1 MSRI Summer Graduate Workshop on Cluster Algebras

Note Title 7/29/2011

Cluster Algebras & Teichmüller Theory

1) Intro to cluster algs from surfaces
2) Tagged arcs and punctured surfaces
3) Graph Theoretic Formulas for Cluster Variables in surfaces
4) Laminations and Crash Course on Hyperbolic Geometry
5) Teichmüller theory and Matrix Product Formulas

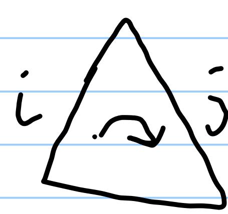
Lecture 1 (Cluster Algebras from surfaces):

Recall Triangulations from Last week's Lectures.

Given a triangulation T of the $(n+3)$-gon number the diagonals arbitrarily $1, 2, \ldots, n$. Number the sides of T by $(n+1), \ldots, (2n+3)$.

The edge-adjacency matrix $\tilde{B} = (b_{ij})$ of T is the $(2n+3) \times n$ matrix with entries

$b_{ij} = \begin{cases} 1 & \text{if} \end{cases}$



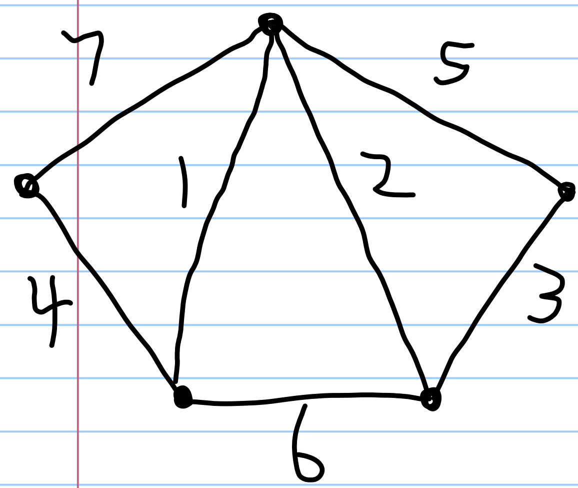
in T,

-1 if

in T,

0 otherwise

2 Example:



$$\tilde{B} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 1 & 0 \end{array}\right]$$

In this mini-course, we discuss cluster algebras that can be constructed by a similar process. These are known as cluster algebras arising from surfaces.

Def (Fomin–Shapiro–Thurston)

Let S be an orientable Riemann surface with a marking M.

We associate to each ideal triangulation of (S, M) a seed for a cluster algebra.

In above example, seed is

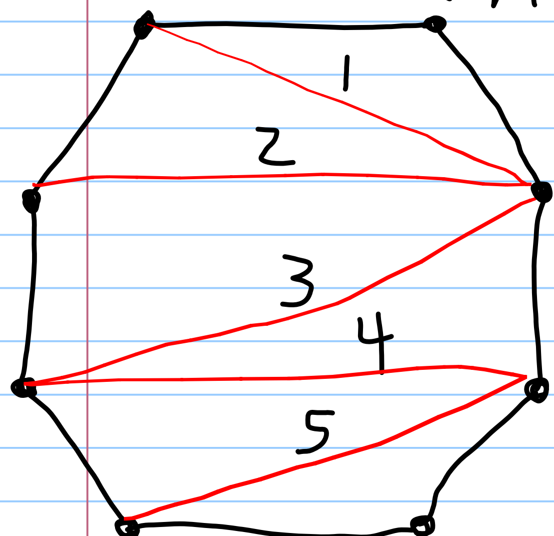
$$\Big(\underbrace{\{x_1, x_2\}}_{\text{initial cluster}}, \underbrace{\tilde{B}}_{\text{edge-adjacency matrix}} \Big)$$

(cluster algebra of geometric type)

= diagonals of pentagon

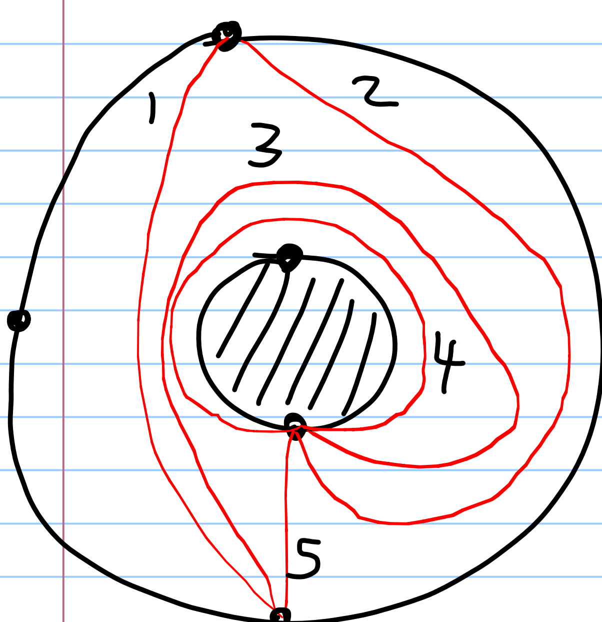
③ More examples

1) Polygon $\left(\begin{array}{l} S = \text{genus } 0, \text{ one boundary component} \\ M \subset \partial S \end{array}\right)$



