Lecture 3: Cluster variable formulas for tagged arcs (for cluster algebras from surfaces)

(1) Recall from last time that Fomin-Shapiro-Thurston construct a cluster algebra for any marked surface (S, M).

If S is of genus g,
 # boundary components of S is b
 $\# M \cap \partial S = c$, and
 $\# M \cap (S - \partial S) = p$ [# punctures],

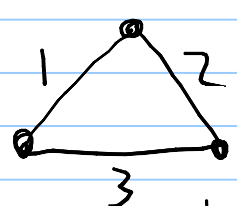
then $\mathcal{A}(S,M)$ is a cluster algebra of rank $n = 6g + 3b + 3p + c - 6$.

In particular, any maximal collection of non-intersecting, non-homotopic arcs or tagged arcs (triangulation or tagged triangulation) is of size n.

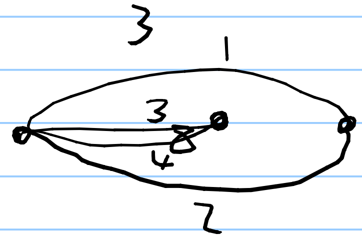
A tagged triangulation T $\longleftrightarrow$ cluster algebra seed

tagged arc $\tau_i \longleftrightarrow x_i$ in initial cluster

adding up contributions from <u>puzzle pieces</u> $\longleftrightarrow$ exchange matrix B.

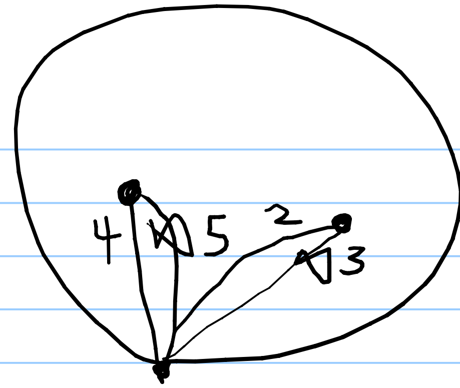


$\longleftrightarrow$ $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$



$\longleftrightarrow$ $\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

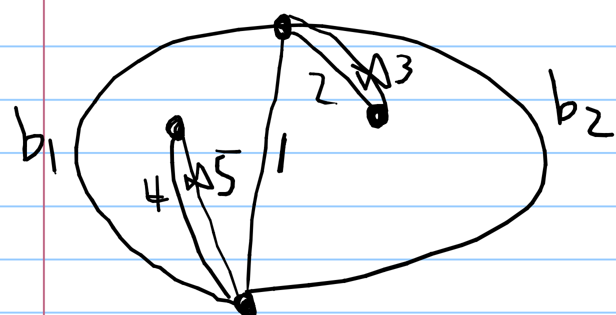
(2)



$\longleftrightarrow \begin{bmatrix} 0 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$

Example: $(\widetilde{D_5})$

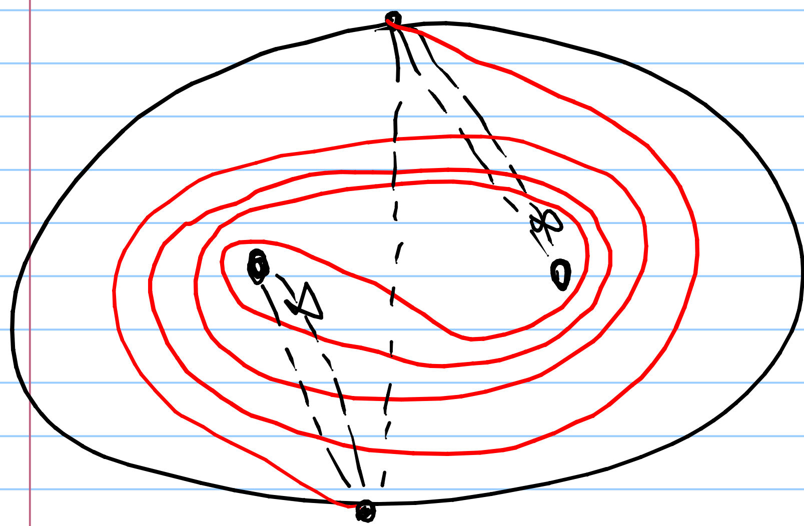
T



$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ b_1 \\ b_2 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & & & 1 & 1 & -1 & \\ & & & & & & \\ & & & & & & \\ -1 & & & 0 & 0 & 1 & \\ -1 & & & 0 & 0 & 1 & \\ 1 & & & -1 & -1 & 0 & \\ & & & & & & \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & b_1 & b_2 \end{matrix} \end{array} + \begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & & & & -1 \\ -1 & 0 & 0 & & & & 1 \\ -1 & 0 & 0 & & & & 1 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 1 & -1 & -1 & & & & 0 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & b_1 & b_2 \end{matrix} \end{array}$$

