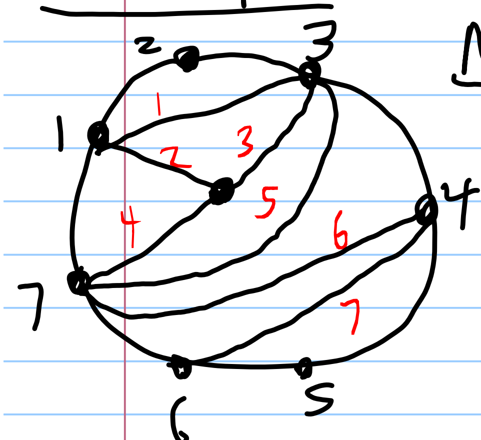
① Lecture 2: Tagged Triangulations and more on Cluster Algs from surfaces

Note Title 8/4/2011

Today: 1) Tagged Arcs and the Tagged Arc complex
2) Denominator vectors
3) Block decomposability

We now allow $M \cap \partial S \neq \emptyset$, i.e. interior marked points, known as punctures.

Example: Once-Punctured n-gon

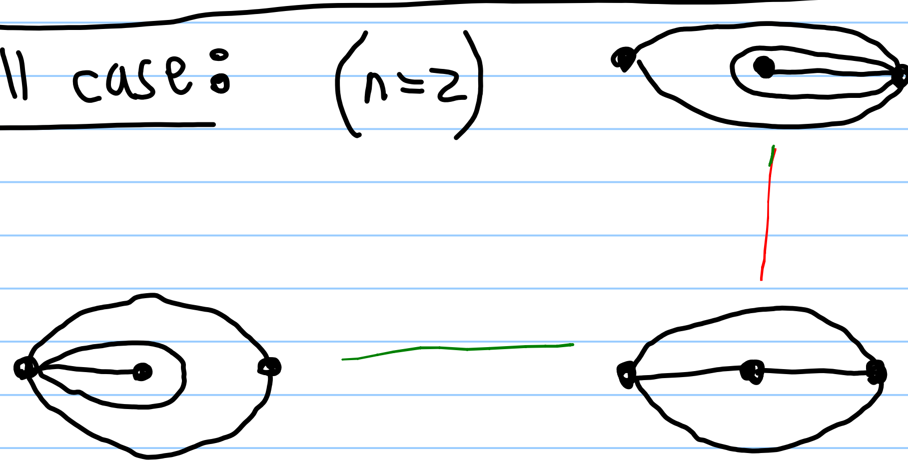


Notice: if $|M| = n+1$, an ideal triangulation contains n arcs.

$c = \# M \cap \partial S$
$p = \# M \cap (S - \partial S)$
$= \#$ punctures

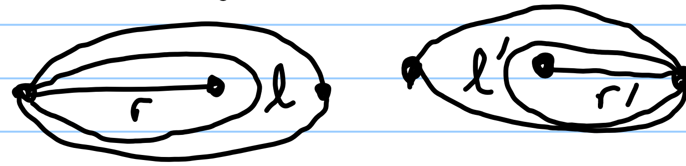
$n = |T| = 6g + 3b + 3p + c - 6$

Small case: $(n=2)$

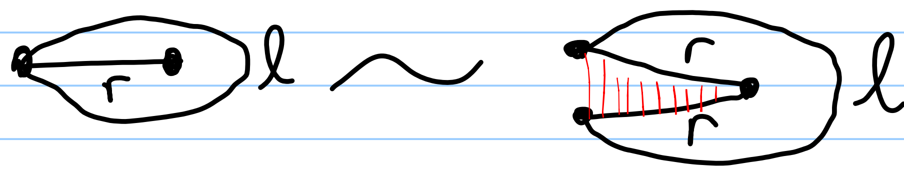


(2) In unpunctured case, ideal triangulation $T \longleftrightarrow$ cluster, and we could flip any arc $\tau \in T$.

In above example, cannot flip arc r or r'



Such a configuration is known as a self-folded quadrilateral, and



is a self-folded triangle.

The exchange graph for a triangulation of a punctured surface is therefore incomplete.

Example

once-punctured 3-gon

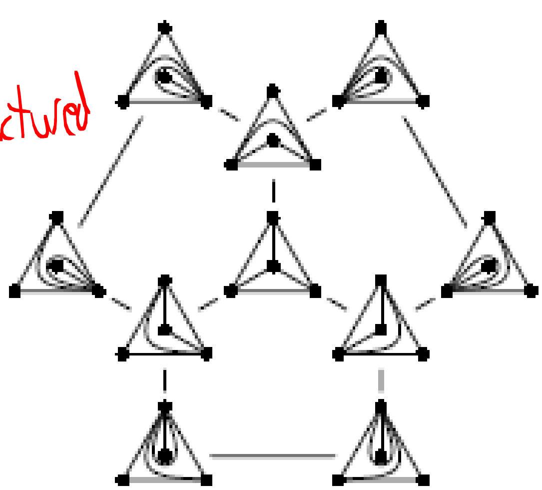
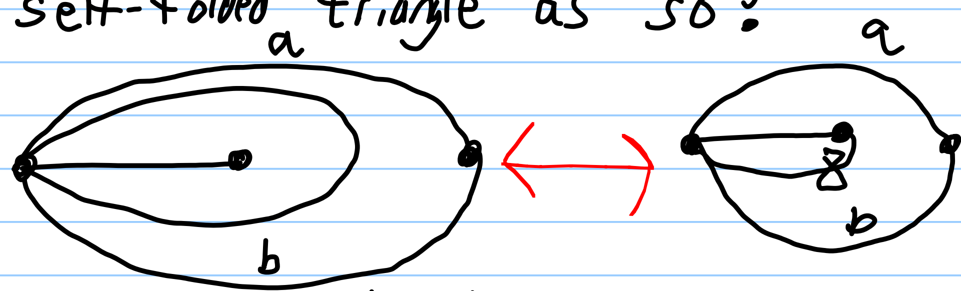