$|M| = 8$

2) Annulus $\left(\begin{array}{l} S = \text{genus } 0, \text{ two boundary components,} \\ M = M_1 \sqcup M_2 \subset \partial S \end{array}\right)$

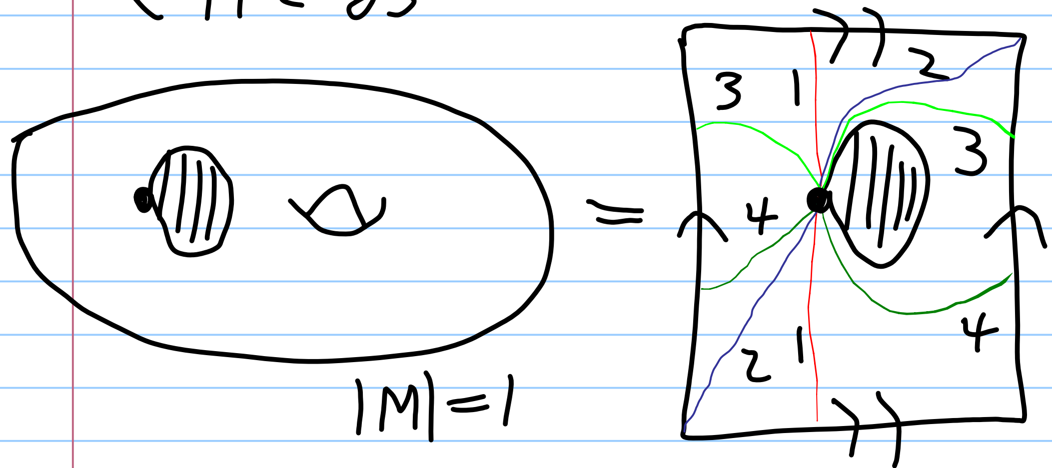


$|M_1| = 2$

$|M_2| = 3$

3) Torus with boundary

(S = genus 1, one boundary comp.)
($M \subset \partial S$)



$|M| = 1$

Notice that in all of today's examples, we have $M \subset \partial S$.

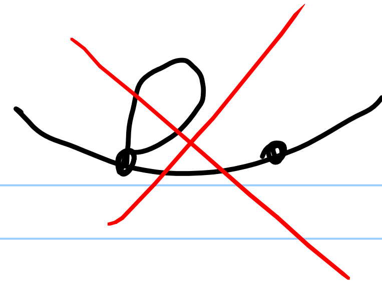
Marked surfaces satisfying this condition are known as <u>unpunctured</u>.

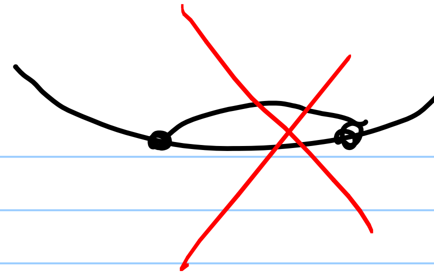
We will assume all our surfaces have this property for the remainder of this lecture.

We now make the notion of <u>ideal triangulation</u> precise:

<u>Def:</u> An <u>arc</u> γ is (S, M) is a curve, considered up to isotopy, such that

- endpoints of $\gamma \in M$,
- γ does not intersect itself, although its endpoints may coincide,
- $\mathrm{Int}(\gamma)$ is disjoint from M and ∂S,
- γ does not cut out an unpunctured monogon or unpunctured digon.





Def: An *ideal triangulation* T of a *marked surface* (S, M) is a maximal collection of arcs s.t. no pair of which
(i) are isotopic to one another,
or (ii) cross each other.

The size $|T|$ of an ideal triangulation only depends on the choice of S and M, and not the choice of T itself.

In particular, given a marked surface (S, M),

$$n = |T| = 6g + 3b + m - 6$$

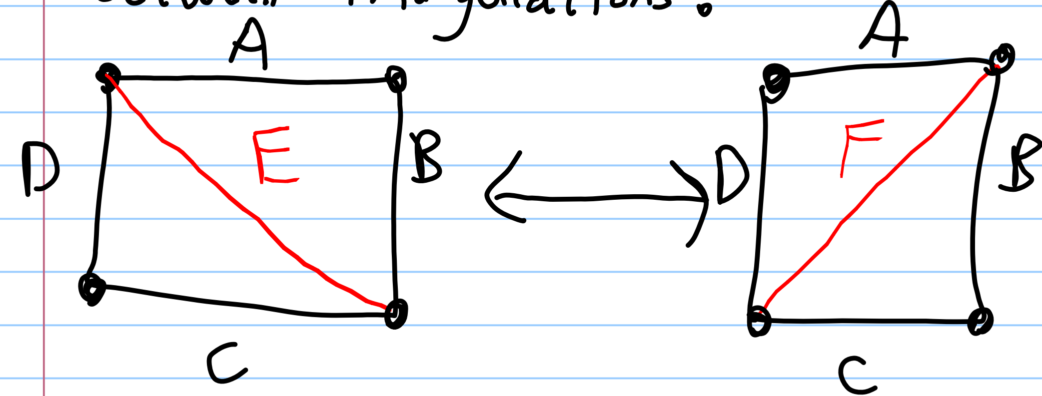
where
- $g = \text{genus}(S)$,
- $b = \#$ boundary comps. in S,
- $m = \# M$.

