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MSRI School Lecture #2: Tot pos gps & cluster algebras

Reference: Fomin + Zelevinsky "Total positivity: tests + parametrization"
Papers on double Bruhat cells

Def: A matrix is totally positive (resp., non-negative) if all its minors are positive (resp., non-neg.) real numbers.

1930's: systematic study of these matrices by Schoenberg, Gantmacher, Krein, Whitney.
Since then, this field has been linked to:
- oscillations in mechanical systems
- stochastic processes
- planar resistor networks ...

1994: Lusztig found a surprising connection between total positivity & canonical bases in quantum groups. ⇝ This led to his introduction of the totally positive & totally nonnegative parts $G_{>0}$, $G_{\geq 0}$ in every real reductive group. Similarly he introduced the totally pos. & non-neg parts of any generalized partial flag variety G/P.

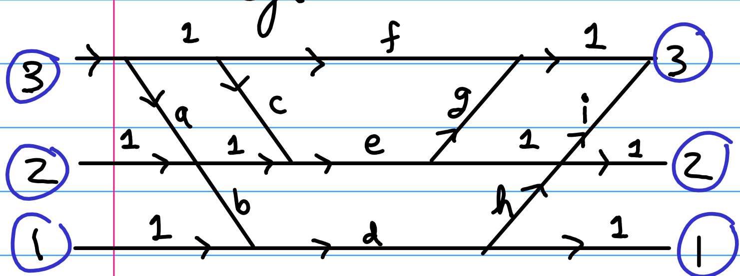
1996-2001: Fomin & Zelevinsky, also Berenstein-Fomin-Zelevinsky further developed Lusztig's theory of total positivity in G & tried to understand Lusztig's dual canonical basis in a "concrete" way. This work led to the introduction of cluster algebras by Fomin + Zelevinsky in 2002.

Today: Explain how the study of total positivity in $G \rightsquigarrow$ cluster algebras. We'll look at the case $G = SL_r$, where $G_{>0}$ and $G_{\geq 0}$ recover totally positive and non-negative matrices (w/ determinant 1)

Questions one might ask:

1. How can we parameterize the set of all elements in $G_{>0}$? in $G_{\geq 0}$?
2. How many minors must we test to deduce that a matrix $M \in G_{>0}$? Which minors?

1. There is a general procedure for producing totally nonnegative matrices. Fix a planar network — an acyclic directed planar graph Γ whose edges have weights.



The weight of a directed path in Γ is defined to be the product of the weights of the edges.
The weight matrix $x(\Gamma)$ is an $n \times n$ matrix (a_{ij}) where
a_{ij} = sum of all weights from i to j.

Here,
$$x(\Gamma) = \begin{pmatrix} d & dh & dhi \\ bd & bdh + e & bdhi + eg + ei \\ abd & abdh + ae + ce & abdhi + (a+c)e(g+i) + f \end{pmatrix}$$

Exercise: Make this more precise

Lemma (Lindstrom – Gessel–Viennot): All minors of such a matrix are polynomials in the edge weight w/ positive coefficients. (There is a combinatorial interpretation for Δ_{IJ} as the sum of weights of all vertex-disjoint paths from the sources I to the sinks J.)

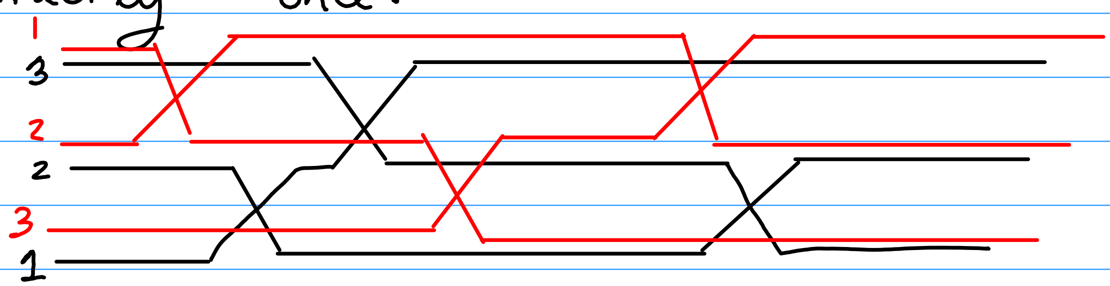
So if each edge weight is in $\mathbb{R}_{>0}$, $x(\Gamma)$ is totally nonnegative.
Moreover

Theorem (A. Whitney '52, Fomin + Zelevinsky) The map $(\mathbb{R}_{>0})^9 \to 3\times 3$ matrices given by $(a,b,c,\dots,i) \mapsto x(\Gamma)$ is a bijection from $(\mathbb{R}_{>0})^9 \longrightarrow$ totally positive 3×3 matrices.
(and the obvious generalization works for $n\times n$ matrices)

Planar networks are a useful tool for parameterizing totally positive matrices & related varieties.