Figure 7. The graph $E^{\circ}(S, M)$ for a once-punctured triangle.

③ Fomin-Shapiro-Thurston complete such exchange graphs to an n-regular one by using <u>tagged arcs</u>.

<u>Quick Idea</u>: We can turn an ideal triangulation into a tagged triangulation by replacing every self-folded triangle as so:

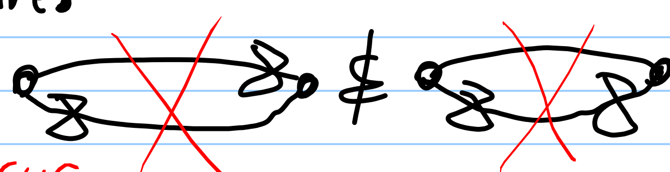


we leave all other arcs as they were.

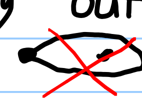
Observe that such a tagged triangulation satisfies

i) no two tagged arcs cross each other,

ii) no two tagged arcs are isotopic except  p is now allowed as long as p is a puncture.

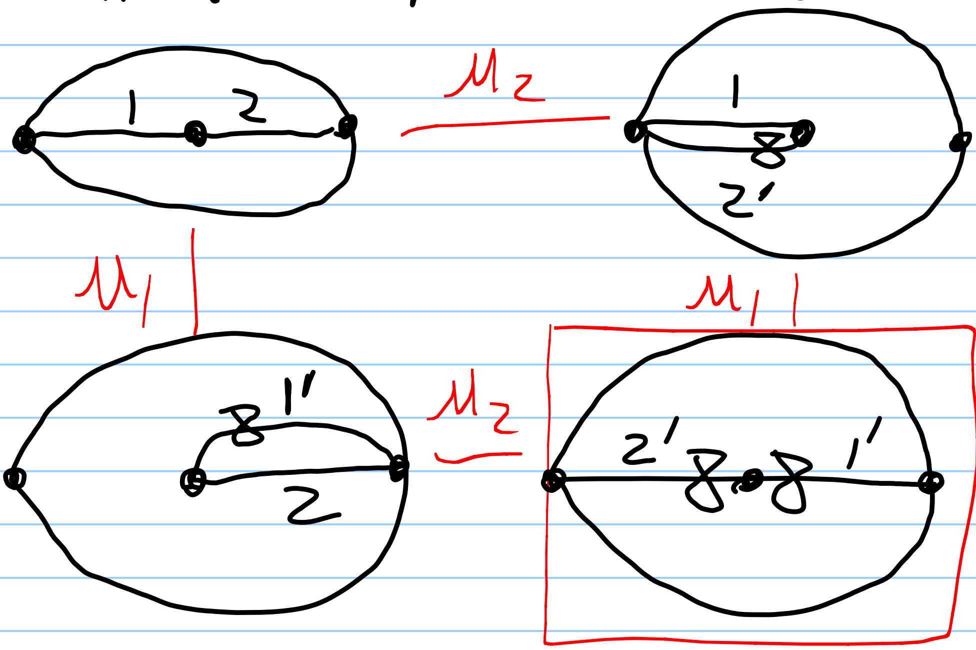
Note:  do not occur.

iii) except for the above, we do not have the local configuration 

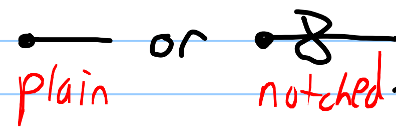
iv) we disallow arc cutting out a once punctured monogon 

④ We define flips of tagged arcs as follows:

In a once-punctured 2-gon

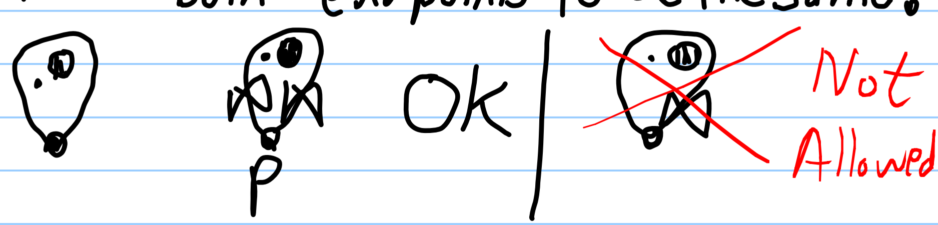


Let me now be more precise:

Def: A tagged arc is an ordinary arc, except for one cutting out a once-punctured monogon, with a decoration at each of its endpoints ()

s.t. Notched endpoints are only allowed at punctures.