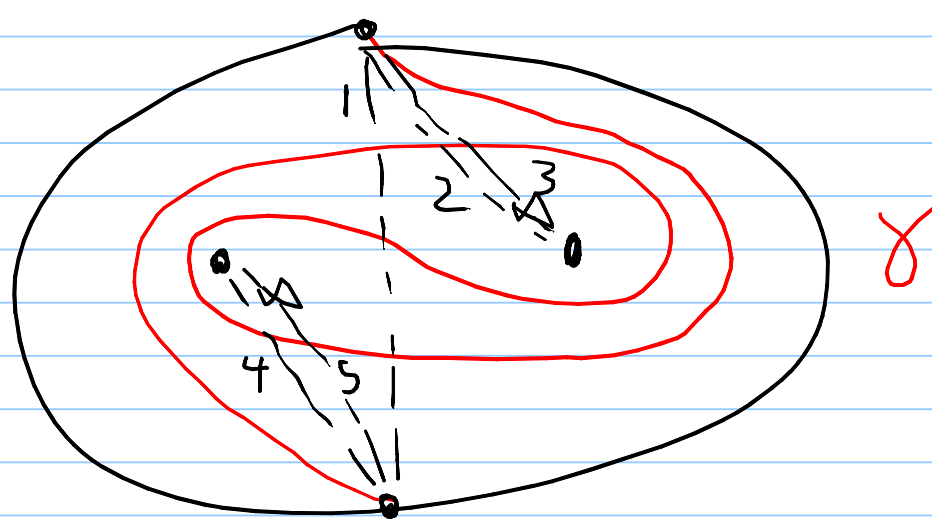
$$= \begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & b_1 & b_2 \end{matrix} \end{array} \qquad B_T$$

③ Consider the arc γ

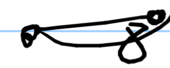


While it is a Theorem of FST that tagged arc complex is connected and thus we could find a sequence of flips so that γ was in a new triangulation, in practice it is difficult to compute cluster variable x_γ that way.

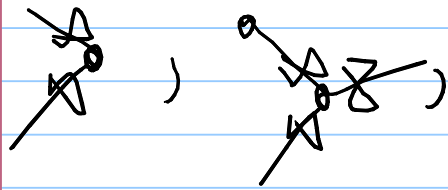
Instead we describe a comb. approach, although we will use a smaller example as an illustration:

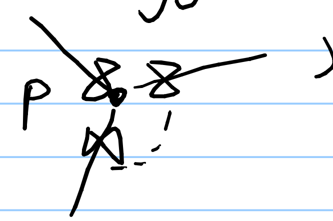


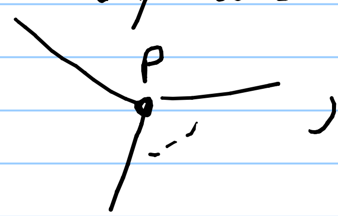
④ <u>Simplifying assumption</u>:

We will let T be a tagged triangulation where no tagged arcs have notches except for 

In other words, we will assume

 does not appear in T.

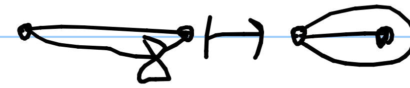
We make this assumption w.l.o.g. Because if B_T is the exchange matrix corresponding to a tagged triangulation containing  and T^P is the same tagged triangulation, it locally looks like

 then $B_{T^P} = B_T$.

<u>Pf:</u> Look at second and third puzzle pieces.