We now generalize the edge-adjacency matrix for marked surfaces:

Given an ideal triangulation T of (S, M), decompose it into Δ's.

Then
$$\tilde{B}_T := \sum_{\substack{\Delta \in T \\ \text{of } (S,M)}} B_\Delta,$$
where B_Δ defined as above for polygons.

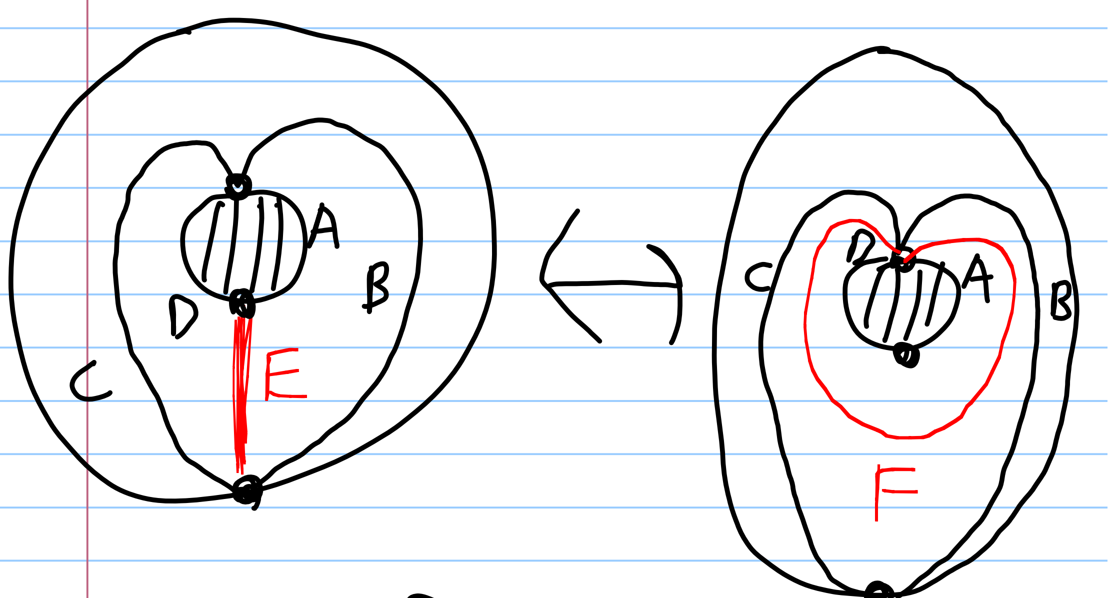
6) Recall for polygons, when constructing the associahedron, there was a flip involution between triangulations:

