In the background, the lattice theory of the weak order plays a fascinating, but still mysterious role, through the *Cambrian (semi)lattice*. Most of the results quoted here are joint with David Speyer.

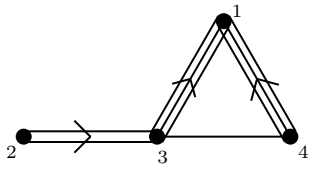
2 Sortable elements and Cambrian (semi)lattices

Review: From B to A to W

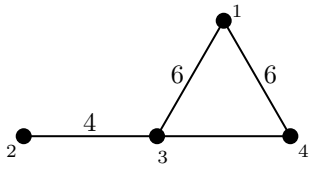
Given a B, we make A:

$$B = \begin{bmatrix} 0 & 0 & -3 & 3 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & -3 & -3 \\ 0 & 2 & -1 & 0 \\ -1 & -2 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} = A$$

Given A, we make a Dynkin diagram.



From there, a Coxeter diagram.

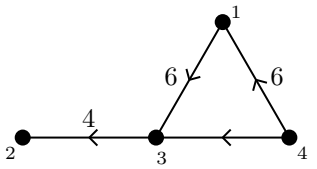


From B to an oriented Coxeter diagram

There is still information left in B.

$$B = \begin{bmatrix} 0 & 0 & -3 & 3 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

We orient each edge of the diagram $i \to j$ if $b_{ij} < 0$.



A *Coxeter element* c of W is an element represented by a word $s_1 s_2 \cdots s_n$ where $S = \{s_1, \ldots, s_n\}$ and $n = |S|$. If the oriented diagram is acyclic, then we say B is *acyclic*, and B defines a Coxeter element. (Arrows in diagram point left in word). In the example, $c = s_2 s_3 s_1 s_4$. *Important*: If A is of finite type, then B is acyclic. Think: "$B = A + c$."

Sorting words

For the rest of the lecture, we will assume B is acyclic and take c to be the Coxeter element defined by B. Fix some reduced word $s_1 \cdots s_n$ for c. Form an infinite word

$$c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$$

The *c-sorting word* for w is the lexicographically first (i.e. leftmost) subword of c^∞ which is a reduced word for w.

Example: $W = B_4$

$$s_1 \overset{4}{\text{———}} s_2 \text{———} s_3 \text{———} s_4$$

For $c = s_1 s_2 s_4 s_3$,

$$c^\infty = s_1 s_2 s_4 s_3 | s_1 s_2 s_4 s_3 | s_1 s_2 s_4 s_3 | \cdots$$

The element $w = s_4 s_2 s_1 s_2 s_3 s_2 s_1 s_2 s_1$ has c-sorting word $s_1 s_2 s_4 s_3 | s_1 s_2 s_3 | s_1 s_2$.

Sorting words in S_{n+1}

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π. Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a *reduced word* for π. Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a *sorting word* for π. (C.f. "bubble sort.")

Example: $W = S_4$, $\quad c = s_1 s_2 s_3$, $\quad \pi = 4231$. $\quad$ Sorting word is $s_1 s_2 s_3 | s_2 | s_1$.

Sortable elements of a Coxeter group W

In general, to find the c-sorting word for $w \in W$: Try the generators cyclically according to c. Place a *divider* "|" every time a pass through S is completed. A c-sorting word can be interpreted as a sequence of sets (sets of letters between *dividers* "|"). If the sequence is nested then w is *c-sortable*.

Example: $\pi = 4231$ with c-sorting word $s_1 s_2 s_3 | s_2 | s_1$ π is not c-sortable because $\{s_1\} \not\subseteq \{s_2\}$.

Example: $W = B_2$, $\quad c = s_1 s_2$

c-sortable:	$1,\ s_1,\ s_1 s_2,\ s_1 s_2 \vert s_1,\ s_1 s_2 \vert s_1 s_2,\ s_2$
not c-sortable:	$s_2 \vert s_1,\ s_2 \vert s_1 s_2$

Sortable elements of S_{n+1}

$W = S_{n+1}$, $\qquad c = s_n s_{n-1} \cdots s_1$

The c-sortable elements are the 231-avoiding or stack-sortable permutations.

1	1234	$s_3 s_2 s_1 \vert s_2$	4213
s_3	1243	$s_3 s_2 \vert s_3$	1432
$s_3 s_2$	1423	$s_3 s_1$	2143
$s_3 s_2 s_1$	4123	s_2	1324
$s_3 s_2 s_1 \vert s_3$	4132	$s_2 s_1$	3124
$s_3 s_2 s_1 \vert s_3 s_2$	4312	$s_2 s_1 \vert s_2$	3214
$s_3 s_2 s_1 \vert s_3 s_2 \vert s_3$	4321	s_1	2134

For $c = s_1 s_2 \cdots s_n$, the c-sortable elements are the 312-avoiding permutations. For other Coxeter elements, the condition is more complicated, blending the two avoidance conditions.