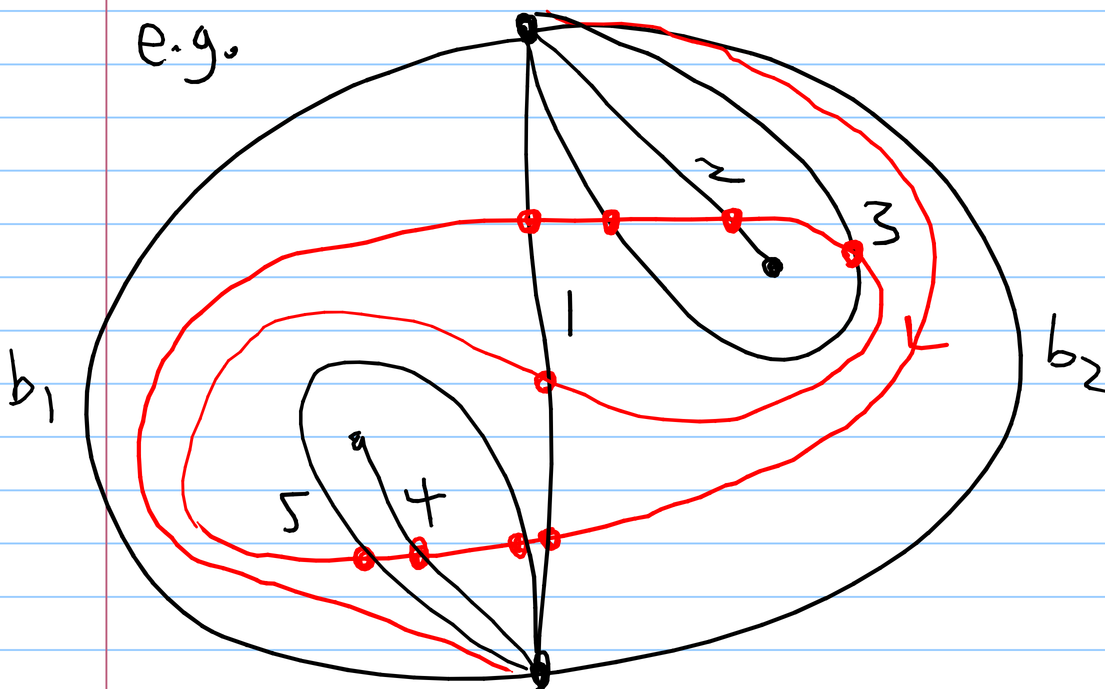
Thus if we want to understand all possible cluster algebras arising from a surface, it suffices to consider those tagged triangulations satisfying the above assumption.

<u>Secondly</u>, tagged triangulations of this form are in bijection with (ideal) triangulations.



5. Thus, providing combinatorial formulas for tagged arcs γ crossing an ideal triangulation T° is sufficient.

e.g.

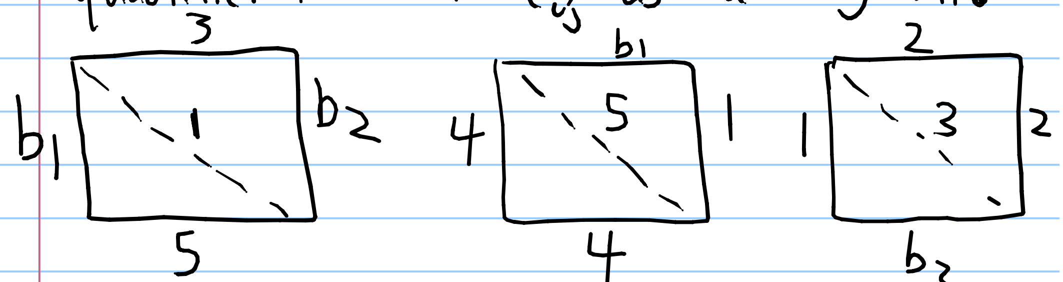


Record crossings of γ in order:

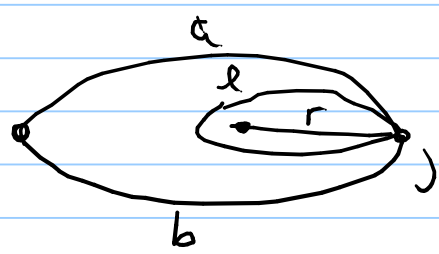
1, 5, 4, 5, 1, 3, 2, 3, 1.