Further, if the endpoints of γ coincide, we force the tagging on both endpoints to be the same:



⑤ **Def**: Two tagged arcs α, β are <u>compatible</u> if

i) α & β do not cross,

ii) α & β are not isotopic unless their tagging differs at exactly one endpoint,

iii) if the endpoints of α and β coincide, they must have the same tagging (with (ii) being the only exception).

Def: A <u>tagged triangulation</u> is a maximal collection of <u>compatible</u> tagged arcs on (S, M).

<u>Example</u>

Once punctured 3-gon

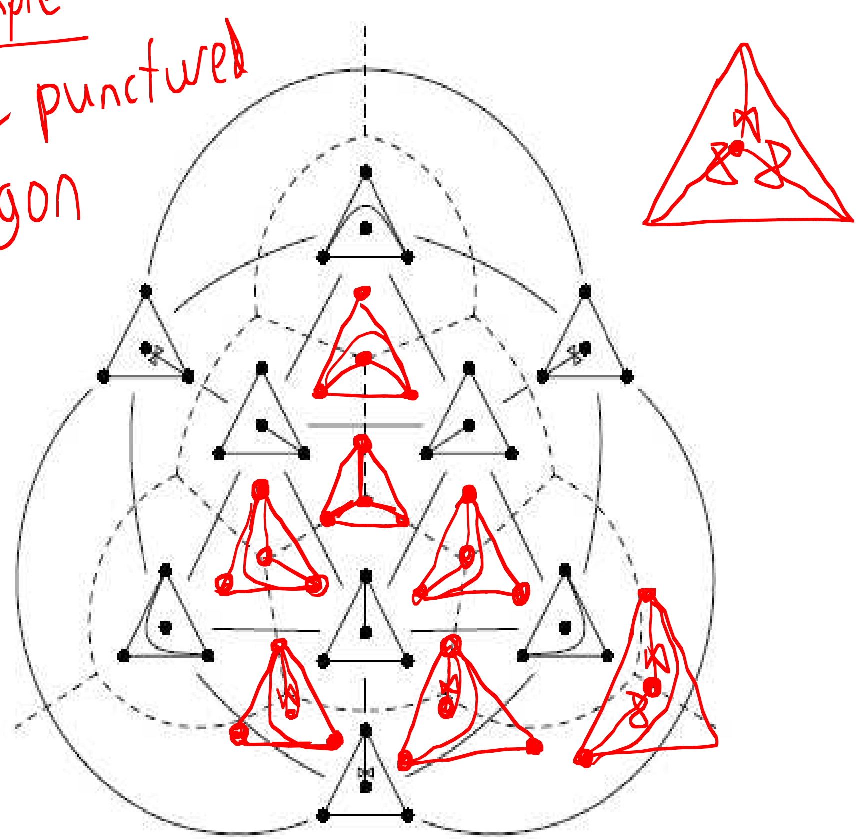


Figure 19. The tagged arc complex of a once-punctured triangle.

⑥ Def (Intersection pairing)

Let α, β be two tagged arcs in (S, M). The intersection number $(\alpha|\beta)$ is defined as follows.

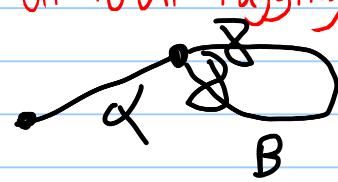
Let α_0, β_0 be untagged versions of α, β.
(no self-intersections, α_0, β_0 intersect minimally)

$C = 0$ unless $\alpha_0 = \beta_0$, in which case $C = -1$

$$(\alpha|\beta) := A + B + C + D$$

A = # intersections of α_0 & β_0 in $S - M$

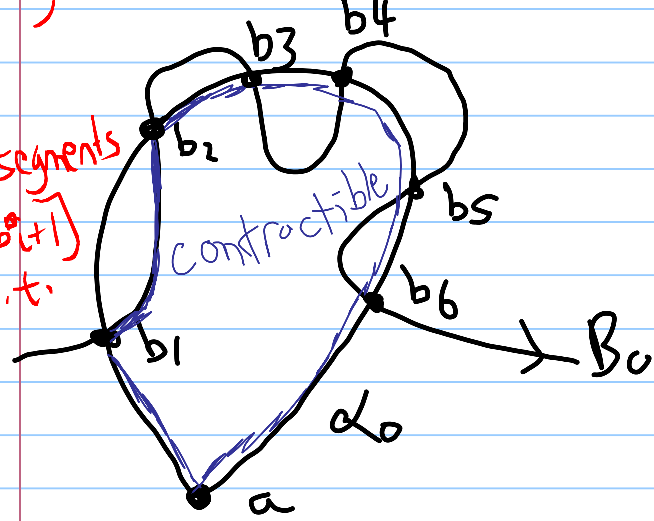
D = # ends of β incident to endpt of α w/ different tagging