Results on sortable elements for finite W

1. For finite W, any c, bijection to W-noncrossing partitions: $w \mapsto \mathrm{cov}(w)$. (SORT)
2. For finite W, any c, bijection to vertices of the generalized associahedron. (SORT)
3. Deep connection to the lattice theory of the weak order on W, via Cambrian lattices. (SC)
4. The Hasse diagram of the c-Cambrian lattice is isomorphic to the exchange graph. (CAMB), (FANS)

(The c-Cambrian lattice is the restriction of the weak order to c-sortable elements. More later.)

Standard parabolic subgroups

Given a subset $J \subseteq S$, the *standard parabolic subgroup* W_J is the subgroup of W generated by J. The subgroup W_J is in particular a Coxeter group with simple generators J. An important case will be when $J = S \setminus \{s\}$ for some $s \in S$. We use the notation $\langle s \rangle = S \setminus \{s\}$.

Example: $W = B_2$. The only non-trivial proper standard parabolic subgroups are the two-element groups generated, respectively, by s_1 and by s_2.

Example: $W = S_{n+1}$. The maximal proper standard parabolic subgroups are as follows:

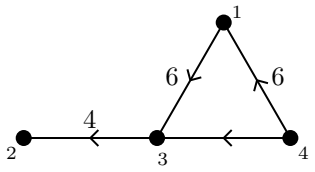
For each i from 1 to n, the subgroup $W_{\langle s_i \rangle}$ is the set of permutations fixing $\{1, \ldots, i\}$ as a set (and therefore fixing $\{i+1, \ldots, n+1\}$ as a set).

Initial and final elements

A given Coxeter element c may have several reduced words. They are all equivalent by transpositions of commuting elements of S. A generator $s \in S$ is *initial* in c if there is a reduced word for c having s as its first letter. Similiarly, s is *final* in c if it is the last letter of some reduced word for c. In either case, the element scs is another Coxeter element.

Example: $W = S_4$ If $c = s_1 s_3 s_2 = s_3 s_1 s_2$ then s_1 and s_3 are initial and s_2 is final. If $c = s_1 s_2 s_3$ then s_1 is initial and s_3 is final.

When we encode Coxeter elements as diagrams, initial generators are sinks and final generators are sources.



Recall that the diagram above encodes $c = s_2 s_3 s_1 s_4$.

Passing from $c \leftrightarrow scs$, for s initial or final, is a *source-sink move* or *BGP reflection functor*.

A recursive characterization of sortable elements

Lemma 5.1. *Let s be initial in c and suppose $w \not\geq s$. Then w is c-sortable if and only if it is an sc-sortable element of $W_{\langle s \rangle}$.*

Lemma 5.2. *Let s be initial in c and suppose $w \geq s$. Then w is c-sortable if and only if sw is scs-sortable.*

Both become obvious on inspection of the definition, and staring at:

$$c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$$

Since the identity element is c-sortable for any c, the lemmas are a recursive characterization of c-sortability, by induction on the length $\ell(w)$ and on the rank of W (the cardinality of S). This form of induction is the *most important* proof technique for sortable elements.

A geometric characterization of sortable elements

The form ω_c has a special relation with the reflection sequences of c-sortable elements. Recall that when $a_1 \cdots a_k$ is a reduced word for some $w \in W$, the *reflection sequence* associated to $a_1 \cdots a_k$ is $t_1, \ldots, t_k$, where $t_i = a_1 a_2 \cdots a_i \cdots a_2 a_1$.

Proposition 5.3 (INF)**.** *Let $a_1 \cdots a_k$ be a reduced word for some $w \in W$ with reflection sequence $t_1, \ldots, t_k$. Then the following are equivalent:*

(i) *$\omega_c(\beta_{t_i}, \beta_{t_j}) \geq 0$ for all $i \leq j$ with strict inequality holding unless t_i and t_j commute.*

(ii) *w is c-sortable and $a_1 \cdots a_k$ can be converted to a c-sorting word for w by a sequence of transpositions of adjacent commuting letters.*

Proposition 5.3 can be proved by induction on the length k of w and the rank $|S|$ of W, using the following three facts:

Exercise 5a. *If s is initial or final in c, then $\omega_c(\beta, \beta') = \omega_{scs}(s\beta, s\beta')$ for all roots β and β'.*

Exercise 5b. *Let s be initial in c and let t be a reflection in W. Then $\omega_c(\alpha_s, \beta_t) \geq 0$, with equality only if s and t commute.*

Exercise 5c. *Let $J \subseteq S$ and let c' be the Coxeter element of W_J obtained by deleting all the letters in $S \setminus J$ from a reduced word for c. Let V_J be the subspace of V spanned by simple roots corresponding to elements of J. Then ω_c restricted to V_J is $\omega_{c'}$.*

3 The Cambrian framework

Skips

v: a c-sortable element of W $\quad a_1 \cdots a_k$: its c-sorting word. Recall: $c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \ldots$

For each $s_i \in S$, there is a leftmost instance of s_i in c^∞ which is not in the subword of c^∞ corresponding to $a_1 \cdots a_k$. Let $a_1 \cdots a_j$ be the initial segment of $a_1 \cdots a_k$ consisting of those letters that occur in c^∞ before the omission of s_i. Say $a_1 \cdots a_k$ *skips* s_i after $a_1 \cdots a_j$. If $a_1 \cdots a_j s_i$ is a reduced word, then this is an *unforced skip*. Otherwise it is a *forced skip*. Define

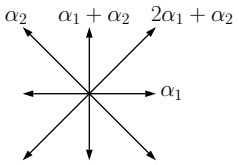
$$C_c^{s_i}(v) = a_1 \cdots a_j \cdot \alpha_i.$$

This is a positive root if and only if the skip is unforced. Write $C_c(v)$ for $\{C_c^{s_i}(v) : s_i \in S\}$.