Construction: For any untagged arc γ, we construct a snake graph G_γ as follows:

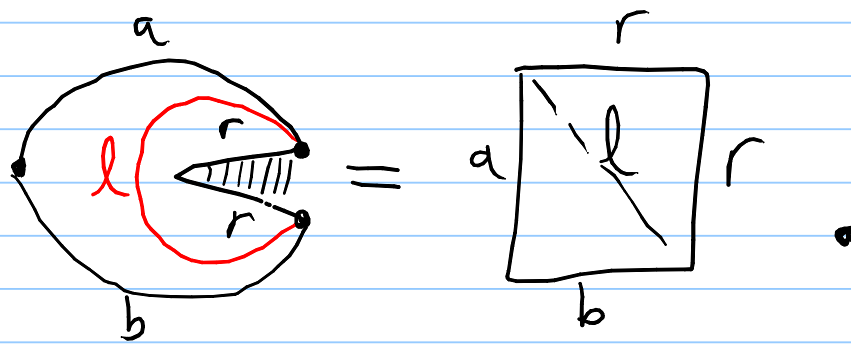
For each crossing γ w/ τ_{i_j} in the ideal triang. T°, construct tile G_{i_j} by forming quadilateral with τ_{i_j} as a diagonal:



(6) Notice that in self-folded quad,

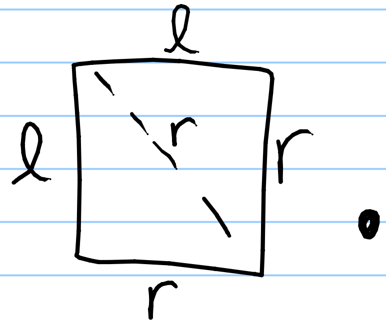


, tile for ℓ is



.

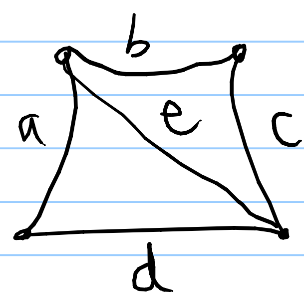
By convention, we define the tile for inscribed arc r to be



. (we lift to a cover to see this)

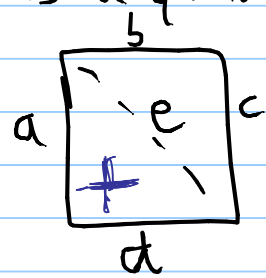
Def: Relative orientation

If

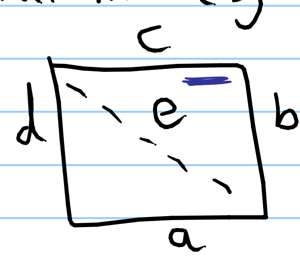


[possibly with edge or vertex identifications]

is a quadrilateral in (S, M), then



or

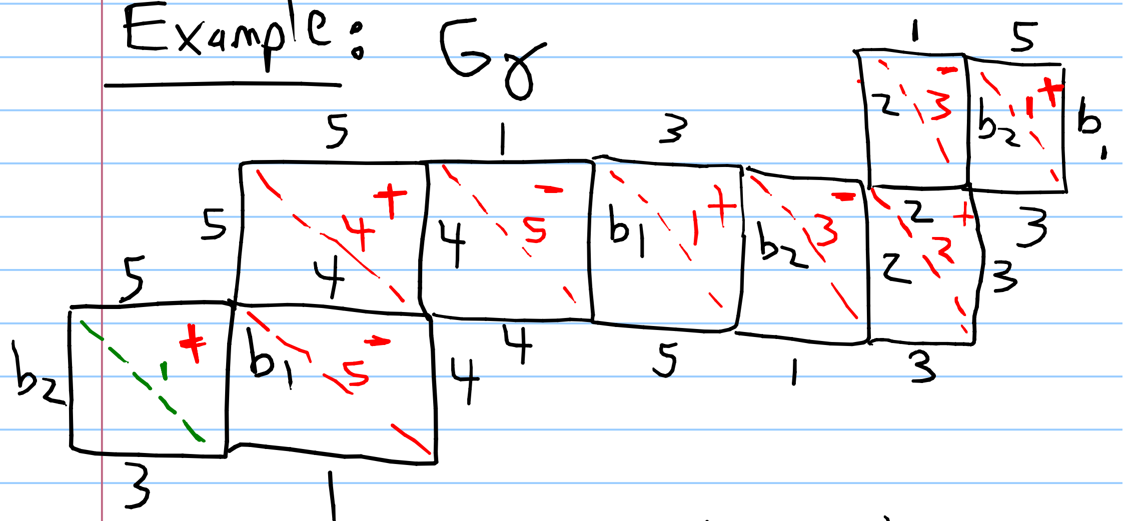


are both tiles for arc e.