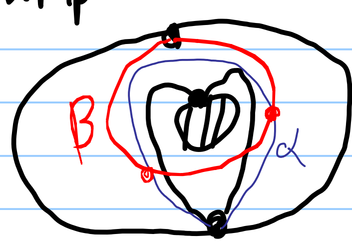
$\Rightarrow D = 2$

$B = 0$ except for configurations like

$B = -$#segments $[b_i, b_{i+1}]$ s.t. contractible



Example where $B \neq 0$ in Ex 1.3



$$(\alpha|\beta) = 2 - 1 + 0 + 0 = 1$$

7 **Thm (Fomin-Shapiro-Thurston)**

If (S,M) is any marked surface, with or without punctures, then

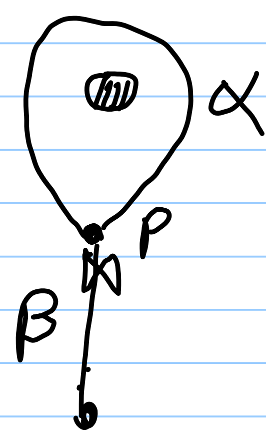
$$\{\text{tagged arcs } \gamma\} \longleftrightarrow \{\text{cluster variables } x_\gamma\}$$

Tagged Arc Complex $\cong$ Cluster Complex

Given initial tagged triangulation $T = \{\tau_1, \tau_2, \ldots, \tau_n\}$, then

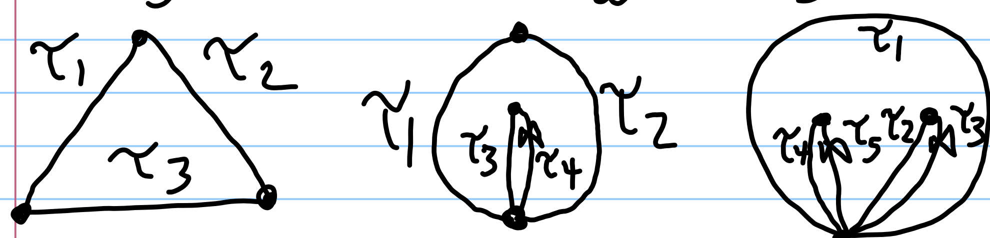
$$\operatorname{denom}(x_\gamma) = \prod_{i=1}^{n} x_{\tau_i}^{(\tau_i | \gamma)}.$$

Remark: In $(\alpha | \beta) \neq (\beta | \alpha)$

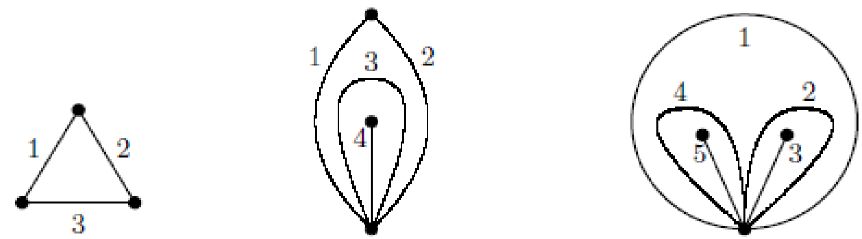


Puzzle Pieces and Block Decomposability

Consider the following three local configurations in a tagged triangulation:



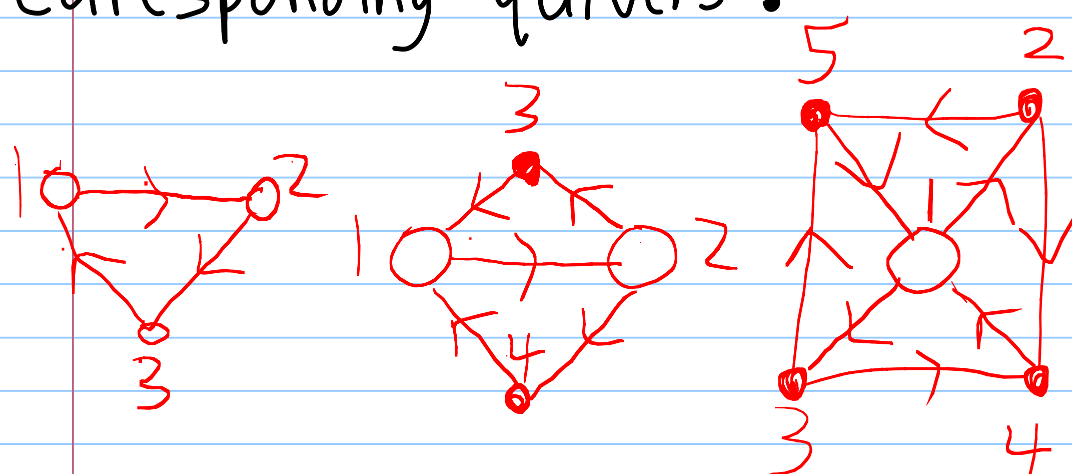
(8) We compute edge-adjacency matrices by considering ideal triangulations



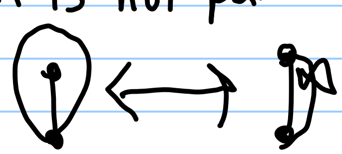
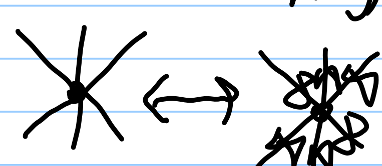
$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

Figure 9. The three puzzle pieces and signed adjacencies within each.