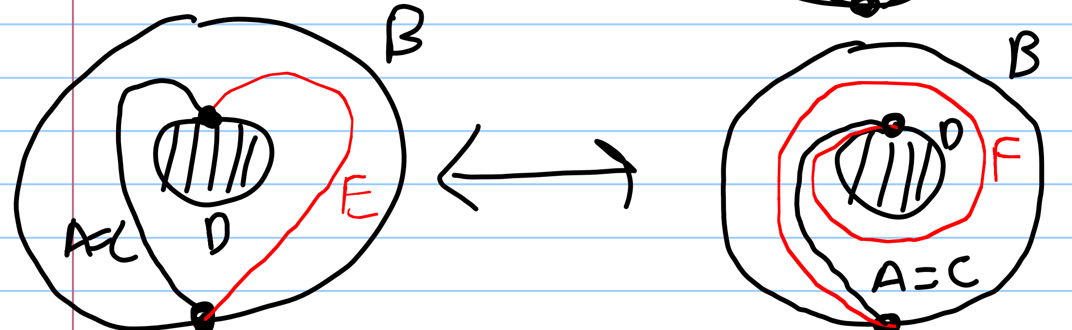


Even satisfy <u>Ptolemy Relation</u>

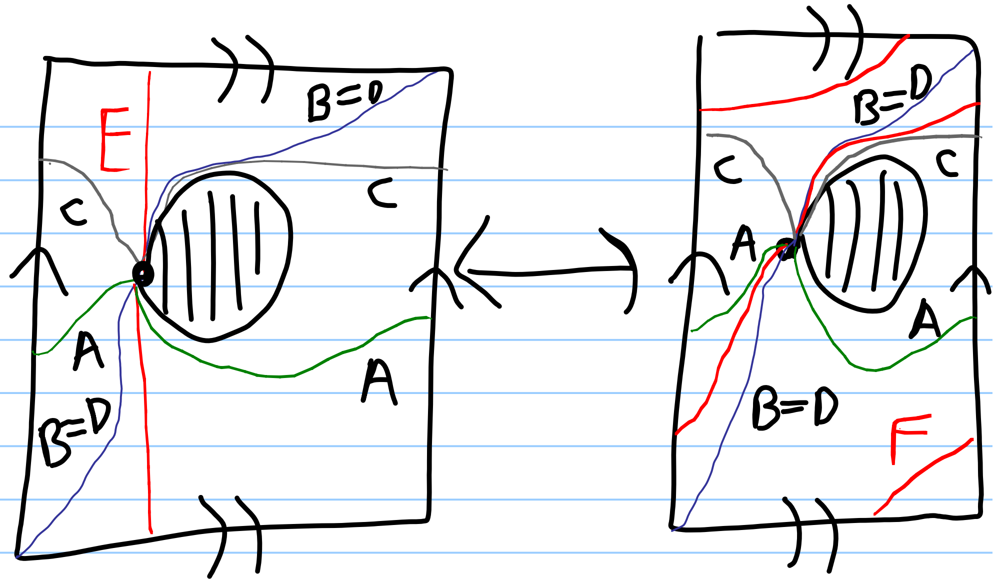
$$EF = AC + BD$$

Same flips occur in ideal triangulations:





7



Moral: When flipping a quadrilateral, will sometimes need to identify arcs.

Based on earlier work by Fock-Goncharov and Gekhtman-Shapiro-Vainshtein, we have the following:

Thm (Fomin-Shapiro-Thurston)

Let S be an orientable Riemann surface with a marking M.

$\ast$ (For now, assume $M \subset \partial S$)

i) Then there is a dictionary

initial cluster algebra seed Σ_T $\longleftrightarrow$ ideal triangulation T

initial cluster variable x_i $\longleftrightarrow$ initial arc $\tau_i \in T$

other cluster variables $\longleftrightarrow$ other arcs in (S, M)

mutation μ_K $\longleftrightarrow$ flipping arc τ_K in quad

8

ii) The cluster complex for the associated cluster algebra $\mathcal{A}_T$, with seed Σ_T, is the dual to the arc complex of (S,M).

iii) Both complexes are connected, and the arc complex is the clique complex of its 1-skeleton.

Some Cluster Algebras from surfaces have already been seen:

e.g. Cluster algebra of n-gon $\longleftrightarrow$ type A_{n+3}

" " (n_1,n_2)-annulus $\longleftrightarrow$ type $\widetilde{A}_{n_1,n_2}$ (affine A)

When we allow punctures, i.e., marked points in Int(S), we get other familiar types D_n & $\widetilde{D}_n$.

[Punctures next lecture.]

Let us finish today by talking about type A vs. type $\widetilde{A}$.
finite affine

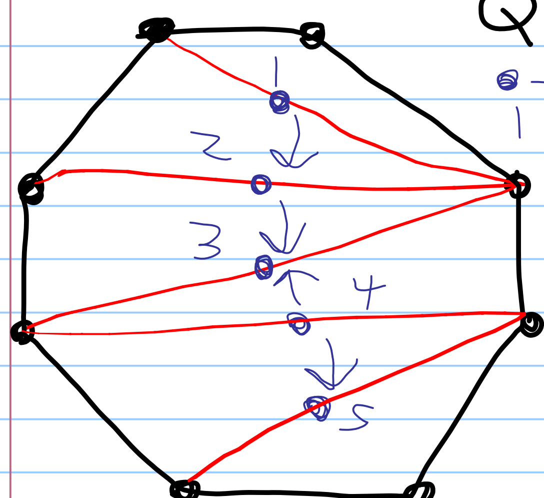
(9) We have a general construction

$$T \longrightarrow \widetilde{B}_T .$$

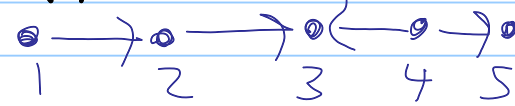
Recall also that if Q is a quiver, we have a map $Q \longrightarrow B_Q$ to an exchange matrix

We can also obtain a quiver Q_T directly from a triangulation T.

Examples

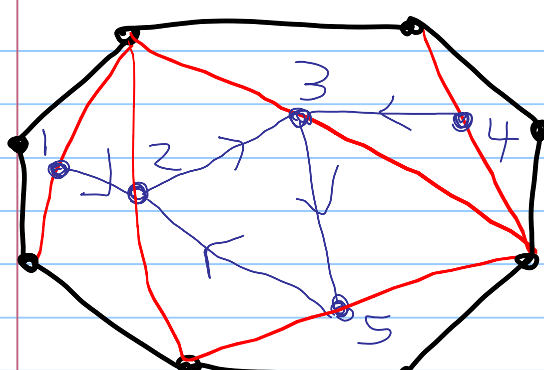


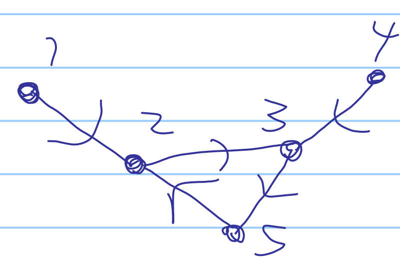
Q_T



arrows in each △ in clockwise order

$Q_{T'}$





$Q_{T'}$ is mutation-equivalent to type A quiver.