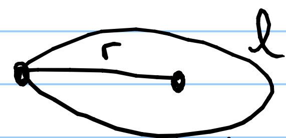
⑦ However we say the first tile has positive relative orientation while the second has negative orient.

We then construct G_γ by gluing tiles (corresp. to crossing points $p_{i_j} \in \gamma_{i_j}$) together so that consecutive tiles have opposite orientations.

Example: G_γ

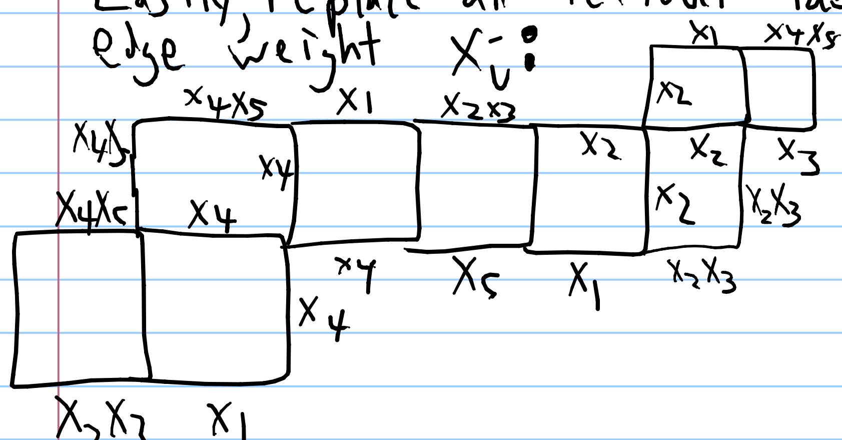


We then obtain $\overline{G_\gamma}$ by replacing all boundaries b_i by edge-weight 1, erasing all diagonals, and replacing any label corresp. to ℓ in



by $x_r x_\ell$.

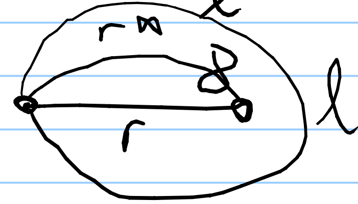
Lastly, replace all leftover labels i by edge weight x_i:



8 Def: We let $\mathrm{cross}(\gamma, T)$ be the **crossing monomial** of γ w.r.t. T°

$$\mathrm{cross}(\gamma, T) = \prod_{\substack{\tau_{ij} \text{ crossed} \\ \text{by } \gamma}} x_{\tau_{ij}},$$

where as before x_ℓ is replaced by $x_r x_{r^{\bowtie}}$.



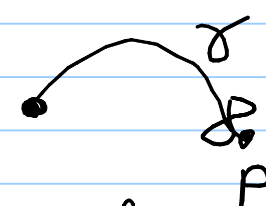
Thm [M– Schiffler–Williams]

If G_γ, $\mathrm{cross}(\gamma, T)$ etc. as above where γ is an unnotched arc, then cluster variable

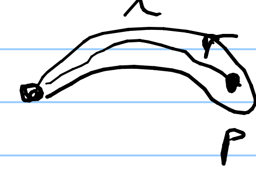
$$x_\gamma = \frac{\sum_{P \text{ perf. matching of } G_\gamma} x(P)}{\mathrm{cross}(\gamma, T)}.$$

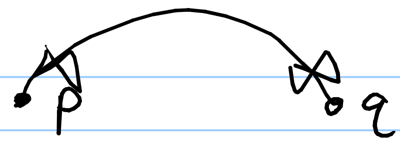
If γ has one notch or two notches we instead use

$$x_\gamma = \frac{x_\ell}{x_r} \quad \text{if}$$

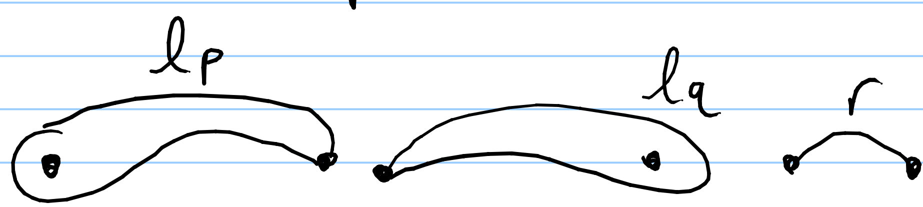


We treat ℓ as if it is any other unnotched arc.