Corresponding quivers:



Any ideal triangulation can be decomposed into these puzzle pieces where we glue two together along an edge (which is not part of a self-folded triangle)

By  or , get exchange matrix for any tagged triangulation also.

9 In terms of quivers, we can erase any vertex ○, i.e. one that does not correspond to an arc in a self-folded triangle, to get six possible blocks:

three shown above already

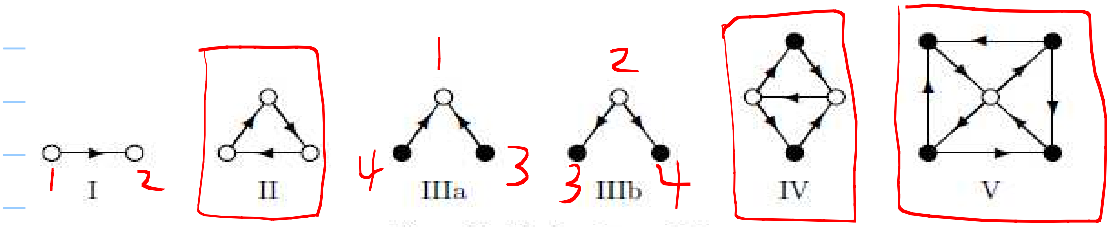
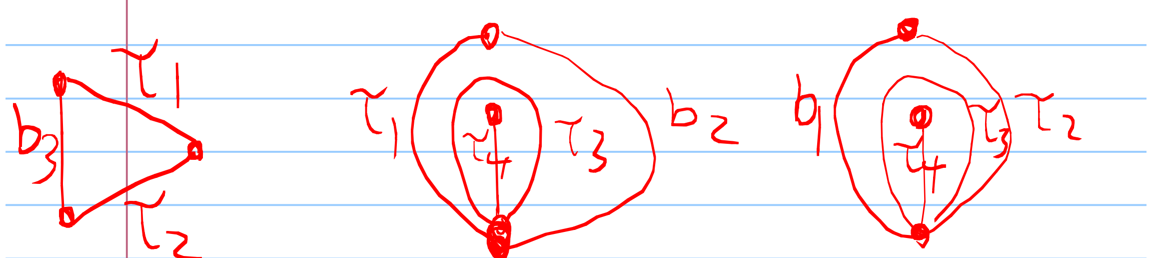


Figure 30. Blocks of types I–V.

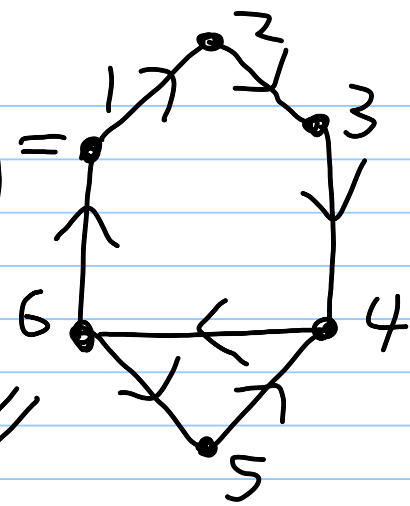


We think of b_3, b_2, b_1 as boundary segments.

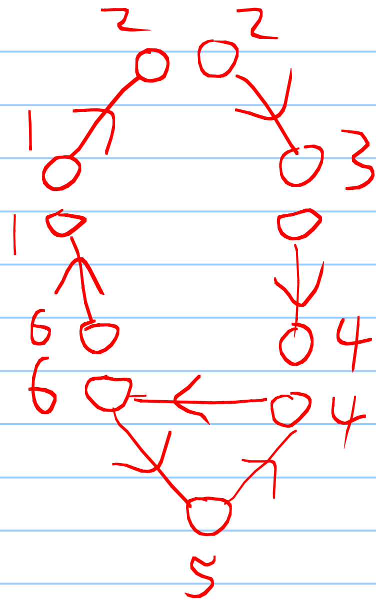
<u>Def:</u> A quiver Q is <u>block-decomposable</u> if we can obtain Q by gluing together copies of the above blocks subject to:

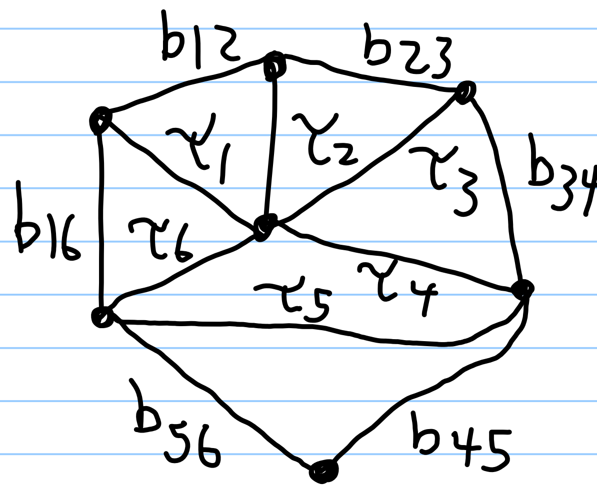
- Only vertices in <u>distinct</u> blocks can be identified.
- A vertex ● cannot be glued to any other vertex
- At most two vertices ○ can be glued together.