Skips example: $W = B_2$, $\quad c = s_1 s_2$**, quad** $c^\infty = s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 \cdots$

c-sortable: $\quad 1,\ s_1,\ s_1 s_2,\ s_1 s_2 | s_1,\ s_1 s_2 | s_1 s_2,\ s_2$

not c-sortable: $\quad s_2 | s_1,\ s_2 | s_1 s_2$



v	s_i	skip		$C_c^{s_i}(v)$
1	s_1	unforced	(s_1 reduced)	α_1
	s_2	unforced	(s_2 reduced)	α_2
s_1	s_1	forced	($s_1 s_1$ not reduced)	$-\alpha_1$
	s_2	unforced	($s_1 s_2$ reduced)	$2\alpha_1 + \alpha_2$
$s_1 s_2$	s_1	unforced	($s_1 s_2 s_1$ reduced)	$\alpha_1 + \alpha_2$
	s_2	forced	($s_1 s_2 s_2$ not reduced)	$-2\alpha_1 - \alpha_2$
$s_1 s_2 s_1$	s_1	forced	($s_1 s_2 s_1 s_1$ not reduced)	$-\alpha_1 - \alpha_2$
	s_2	unforced	($s_1 s_2 s_1 s_2$ reduced)	α_2
$s_1 s_2 s_1 s_2$	s_1	forced	($s_1 s_2 s_1 s_2 s_1$ not reduced)	$-\alpha_1$
	s_2	forced	($s_1 s_2 s_1 s_2 s_2$ not reduced)	$-\alpha_2$
s_2	s_1	unforced	(s_1 reduced)	α_1
	s_2	forced	($s_2 s_2$ not reduced)	$-\alpha_2$

Two facts about skips

Proposition 5.4 (INF). *For s initial in c,*

$$C_c(v) = \begin{cases} C_{sc}(v) \cup \{\alpha_s\} & \text{if } v \not\geq s \\ sC_{scs}(sv) & \text{if } v \geq s \end{cases}$$

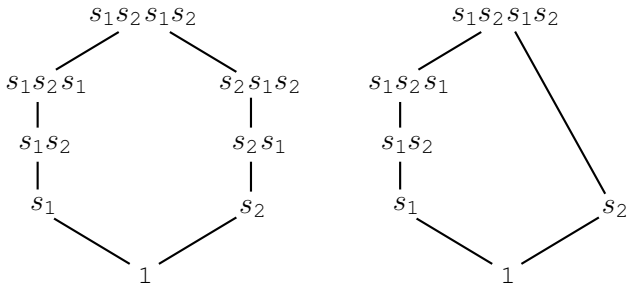
The sets $C_{sc}(v)$ and $C_{scs}(sv)$ are defined by induction on the rank of W or on the length of v.

Recall: A *cover reflection* of $w \in W$ is an inversion t of w such that $tw = ws$ for some $s \in S$.

Proposition 5.5 (INF). *Let v be a c-sortable element. The set of negative roots in $C_c(v)$ is $\{-\beta_t : t \in \mathrm{cov}(v)\}$.*