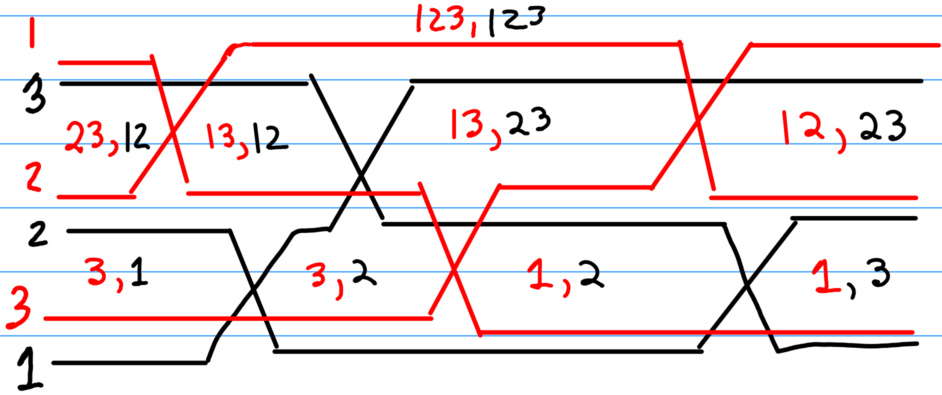
On question 2: (How many, & which minors do we need to test if a matrix is TP?)

Double wiring diagrams (Fomin + Zelevinsky)
Choose two families of piecewise straight lines, each family colored w/ one of two colors, s.t. each pair of lines of like colors intersect exactly once.



Remark: if we look at the set of lines in a fixed color, this encodes a reduced decomposition for the longest permutation $w_0 = (n, n-1, \dots, 2, 1)$.

Assign to each chamber of a diagram a pair of subsets of the set $[1,n] = \{1, \dots, n\}$: each subset indicates which lines of the corresponding color pass below the chamber:



Interpret A, B as the "chamber minor" $\Delta_{A,B}$

rows → columns

Theorem (Fomin + Zelevinsky): Each double wiring diagram — each of which is determined by a shuffle of two reduced decomp's for w_0 — gives rise to the following criterion: an $n \times n$ matrix is totally positive iff all its chamber minors are positive.

Exercise: prove this!

Example above says: A 3×3 matrix M is totally positive iff the following minors are pos:

$\Delta_{123,123}(M)$

$\Delta_{23,12}(M)$ $\quad \Delta_{13,12}(M)$ $\quad \Delta_{13,23}(M)$ $\quad \Delta_{12,23}(M)$