10

Little Exercise: a) Find a mutation sequence to get $Q_{T'}$ to look like Q_T.

b) What about with labels?

We have seen Cluster Algebras of type A_n several times now:

- Plücker coordinates of $Gr(2, n+3)$,
- Denom vectors $\longleftrightarrow$ almost pos. roots for type A_n

... and

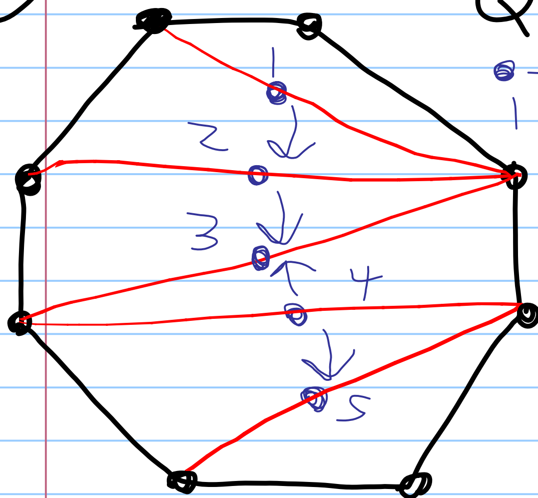
- triangulations of $(n+3)$-gon.

This is a cluster algebra of <u>finite type</u>, meaning there are a <u>finite number of cluster variables</u>.

Let us now look closer at how mutations compare to reflections in a coxeter group.

Let Q_T be an orientation of a type A_n Dynkin Diagram.

Example:



Q_T

$$\underset{1}{\circ} \to \underset{2}{\circ} \to \underset{3}{\circ} \leftarrow \underset{4}{\circ} \to \underset{5}{\circ}$$

arrows in each $\triangle$ in clockwise order

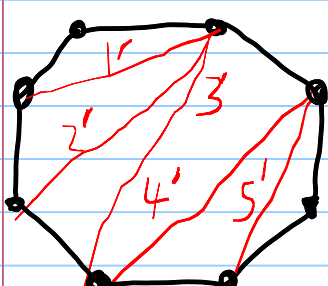
Related to BGP-reflection functors also

We can define a Coxeter element c associated to Q_T by writing down a permutation of the vertices $\{v_1, \dots, v_n\}$ s.t. if we mutate Q_T in order, we are always mutating at a <u>source</u>.

e.g. in above $c = s_1 s_2 s_4 s_3 s_5$

<u>Note</u>: other reduced words for c, such as $s_4 s_5 s_1 s_2 s_3$.

If we mutate T accordingly, we obtain



using either reduced word.

$\boxed{\text{Any observations?}}$

cQ_T also yields Q_T again.

Let us look at cluster algebra of affine type, $\tilde{A}_n$, instead:

12

Coeff-free

Example $\left(\{x_1, x_2\}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\right)$,

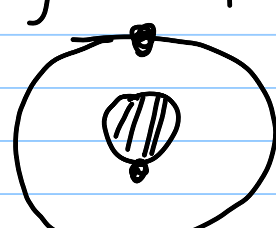
If we let $x_3 = x_1'$, $x_0 = x_2'$, and x_n for $n \in \mathbb{Z}$ defined by alternating mutations,

$$x_n = \begin{cases} \dfrac{P_n(x_1, x_2)}{x_1^{n-2} x_2^{n-3}} & \text{if } n \geq 3 \\[2ex] \dfrac{P_n(x_1, x_2)}{x_1^{-n} x_2^{-n+1}} & \text{if } n \leq 0 \end{cases}$$

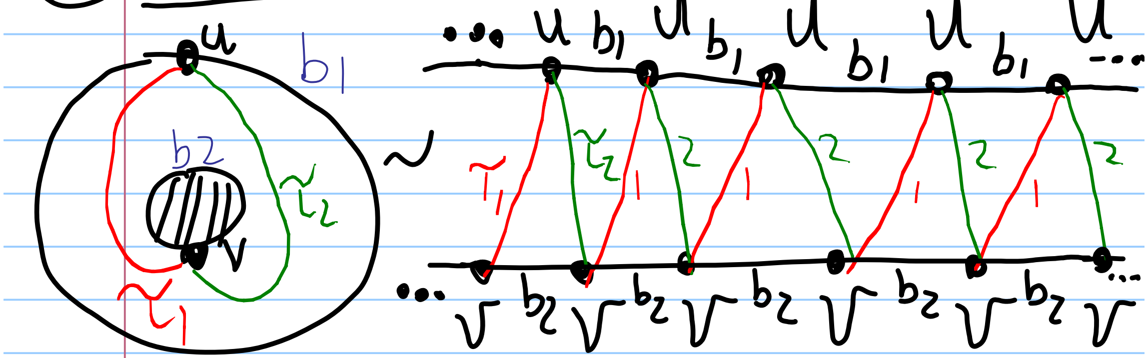
We also saw in SAGE demo, if we let $x_1 = x_2 = 1$, then