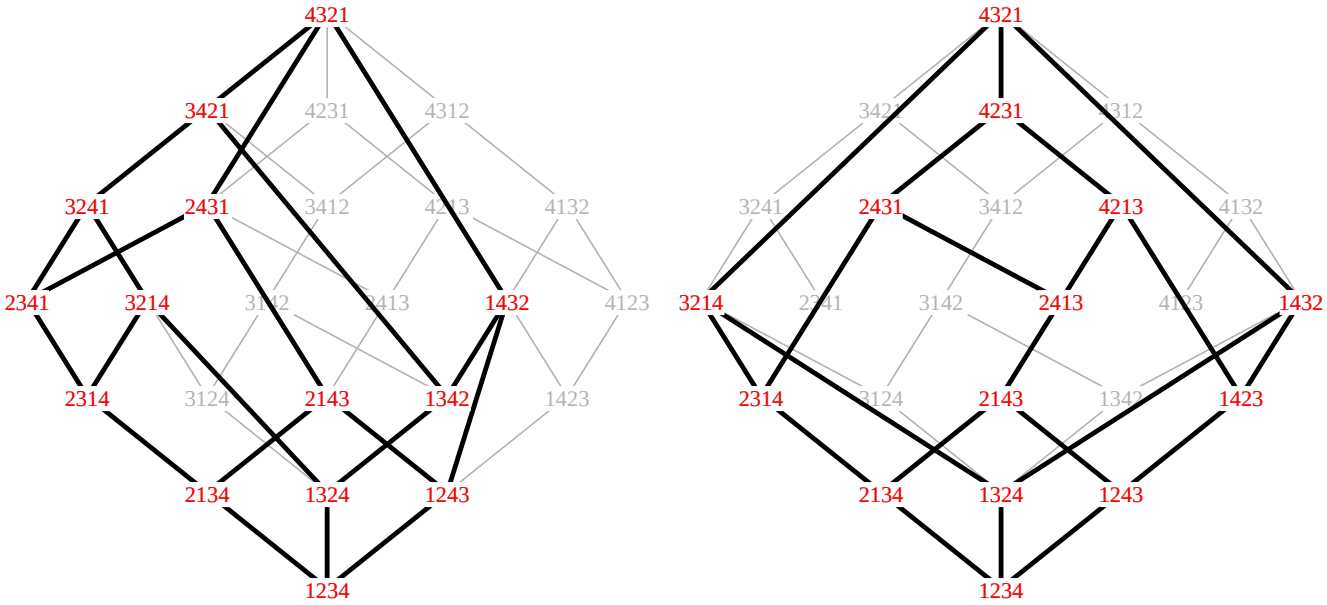
The Cambrian (semi)lattice

The *c-Cambrian semilattice* Camb_c is the subposet of the weak order on W induced by the c-sortable elements. We will also use the symbol Camb_c to denote the undirected Hasse diagram of Camb_c.



When W is finite, this is the c-Cambrian *lattice*. In the case $W = S_{n+1}$ and $c = s_1 \cdots s_n$, the c-Cambrian lattice is the weak order restricted to 231-avoiding permutations, AKA the *Tamari lattice*.

Cambrian lattices in S_4



The Cambrian framework, finite type

Recall: A *(weak) reflection framework* is a pair (G, C) such that

- G is a connected n-regular quasi-graph, and
- C is a labeling of each incident pair by a vector $C(v, e)$ in V satisfying
  - the Base condition,

– the Root condition,

– the Reflection condition, and

– the Euler conditions.

The (undirected) Hasse diagram Camb_c of the c-Cambrian lattice is an n-regular graph. (That takes some checking.) We want to say that the pair (Camb_c, C_c) is a reflection framework. Base condition: v_b is the identity element. Root condition: by definition. Reflection condition and Euler conditions: Need some difficult results about sortable elements.

Even with the Base, Root, Reflection and Euler conditions, we still don't have a framework. The problem: We have n-labels for each vertex, but we don't yet know how to assign a label to each incident pair.

Lemma 5.6. *If $v' \lessdot v$ in the c-Cambrian semilattice, then there exists a unique positive root β such that $\beta \in C_c(v')$ and $-\beta \in C_c(v)$.*

Lemma 5.6 lets us label each incident pair in Camb_c: Suppose $v' \lessdot v$ in Camb_c with $v' = \pi^c_{\downarrow}(tv)$ and write e for the edge v—v'. We label the incident pair the incident pair (v', e) by the positive root β from Lemma 5.6 and label (v, e) by $-\beta$. We re-use the symbol C_c for this labeling of incident pairs.

Theorem 5.7 (FRM)**.** *If A is of finite type, then (Camb_c, C_c) is a complete, exact, well-connected polyhedral reflection framework for B.*

Recall what this means:

Complete: n-regular *graph* (not quasi-graph).

Exact: Implies $\mathrm{Camb}_c \cong \mathrm{Ex}_\bullet(B)$.

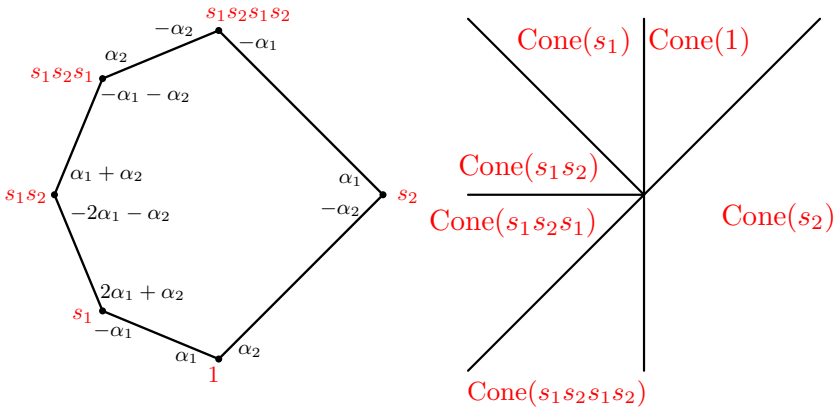
Well-connected polyhedral: $\mathrm{Cone}(v) = \bigcap_{e \in I(v)} \{x \in V^* : \langle x, C^\vee(v, e)\rangle \geq 0\}$. Polyhedral means the collection of all these cones, and their faces, is a fan. Well-connected is a local connectivity condition.

We call this fan the *c-Cambrian fan*.