(9) if $\gamma =$  ,

$$X_\gamma = \frac{X_{\ell_p} X_{\ell_q}}{X_r^{\,3}}$$



Remark: There are also alternative combinatorial expressions for X_γ when $\gamma =$ or $\gamma =$ that show that these quotients are positive expansions.

γ - symmetric & pairs of γ-compatible matchings.

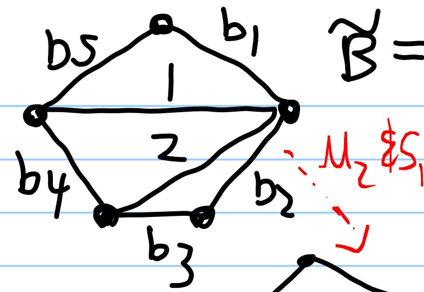
Cor: Proves the positivity conj of Fomin-Zelevinsky for cluster algebras from surfaces.

We also can get formulas for cluster variables in cluster algebras with principal coefficients.

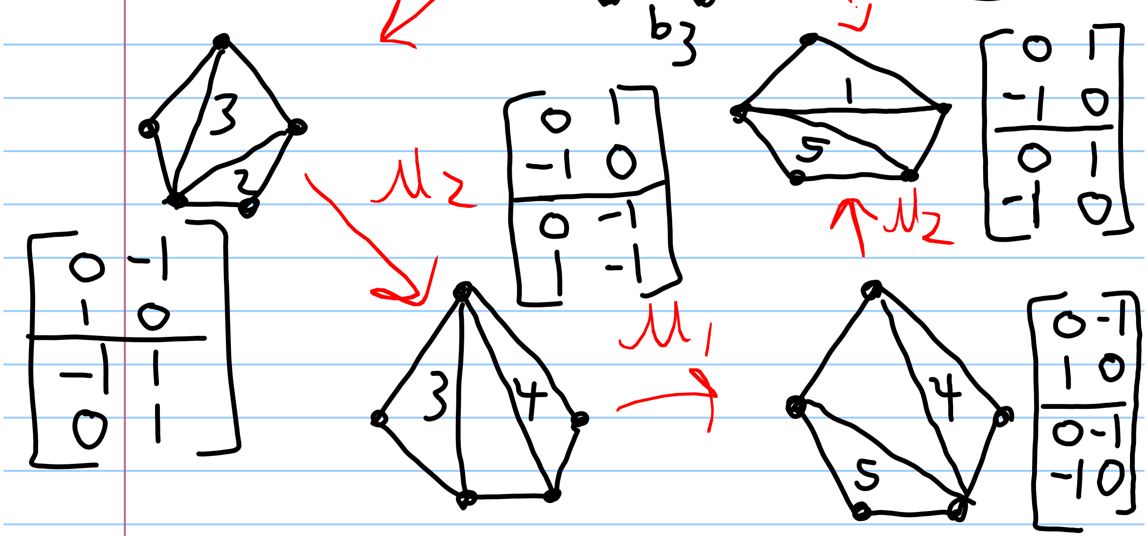
See [MSW] or slides from my webpage for more details.

Involves heights of perfect matchings.

10 Example



$$\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{bmatrix}$$



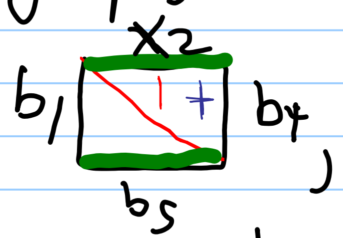
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \hline -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline 0 & -1 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \hline 0 & -1 \\ -1 & 0 \end{bmatrix}$$