10 **Example**: $Q_1 =$

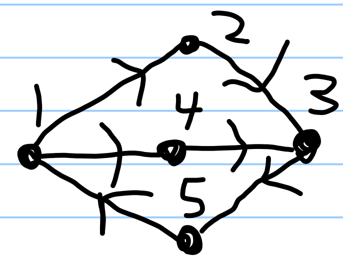


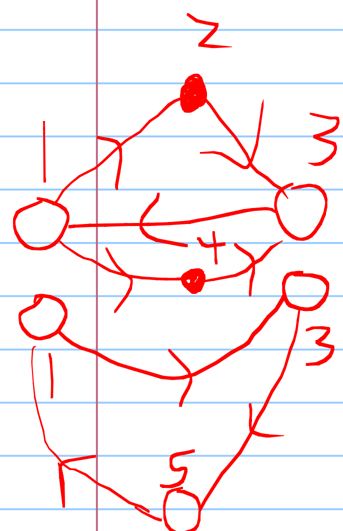
is block decomposable

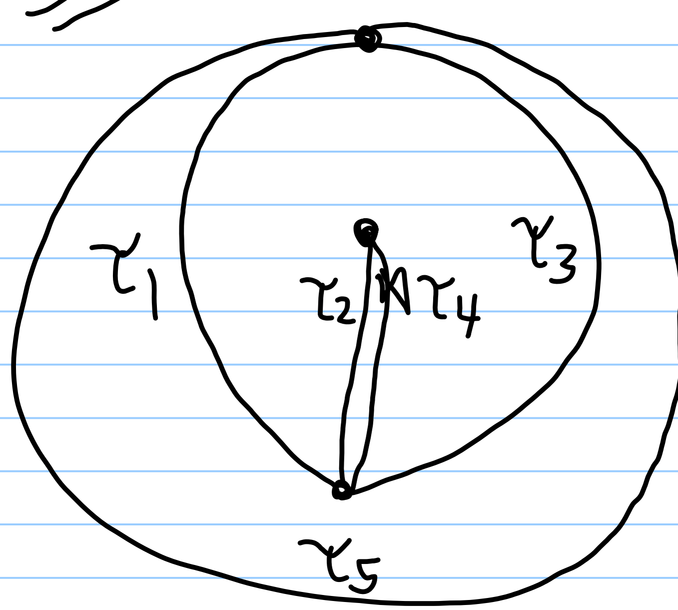




So is $Q_2 =$







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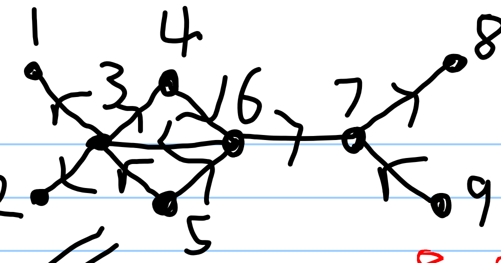
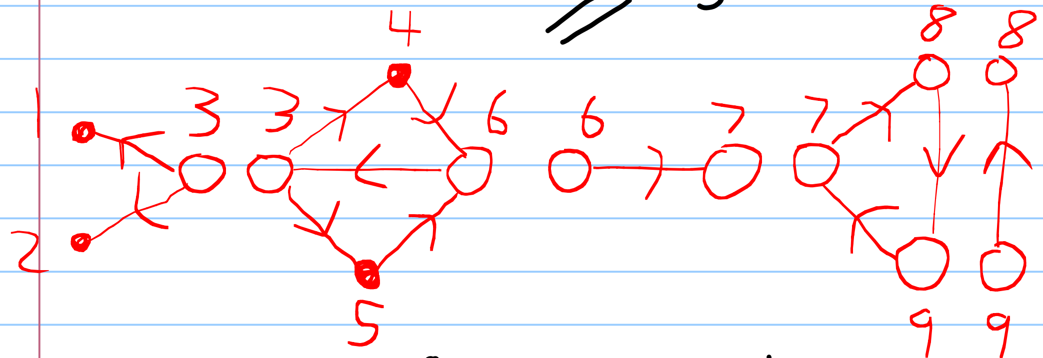
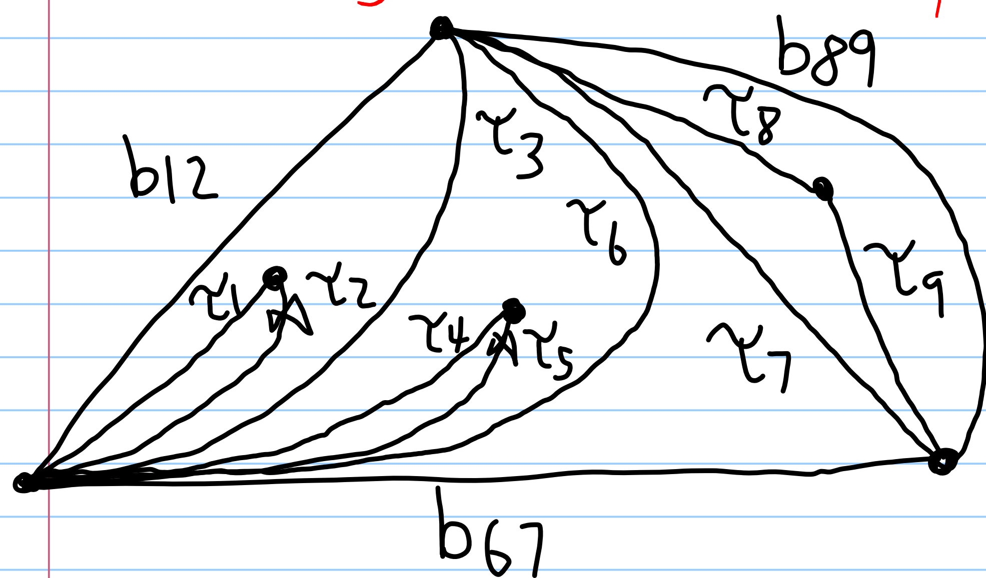
Another example: $Q_3 =$