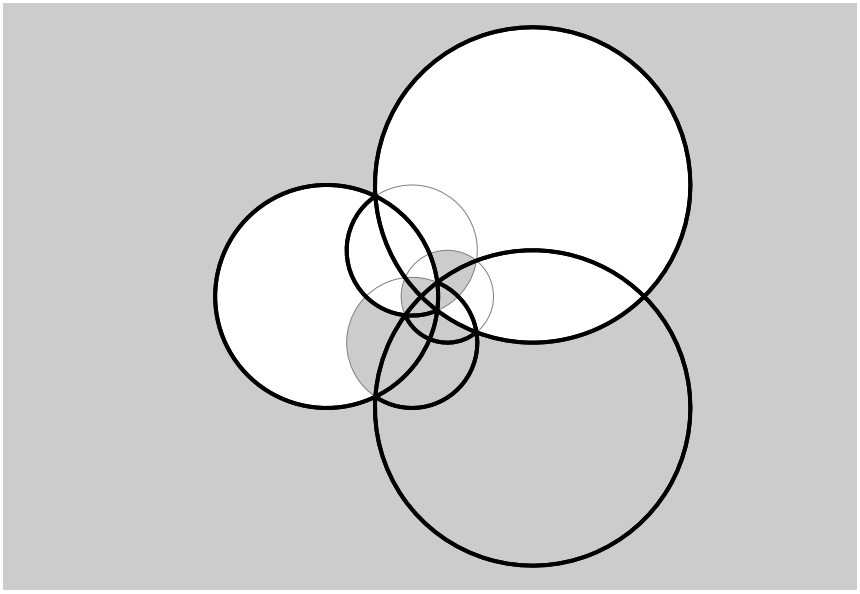
The Cambrian framework & fan for our favorite example

$B = \left[\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix}\right]$, $\quad A = \left[\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right]$, $\quad m(s_1, s_2) = 4$, $\quad c = s_1 s_2$

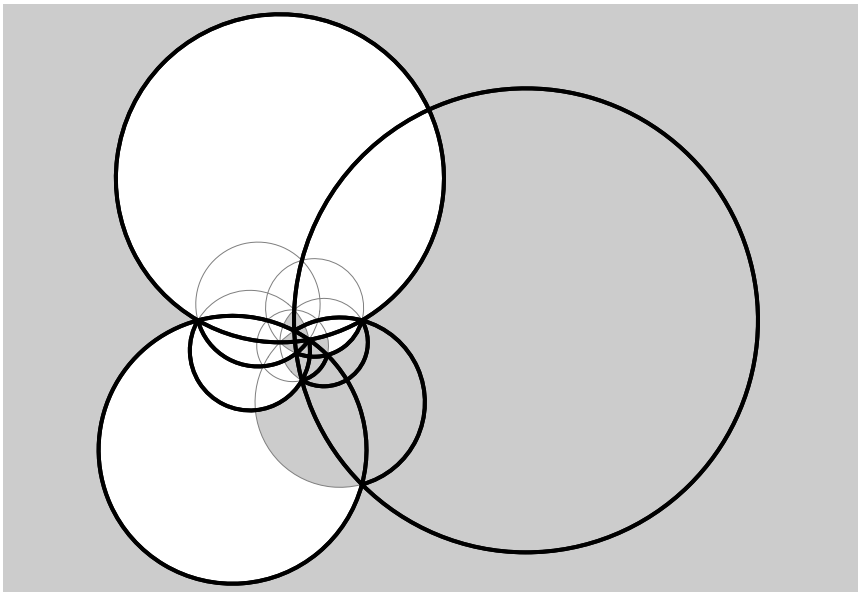
Recall: We orient each edge of the diagram $i \to j$ if $b_{ij} < 0$. To define c, arrows in diagram point left in the word for c.