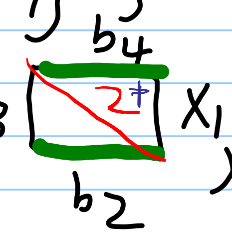
$$x_3 = \frac{y_1 + x_2}{x_1}, \quad x_4 = \frac{y_1 y_2 x_1 + y_1 + x_2}{x_1 x_2},$$

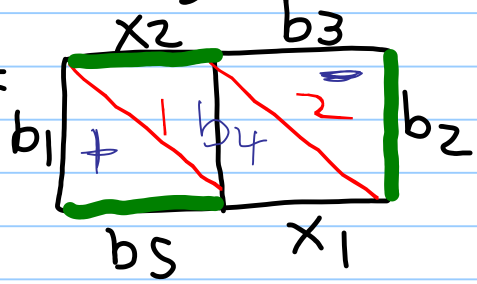
$$x_5 = \frac{y_2 x_1 + 1}{x_2} = \frac{y_2 + x_4}{x_3}$$

Exercise: verify this equality.

Make graphs for τ_3, τ_4, τ_5:

$G_3 =$ 

$G_5 =$ 

$G_4 =$ 

Minimal Matchings in **Green**

Correspond to terms w/o y_i in x_γ.

We "twist" tiles to get other matchings/terms.

Question: How large a class of cluster algebras is the family of cluster algebras from surfaces?

Answer: Recall that any exchange matrix coming from a cl. alg. of a surface has entries bounded in $\{-2,-1,0,1,2\}$.

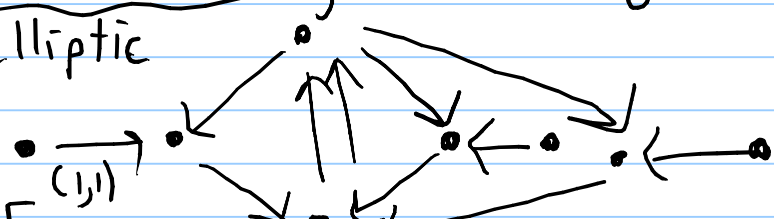
$\Rightarrow$ Each such cluster algebra is of finite mutation type

In fact, by Thm of Felikson-Shapiro-Tumarkin, any skew-symmetric cluster algebra of finite mutation type is

i) of rank 2,
ii) comes from a surface, or
iii) is mutation equivalent to one of eleven exceptional cases:

$\underbrace{E_6, E_7, E_8}_{\text{finite type}}$, $\underbrace{\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8}_{\text{affine}}$,

$\underbrace{E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}}_{\text{elliptic}}$, or X_6 or X_7.



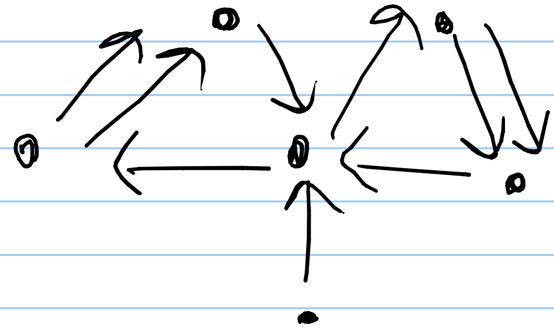
$E_6^{(1,1)}$

$E_7^{(1,1)}$

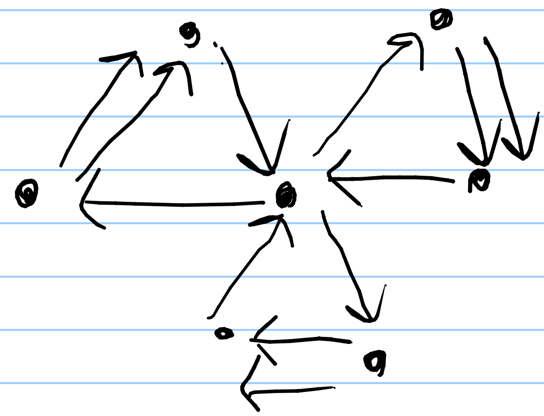
12

$E_8^{(1,1)}$

X_6



X_7



X_6 & X_7 found by Derksen-Owen in an REU.

It is shown that these 11 exceptional cases do not come from surfaces by block decomp.'s

Idea: A triangulation must be formed by gluing puzzle pieces together (and possibly deleting boundary arcs from quiver/exchange matrix).

These 11 cannot be decomposed this way.