$$\{x_3, x_4, x_5, \ldots\} = \{2, 5, 13, 34, 89, \ldots\}$$

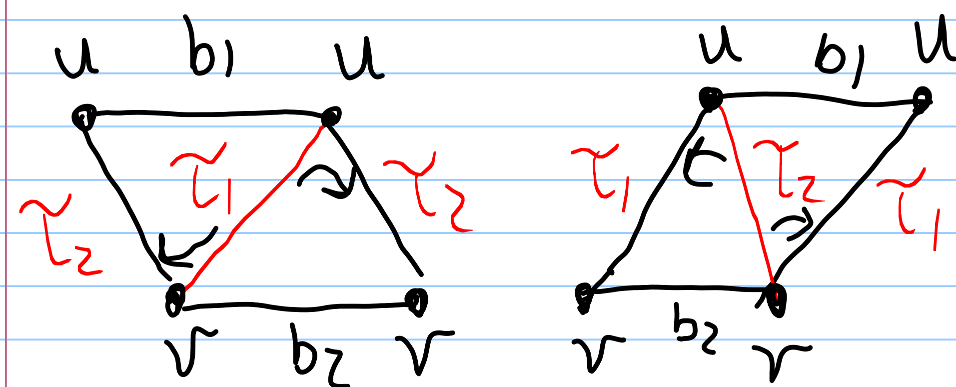
We can also realize this cluster algebra using ideal triangulations of annulus with $|M_1| = 1$, $|M_2| = 1$.



(13) Example seed



Fundamental Domains

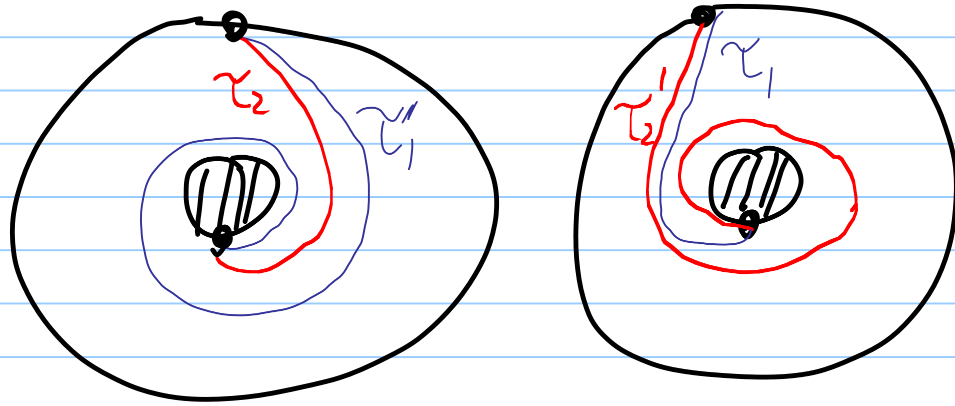


So edge-adjacency matrix

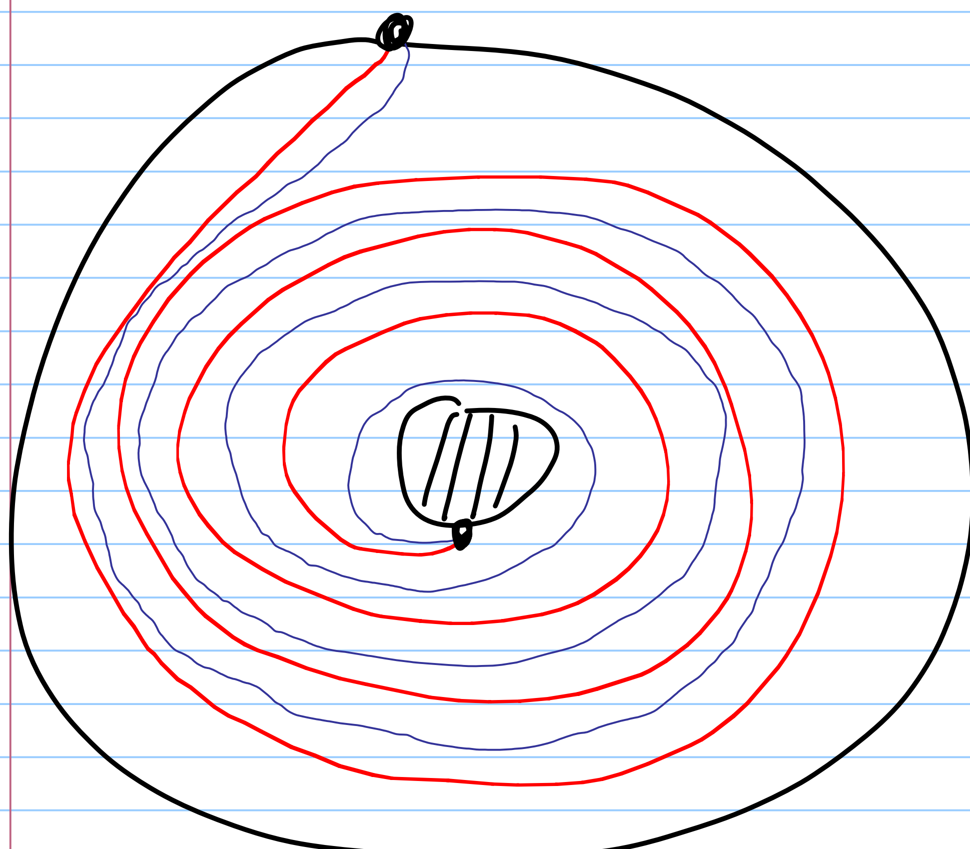
$$\tilde{B} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 2 \\ 2 & -2 & 0 \\ b_1 & 1 & -1 \\ b_2 & 1 & -1 \end{array} \qquad B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