Example: A_3, $\quad c = s_1 s_2 s_3$



Example: B_3, $\quad c = s_1 s_2 s_3$



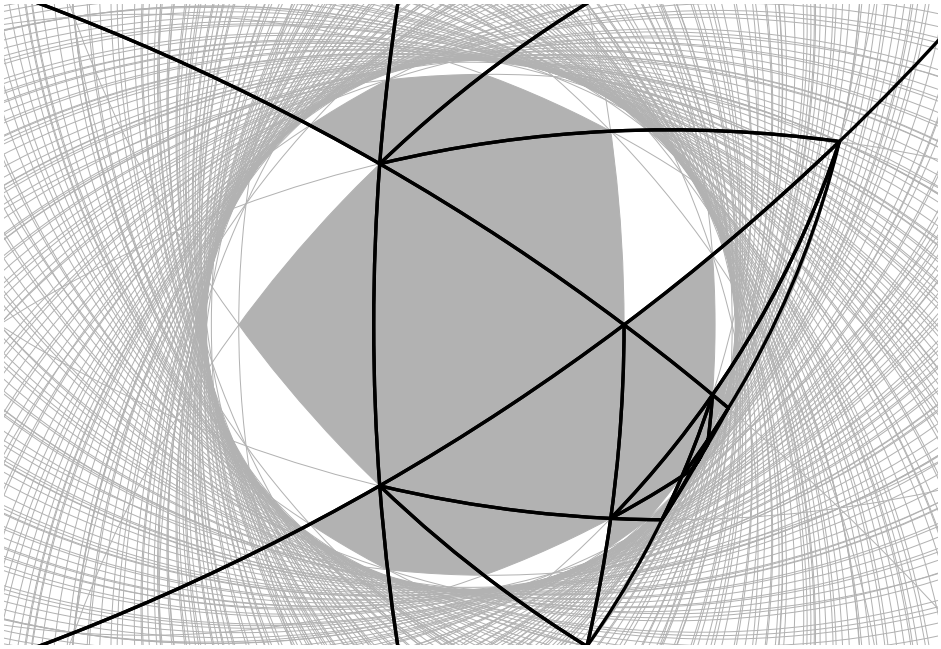
The Cambrian framework, *in*finite type

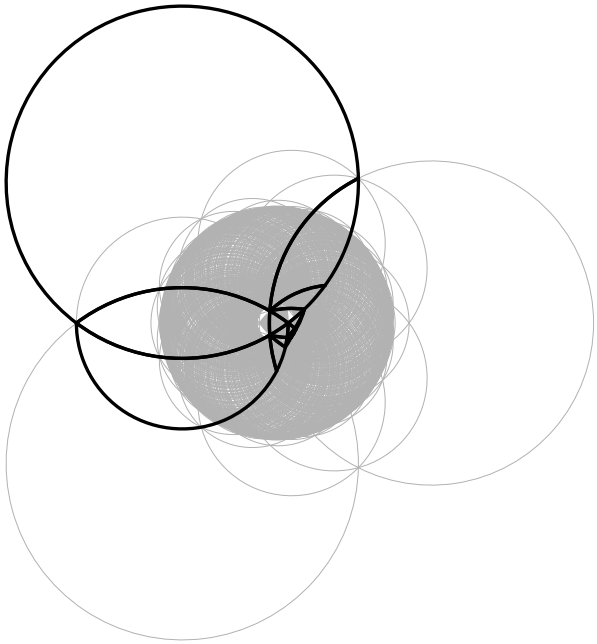
When A is of infinite type (so W is infinite), Camb_c is not n-regular, but no vertex has degree $> n$. We do, however, have n labels for each vertex, some attached to edges in Camb_c and some not. We augment Camb_c to be an n-regular quasi-graph, by adding the right number of half-edges to each vertex. The new incident pairs get the remaining labels. We re-use the symbols Camb_c and C_c for this quasi-graph and labels.

Theorem 5.8 (FRM)**.** *The pair* (Camb_c, C_c) *is an exact, well-connected polyhedral reflection framework for the exchange matrix* B*. It is complete if and only if* A *is of finite type.*

Essential reason for incompleteness in the infinite case: Each $\mathrm{Cone}(v)$ contains vD, and so intersects the Tits cone. Typically, there are **g**-vector cones that don't intersect the Tits cone.

An infinite Cambrian fan





Cambrian frameworks of *affine* Cartan type

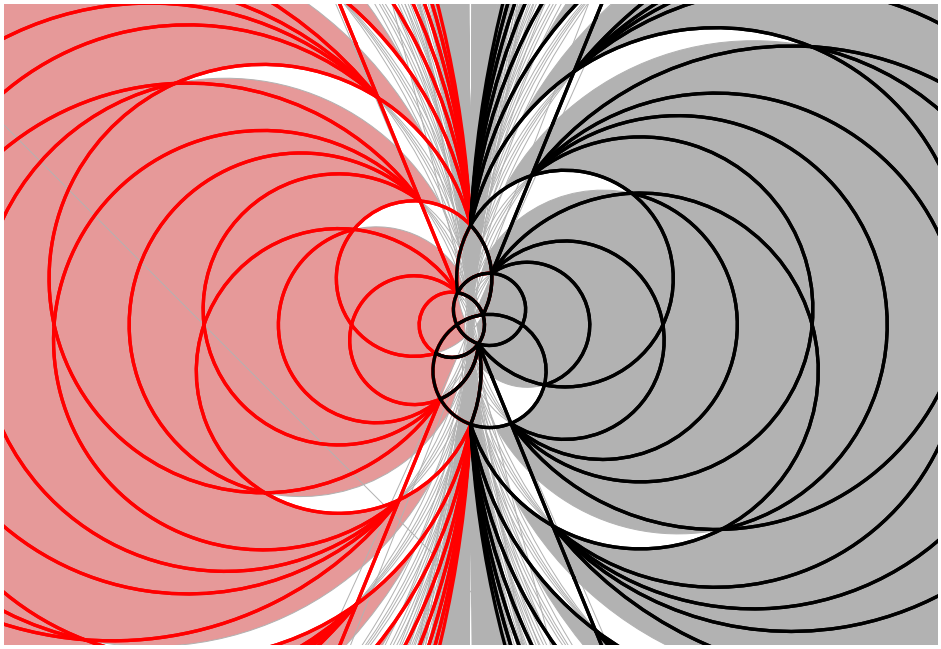
A Cartan matrix is of *affine type* if $\overline{\mathrm{Tits}(A)}$ is a halfspace. Let B be an acyclic exchange matrix defining a Cartan matrix A of affine type, a Coxeter group W, and a Coxeter element c. Write $\mathcal{DF}_c$ for the union of the collection of the faces of the Cambrian fan $\mathcal{F}_c$ and the faces of $-\mathcal{F}_{c^{-1}}$ (the image of the Cambrian fan $\mathcal{F}_{c^{-1}}$ under negation).