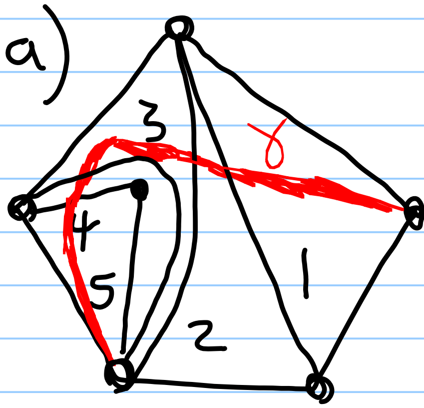
Recent sequel also classifies finite mutation non-skew-symmetric cases.

Lecture 3 Exercises

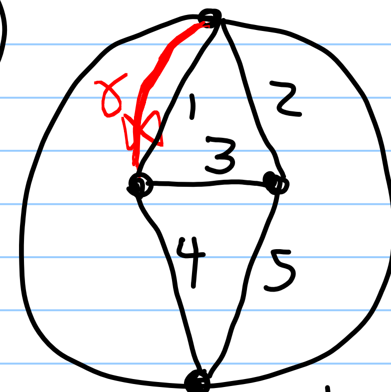
3-1 Verify pentagonal example on page 10. Compute $\widetilde{B}$ matrices, x_4 and x_5. Verify the weight/height formula for this example.

3-2 Compute x_γ (with principal coefficients) for the following examples

a)

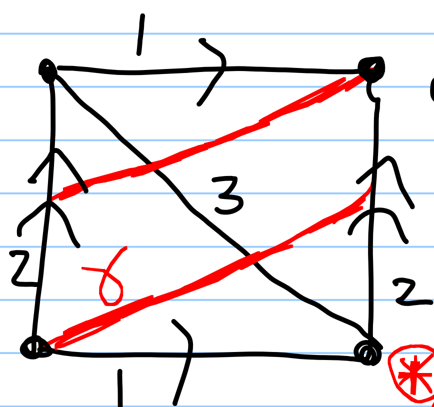


b)



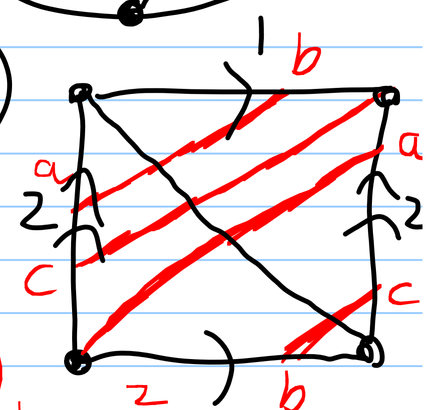
In once-punctured torus

c)



slope $\frac{1}{2}$

d)

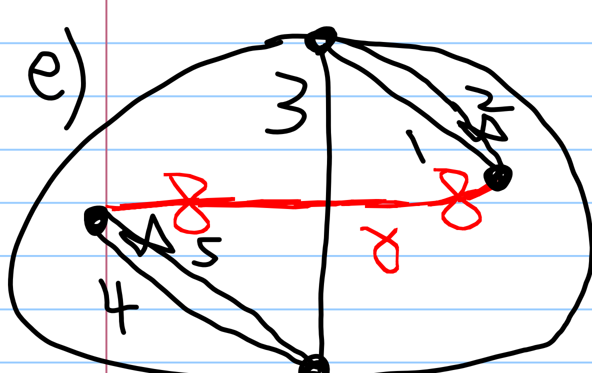


slope $\frac{2}{3}$

(*) At least draw graph and compute a few terms.

How many terms are in this example?

e)



3-3 The triangulation T in 3-2 (a) corresponds to a cluster algebra of finite type.

a) Write down Q_T and identify its type.

b) Write down a corresponding Coxeter element $c = s_{i_1} s_{i_2} \cdots s_{i_S}$

c) Apply mutation $\mu_{i_1}, \ldots, \mu_{i_S}$ to T.

Describe how c acts on T in this case.

d) Try this with another orientation of Q_T and the corresponding triangulation.

e) Describe how this procedure could be used to obtain (efficiently) all cluster variables in $\mathcal{A}(Q_T)$. (assuming Q_T is acyclic of finite type)

f) Does this remind you of any construction from Ralf Schiffler's lectures?