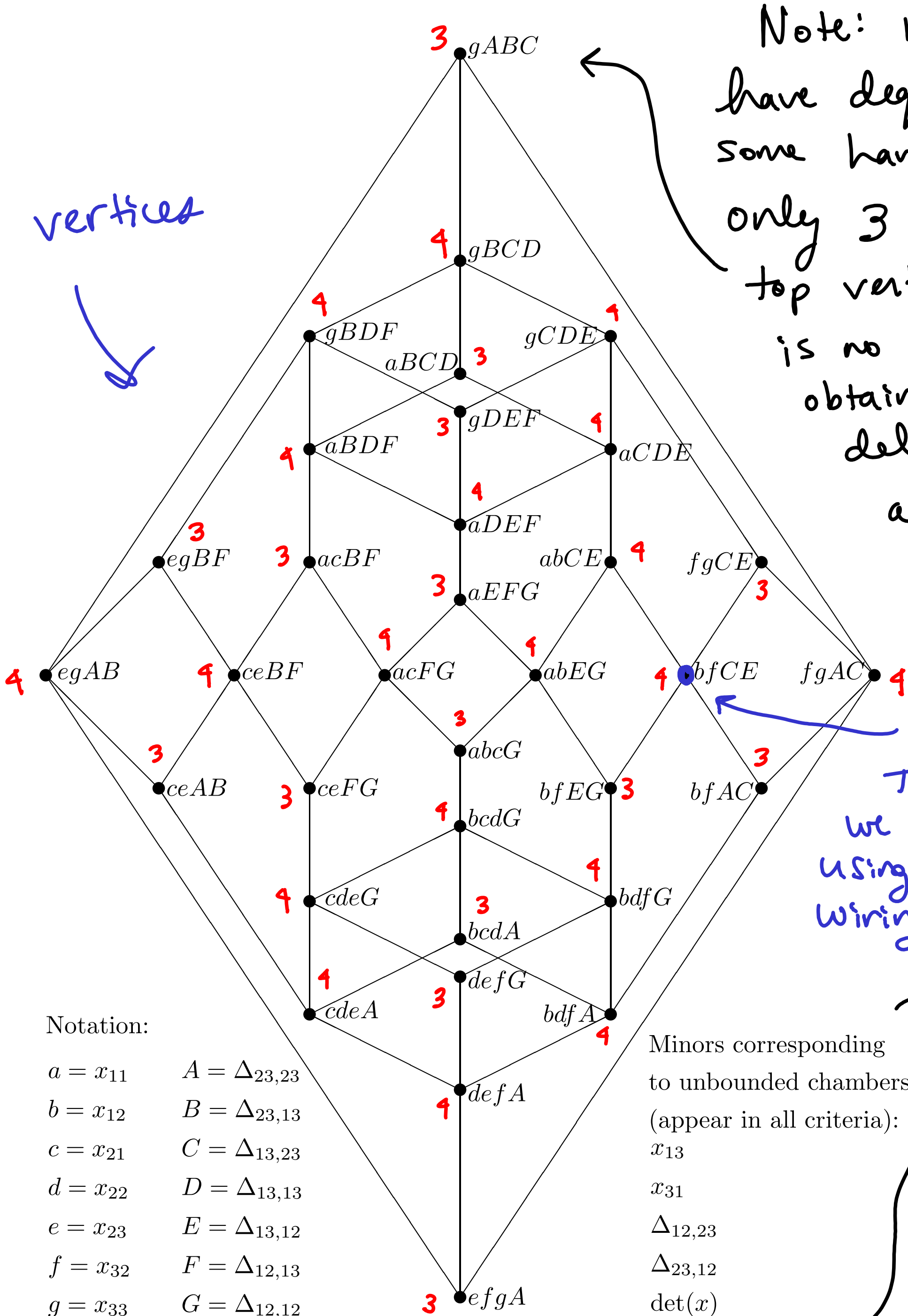
$\Delta_{3,1}(M)$ $\quad \Delta_{3,2}(M)$ $\quad \Delta_{1,2}(M)$ $\quad \Delta_{1,3}(M)$

We get a lot of TP criteria this way. Let's make a chart showing all of them.

Here is an "exchange graph" showing TP criteria for GL_3. I've drawn in the degree of each vertex.

Note: many have degree 4, but some have degree only 3 — eg for top vertex, there is no other vertex obtained by deleting g & adding a new minor.

34 vertices



this is the TP criteria we saw earlier, using the double wiring diagram.

analogous to "frozen" coefficient variables

FIGURE 8. Total positivity criteria for GL_3

two arrangements Arr(**i**) and Arr(**i**′) whose isotopy types are adjacent in the graph

Ex: For each edge in this graph, find an algebraic relation that relates the variables of the 2 corresponding "clusters"

Fomin & Zelevinsky realized that perhaps they were just looking at a piece of a bigger graph — that there should be some "mutation" procedure to go from each TP criteria ("clusters") to 4 others.

In this example (SL_3 — which is basically the same as GL_3), there are actually 50 clusters, so we were missing 16 before. It is of type D_n.

The "coefficient" variables are $x_{13}, x_{31}, \Delta_{12,23}, \Delta_{23,12}$, & the cluster variables are:

(i) the other 14 minors of a 3×3 matrix (except det)
(ii) $x_{12}x_{21}x_{33} - x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32} + x_{13}x_{22}x_{31}$
(iii) $x_{11}x_{23}x_{32} - x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32} + x_{13}x_{22}x_{31}$

16 cluster variables $\leftrightarrow$ almost positive roots of D_n.

Each cluster gives rise to a total positivity criteria: a matrix $x \in SL_3$ is TP iff the 4 elements of the given cluster & the 4 coeff variables are all positive at x.

What about the totally non-negative matrices which are not totally positive? Let's take a step back...

There is a decomposition of G into strata (double Bruhat cells) which is "good" w/ respect to total positivity — Lusztig, Fomin & Zelevinsky.

Notation:

Let $G = SL_{r+1}$, B and B_- two "opposite Borel subgps" $B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $B^- = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$

$H = B \cap B_- = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$ the "maximal torus",

$W = \mathrm{Norm}_G(H)/H$ the Weyl gp, which for $G = SL_{r+1}$ is the symmetric group S_{r+1}.