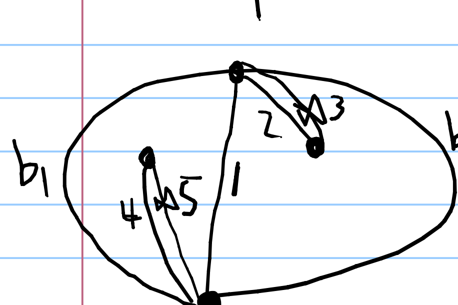






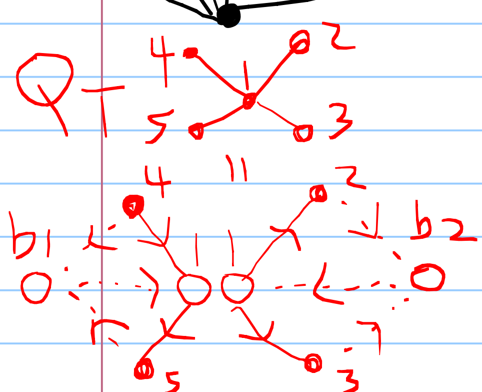
Another example (if time)

T



$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ b_1 \\ b_2 \\ \\
\end{array}
\begin{array}{c}
\begin{bmatrix}
0 & & & 1 & 1 & -1 & \\
 & & & & & & \\
 & & & & & & \\
-1 & & & 0 & 0 & 1 & \\
-1 & & & 0 & 0 & 1 & \\
1 & & & -1 & -1 & 0 & \\
 & & & & & &
\end{bmatrix} \\
\begin{matrix} 1 & 2 & 3 & 4 & 5 & b_1 & b_2 \end{matrix}
\end{array}
+
\begin{array}{c}
\begin{bmatrix}
0 & 1 & 1 & & & & -1 \\
-1 & 0 & 0 & & & & 1 \\
-1 & 0 & 0 & & & & 1 \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
1 & -1 & -1 & & & & 0
\end{bmatrix} \\
\begin{matrix} 1 & 2 & 3 & 4 & 5 & b_1 & b_2 \end{matrix}
\end{array}
$$

Q_T



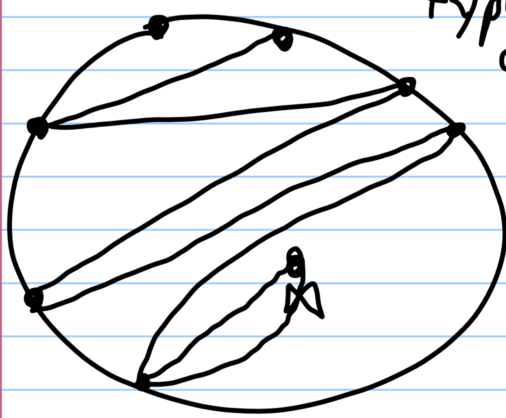
$$
= \begin{array}{c}
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix} \\
\begin{matrix} 1 & 2 & 3 & 4 & 5 & b_1 & b_2 \end{matrix}
\end{array}
\quad B_T
$$

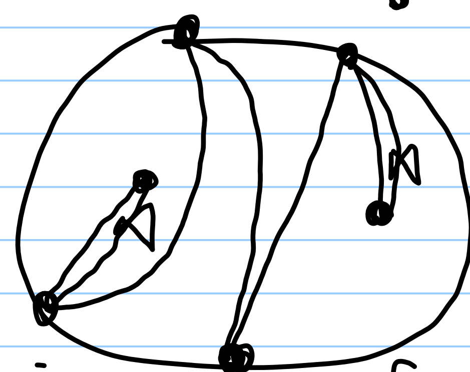
Lecture 2 Exercises and Handout

2-1: a) Consider the tagged arc complex for the once-punctured 3-gon. What previously seen simplicial complex is this?

b) Do you get a similar result for the tagged arc complex of a once-punctured n-gon? Why or Why not?