Theorem 5.9 (INF)**.** *The collection $\mathcal{DF}_c$ of cones is a simplicial fan.*

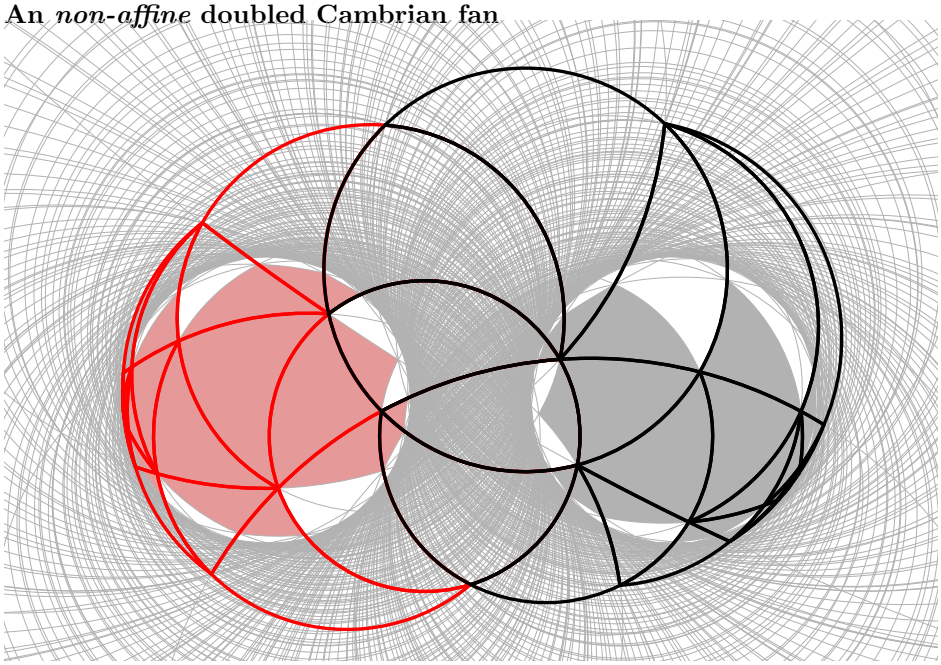
We call $\mathcal{DF}_c$ the *doubled Cambrian fan*. Theorem 5.9 lets us "glue" Camb_c to "$-\mathrm{Camb}_{c^{-1}}$" to get a framework $(\mathrm{DCamb}_c, \mathbb{C}_c)$.

Theorem 5.10 (INF)**.** *If B is acyclic and of affine Cartan type, then $(\mathrm{DCamb}_c, \mathbb{C}_c)$ is a complete, exact, well-built reflection framework.*

An affine *doubled* Cambrian fan



An *non-affine* doubled Cambrian fan



Consequences for structural conjectures

Corollary 5.11. *If B is of finite or affine Cartan type, then Conjectures 2.12–2.17 hold for B.*

Most of these are proven in finite type, but this seems to be the first proof of Conjecture 2.16 in finite type. These all seem to be new in general affine type.

2.12: Each F-polynomial has constant term 1.

2.13: Each F-polynomial has a unique max.-degree monomial.

2.14: For each cluster, the $\mathbf{g}$-vectors are a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

2.15: Different cluster monomials have different $\mathbf{g}$-vectors.

2.16: In the principal-coefficients case, if seeds have equivalent extended exchange matrices, then the seeds are equivalent. 2.17: The rows of the bottom half of principal-coeff extended exchange matrices are sign-coherent.

Denominator vectors in frameworks

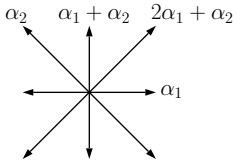
Let v be c-sortable with c-sorting word $a_1a_2\cdots a_k$. Given $s_i \in S$, the *last reflection* for s in v is $a_1\cdots a_{j-1}a_ja_{j-1}\cdots a_1$, where a_i is the rightmost occurrence of s in $a_1a_2\cdots a_k$. Define $\mathrm{cl}_c^{s_i}(v)$ to be the positive root associated to this last reflection. That is $\mathrm{cl}_c^{s_i}(v)$ is $a_1\cdots a_{j-1}\alpha_j$. If s doesn't occur in $a_1a_2\cdots a_k$, define $\mathrm{cl}_c^{s_i}(v) = -\alpha_i$. Define $\mathrm{cl}(v) = \{\mathrm{cl}_c^{s_i}(v) : i = 1,\ldots,n\}$.

Example: $W = B_4$, $\quad c = s_1s_2s_4s_3$, quad $v = s_1s_2s_4s_3|s_1s_2s_3|s_1s_2$

$\mathrm{cl}_c(v) = \{s_1s_2s_4s_3s_1s_2s_3\alpha_1, s_1s_2s_4s_3s_1s_2s_3s_1\alpha_2, s_1s_2s_4s_3s_1s_2\alpha_3, s_1s_2\alpha_4\}$.

Theorem 5.12 (SORT)**.** *If B is of finite Cartan type, then $\mathrm{cl}_c(v)$ is the set of denominator vectors in* $\mathrm{Seed}(v)$.