G has two Bruhat decompositions (into double cosets w/ respect to B and B_-):

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_- v B_- .$$

The double Bruhat cell $G^{u,v} := BuB \cap B_- v B_-$.

This is not actually a cell — but it is biregularly isomorphic to a Zariski open subset of an affine space of dimension $r + \ell(u) + \ell(v)$ — described by saying certain minors must or must not vanish.

We have $$G = \bigcup^{*}_{\substack{u \in W \\ v \in W}} G^{u,v}$$ disjoint union

In what sense is this decomposition good w/ respect to total positivity?

If we define $G^{u,v}_{>0} = G^{u,v} \cap G_{\geq 0}$ then

<u>Theorem:</u> $G^{u,v}_{>0} \cong \mathbb{R}^{r+\ell(u)+\ell(v)}_{>0}$ (Lusztig)

Further, $G^{w_0,w_0}_{>0} = G_{>0}$, the set of TP matrices (w/ det 1)

This gives a decomposition $G_{\geq 0} = \bigcup_{u,v \in W}^{*} G^{u,v}_{>0}$ ← these are all homeomorphic to open balls

Lusztig proved this theorem by giving a parameterization using rational functions that are not necessarily regular on $G^{u,v}$. One might hope for something more...

<u>Def:</u> A TP-basis for $G^{u,v}$ is a collection of regular functions $F = \{f_1, \dots, f_m\} \subset \mathbb{C}[G^{u,v}]$ s.t.:

(i) $f_1, \dots, f_m$ are algebraically independent & generate the field of rational functions $\mathbb{C}(G^{u,v})$. In particular, $m = r + \ell(u) + \ell(v)$

(ii) The map $(f_1, \dots, f_m): G^{u,v} \to \mathbb{C}^m$ restricts to a biregular isomorphism $U(F) \to (\mathbb{C}_{\neq 0})^m$ where $U(F) = \{x \in G^{u,v} : f_k(x) \neq 0 \ \forall \ k \in [1,m]\}$

"$G^{u,v}$ looks a lot like $\mathbb{C}^m$"

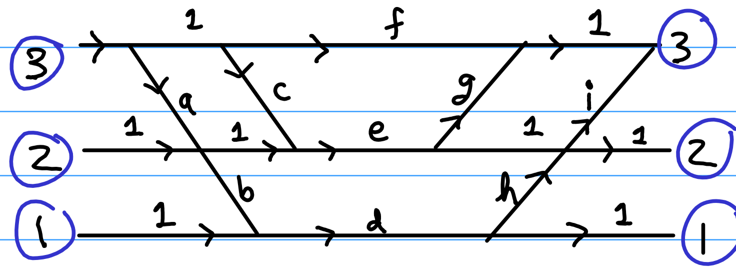
(iii) The map $(f_1, \dots, f_m): G^{u,v} \to \mathbb{C}^m$ restricts to an isomorphism $G^{u,v}_{>0} \to \mathbb{R}^m_{>0}$.

"$f_1, \dots, f_m$ provide a <u>total positivity criterion</u> in $G^{u,v}$: an element $x \in G^{u,v}$ is totally non-negative iff $f_k(x) > 0 \ \forall \ k \in [1,m]$"

Fomin + Zelevinsky found a large number of total positivity criteria for testing whether a matrix $x \in G^{u,v}$ is totally nonnegative. Their construction uses a version of the double wiring diagram we saw before, with reduced decompositions for u and v replacing the two reduced decompositions for w_0 and w_0.

Exercises:

1. By using this network & its weight matrix & looking at examples, try to guess a combinatorial formula for all of the minors of the weight matrix in terms of path families.