If we mutate, we flip a diagonal $\rightsquigarrow \mu_1(B) = \mu_2(B) = -B$

14 On annulus with $|M_1| = |M_2| = 1$,



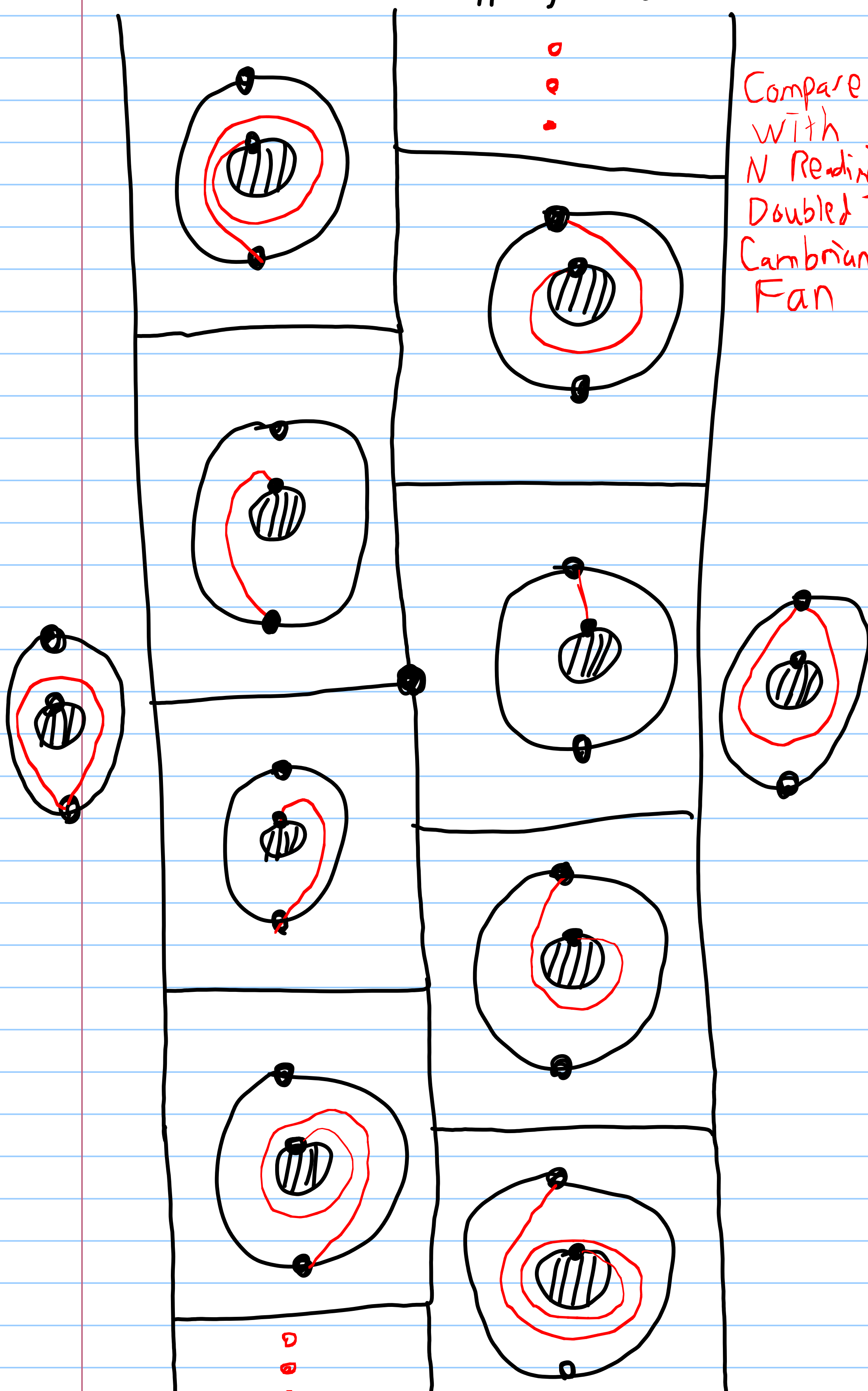
Can keep mutating/Flipping



Exchange matrix still $\pm \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

15 Arc Complex (Dual of Cluster Complex)

Annulus with $|M_1|=1$, $|M_2|=2$



Compare with N Reading's Doubled Cambrian Fan

16 Moral: Infinite number of cluster variables in this case. However edge-adjacency matrices have $0, \pm 1$, or ± 2 as every entry.