Denominators example: $W = B_2$**,** $\quad c = s_1s_2$



v	s_i	$\mathrm{cl}_c^{s_i}(v)$
1	s_1	$-\alpha_1$
	s_2	$-\alpha_2$
s_1	s_1	α_1
	s_2	$-\alpha_2$
s_1s_2	s_1	α_1
	s_2	$s_1\alpha_2 = 2\alpha_1+\alpha_2$
$s_1s_2s_1$	s_1	$s_1s_2\alpha_1 = \alpha_1+\alpha_2$
	s_2	$s_1\alpha_2 = 2\alpha_1+\alpha_2$
$s_1s_2s_1s_2$	s_1	$s_1s_2\alpha_1 = \alpha_1+\alpha_2$
	s_2	$s_1s_2s_1\alpha_2 = \alpha_2$
s_2	s_1	$-\alpha_1$
	s_2	α_2

Clus. Var.:	x_1,	x_2,	$\frac{x_2+1}{x_1}$,	$\frac{x_1^2+(x_2+1)^2}{x_1^2x_2}$,	$\frac{x_1^2+x_2+1}{x_1x_2}$,	$\frac{x_1^2+1}{x_2}$
Denom. vec.:	$[-1,0]$,	$[0,-1]$,	$[1,0]$,	$[2,1]$,	$[1,1]$,	$[0,1]$

Exercise 5d. *Let s be initial in c, let v be c-sortable and let $r \in S$. Show that*

1. *If $v \not\geq s$ then*
$$\mathrm{cl}_c^r(v) = \begin{cases} -\alpha_s & \text{if } r = s, \text{ or} \\ \mathrm{cl}_{sc}^r(v) & \text{if } r \neq s \end{cases}$$
2. *If $v \geq s$ then $\mathrm{cl}_c^r(v) = \sigma_s(\mathrm{cl}_{scs}^r(sv))$.*

The sets $\mathrm{cl}_{sc}^r(v)$ and $cl_{scs}^r(sv)$ are defined by induction on the rank of W or on the length of v.

A conjecture on denominator vectors and g-vectors

Recall that the *Euler form* E associated to B is

$$E(\alpha_i^\vee, \alpha_j) = \begin{cases} \min(b_{ij}, 0) & \text{if } i \neq j, \text{ or} \\ 1 & \text{if } i = j. \end{cases}$$

We define a map $\nu : V \to V^*$ by setting

$$\nu(\alpha_j) = -\sum_{i\in I} E(\alpha_i^\vee, \alpha_j)\rho_i.$$

When B is acyclic, ν is given by the negative of an upper uni-triangular matrix, and therefore it is invertible. The inverse map, by a standard combinatorial trick, is $\eta : V^* \to V$ by

$$\eta(\rho_j) = -\sum_{i\in I} F(\alpha_i^\vee, \alpha_j)\alpha_j,$$

where

$$F(\alpha_i^\vee, \alpha_j) = \sum(-E(\alpha_{i_0}^\vee, \alpha_{i_1}))(-E(\alpha_{i_1}^\vee, \alpha_{i_2}))\cdots(-E(\alpha_{i_{k-1}}^\vee, \alpha_{i_k})).$$

The sum is over all paths $i = i_0$—i_1—$\cdots$—$i_k = j$ in the complete graph with vertices I. Since B is acyclic, this is a finite sum.

Conjecture 5.13. *If B is acyclic and x is a cluster variable not contained in the initial seed, then* $\mathbf{g}(x) = \nu(\mathbf{d}(x))$. *Equivalently,* $\mathbf{d}(x) = \eta(\mathbf{g}(x))$.

As written, the conjecture relates a vector in the weight lattice to a vector in the root lattice. The conjecture is easily rewritten in terms of the integer vectors. Note that (modulo the condition "not contained in the initial seed"), this is saying that the $\mathbf{g}$-vectors are $\nu_c(\mathrm{cl}_c(v))$. ν_c? To emphasize the dependence on c, and let us think about source-sink moves $c \leftrightarrow scs$ for s initial/final.

The Cambrian framework (or, work by one of you?) lets us prove the conjecture when B is of finite Cartan type. Recall that $\mathbf{g}$-vectors are the dual basis to C_c. We know that C_c and cl_c can be characterized by induction on length and rank. The map ν_c is compatible with this induction, so we argue by this induction. By the same argument, if Conjecture 5.13 is true, then cl maps c-sortable elements to denominator vectors outside of finite type, too, in the (not complete) Cambrian framework.

References

(CAMB) N. Reading, "Cambrian lattices." Adv. Math. **205**.

(SORT) N. Reading, "Clusters, Coxeter-sortable elements and noncrossing partitions." Transactions AMS **359**.

(SC) N. Reading, "Sortable elements and Cambrian lattices." Algebra Universalis **56**.

(FANS) N. Reading and D. Speyer, "Cambrian fans." J. Eur. Math. Soc. (JEMS) **11**

(INF) N. Reading and D. Speyer, "Sortable elements in infinite Coxeter groups." Transactions AMS **363**.

(FRM) N. Reading and D. Speyer, "Combinatorial frameworks for cluster algebras." In preparation.

Exercises, in order of priority

Although the lecture series is now over, and it's hard to say when these exercises could be "due," I've still put them in order of priority for you. The first line still constitutes a minimum immediate goal. It would be profitable to work all of the exercises eventually.

5a, 5b, 5d,

5c.