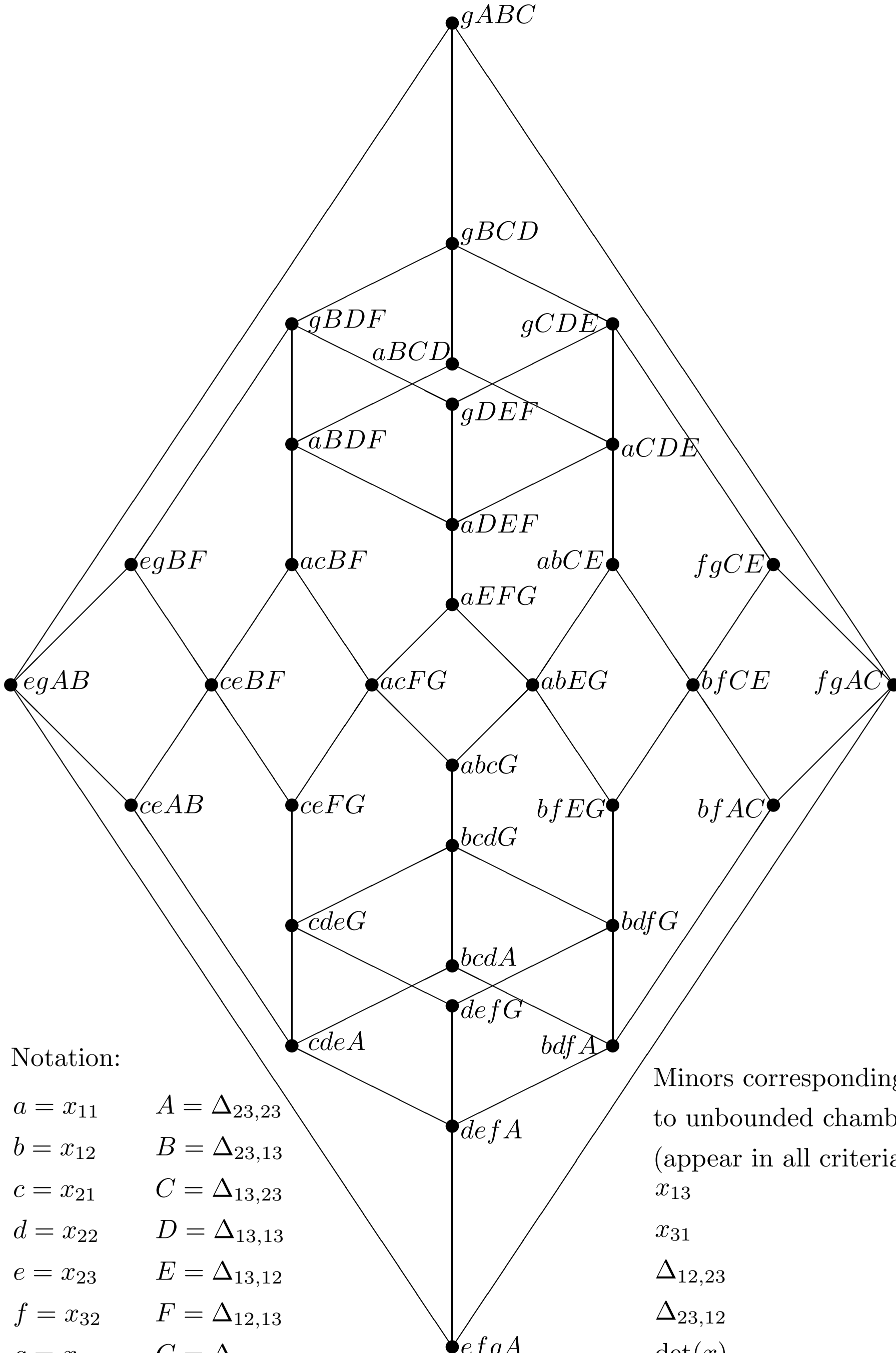
2. Prove that a 3×3 matrix M is totally positive iff the following minors are pos:

$\Delta_{123,123}(M)$

$\Delta_{23,12}(M)$ $\Delta_{13,12}(M)$ $\Delta_{13,23}(M)$ $\Delta_{12,23}(M)$

$\Delta_{3,1}(M)$ $\Delta_{3,2}(M)$ $\Delta_{1,2}(M)$ $\Delta_{1,3}(M)$

3. For each edge in the graph below, find an algebraic relation that relates the 2 corresponding "clusters."



Notation:

$a = x_{11}$ $\quad A = \Delta_{23,23}$

$b = x_{12}$ $\quad B = \Delta_{23,13}$

$c = x_{21}$ $\quad C = \Delta_{13,23}$

$d = x_{22}$ $\quad D = \Delta_{13,13}$

$e = x_{23}$ $\quad E = \Delta_{13,12}$

$f = x_{32}$ $\quad F = \Delta_{12,13}$

$g = x_{33}$ $\quad G = \Delta_{12,12}$

Minors corresponding to unbounded chambers (appear in all criteria):

x_{13}

x_{31}

$\Delta_{12,23}$

$\Delta_{23,12}$

$\det(x)$

FIGURE 8. Total positivity criteria for GL_3

two arrangements $\mathrm{Arr}(\mathbf{i})$ and $\mathrm{Arr}(\mathbf{i}')$ whose isotopy types are adjacent in the graph