2-2: a) Consider the following tagged triangulations of punctured n-gons. What are the types of the corresponding cluster algebras?

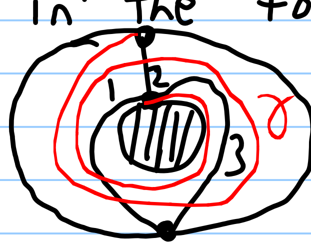




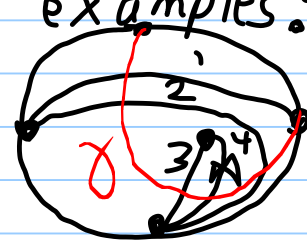
b) One of these is of finite type, the other is not. Which one is which and model an infinite sequence of cluster variables in the later case.

2-3: Compute the denominator vector of x_γ in the following examples:

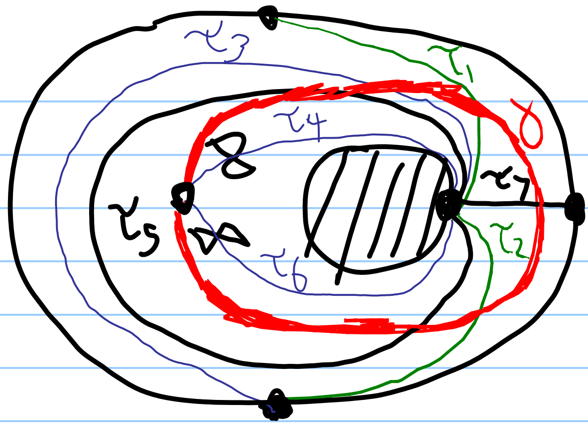
a)



b)

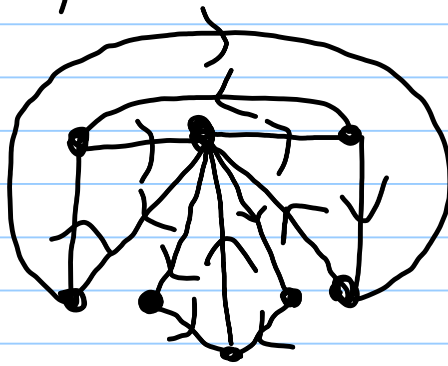


c)

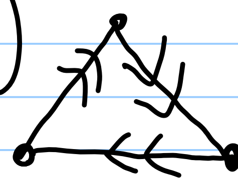


2-4: Are the following quivers associated to a cluster algebra from a surface? If so, try to draw such a surface.

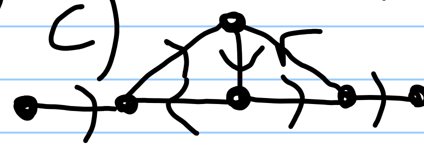
a)



b)



c)



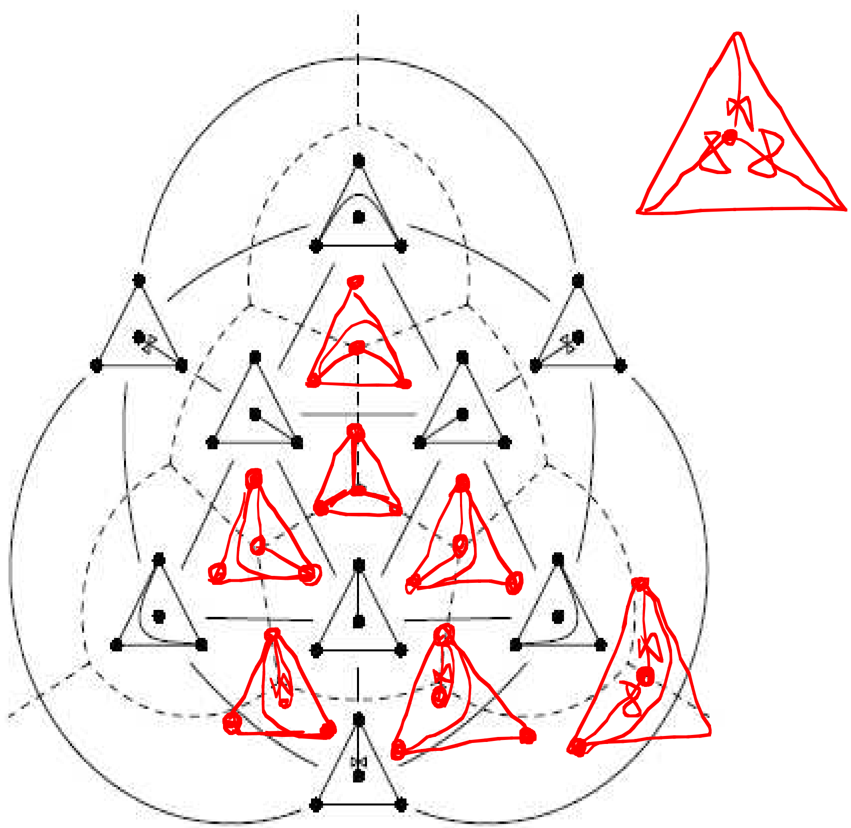


Figure 19. The tagged arc complex of a once-punctured triangle.