$\Rightarrow$ Finite number of possible exchange matrices.

Called Finite Mutation Type.

True for any cluster algebra from a surface.

Thm (Felikson–Shapiro–Tumarkin)

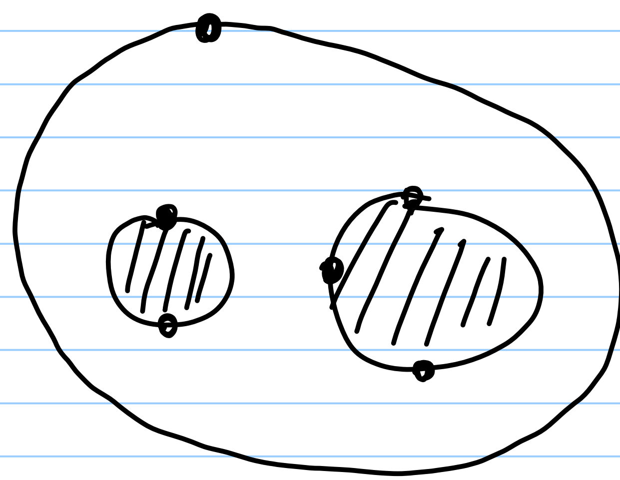
A skew-symmetric cluster algebra of finite mutation type is

1) rank 2,
2) comes from a surface, or
3) is of type $E_6, E_7, E_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{\tilde{E}}_6, \tilde{\tilde{E}}_7, \tilde{\tilde{E}}_8, X_6$ or X_7.

Lecture 1 Exercises

Exercise 1-1: Consider

the following unpunctured surfaces: (i) hexagon, (ii) 10-gon,
iii) annulus with $(|M_1|, |M_2|) =$ $(1,1)$, $(1,2)$, $(2,2)$, $(6,7)$,
iv) pair of pants



v) torus with one boundary and $\# M = 2$ on ∂S.

For each such marked surface,

A) Draw at least two ideal triangulations of (S, M).

B) Verify that $|T|$ is given by the above formula.

C) Compute $\widetilde{B}_T$'s for some of your choices.

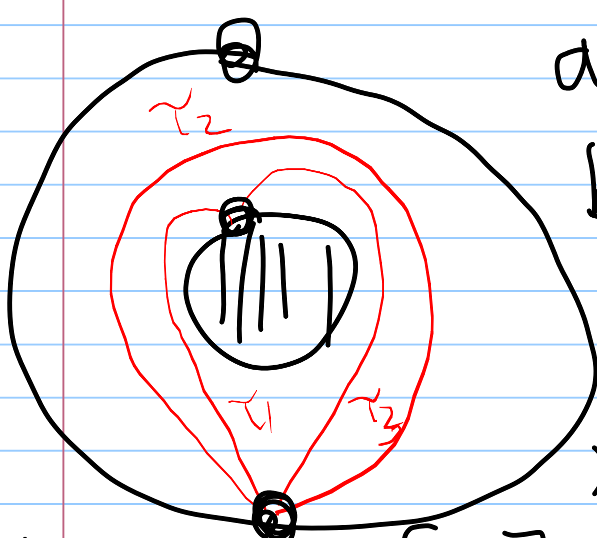
Exercise 1-2:

i) Pick an ideal triangulation T and arc $\gamma \in T$ from those constructed in Exercise 1.

Flip γ to get T' and compute $\widetilde{B}_{T'}$. Compare to $\mu_k(B_T)$ where arc γ is labeled by a k.

ii) Pick ideal triangulations T_1 & T_2 for (S, M) as in Exercise 1. Try to find a sequence of flips so that T becomes T'.

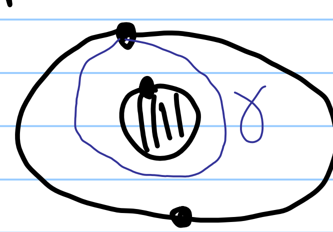
Exercise 1-3: Given triangulation T



a) Compute B_T

b) Q_T

c) The Laurent expansion of x_γ.



d) Let $\widetilde{B}_T = \begin{bmatrix} B_T \\ I \end{bmatrix}$ or otherwise compute F_γ (F-polynomial)

e) Compare maximal term of F_γ to denom(x_γ). (Now try w/ other γ's.)