Note Title 4/25/2011

Lecture 4: Laminations, general coefficient systems, and Hyperbolic geometry

(1) Today, we discuss how to construct cluster algebras from surfaces with general coefficients.

i.e., we want $(m+n)\times n$ extended matrix

$$\widetilde{B} = \begin{array}{c} n \\ m \end{array}\left[\begin{array}{c} B \\ \hline C \end{array}\right] \quad \text{where } C \text{ is general.}$$
$$\qquad\qquad n$$

References: Fomin–Thurston, Cluster Algebras from Surfaces II: Lambda Lengths

For this, we use multi-laminations.

Def: We define an integral unbounded measured lamination L on a marked surface (S, M) as a finite collection of pairwise non-intersecting curves, each of which has no self-intersections, modulo isotopy (relative to M) such that:

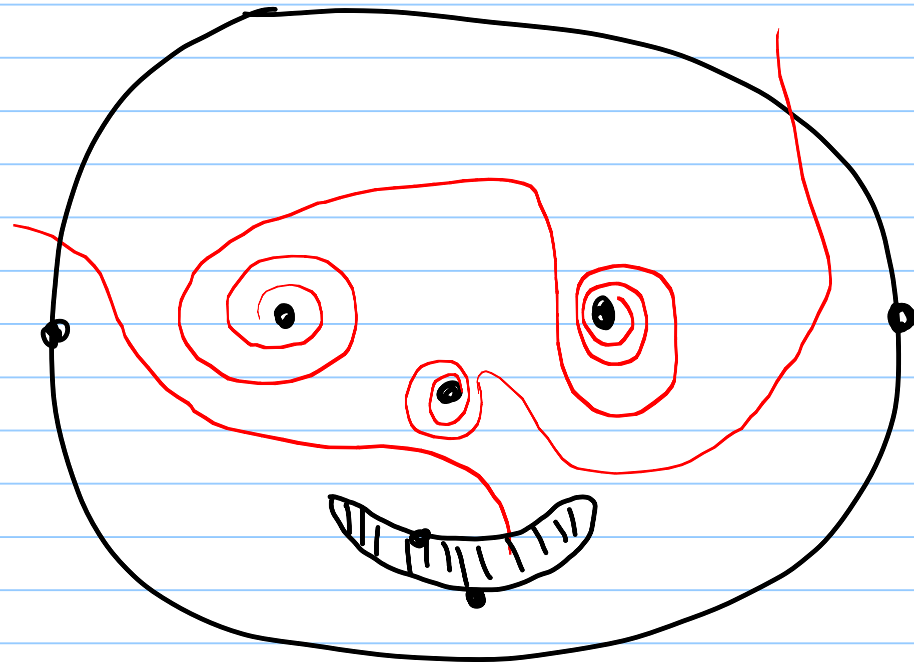
Each curve must be either:

i) a closed curve
ii) a curve connecting two unmarked points on $\partial S - M$
iii) a curve with one endpoint on $\partial S - M$ and one endpoint spiraling into a puncture (clockwise OR counter-clockwise), or
iv) a curve with both endpoints spiralling into a puncture.

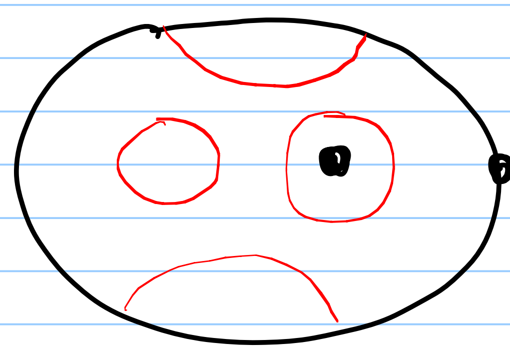
② The following curves are disallowed:

i) a closed curve bounding an unpunctured or once punctured disc,

ii) a curve with two endpoints on ∂S which is isotopic to a boundary arc containg 0 or 1 marked point



Disallowed



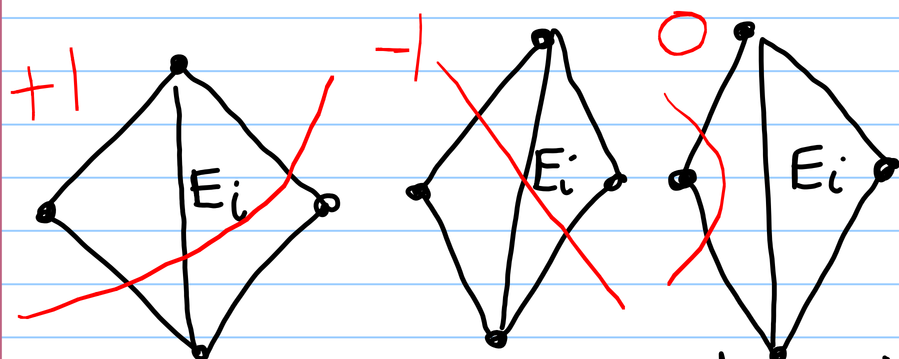
∂S

③ **Def:**

(Tropical)
We assign a <u>Shear Coordinate</u> $\in \mathbb{Z}$ to each arc $E_i \in T$ of a triangulation w.r.t. a choice of lamination:

$$b_{E_i}(T, L) \quad \text{for each } E_i \in T.$$

We look at a quadrilateral inscribing E_i (in ideal triangulation T) for each curve of lamination L cutting through the quadrilateral, we calculate a contribution to the shear coordinate. Adding them all up gives the appropriate contribution.



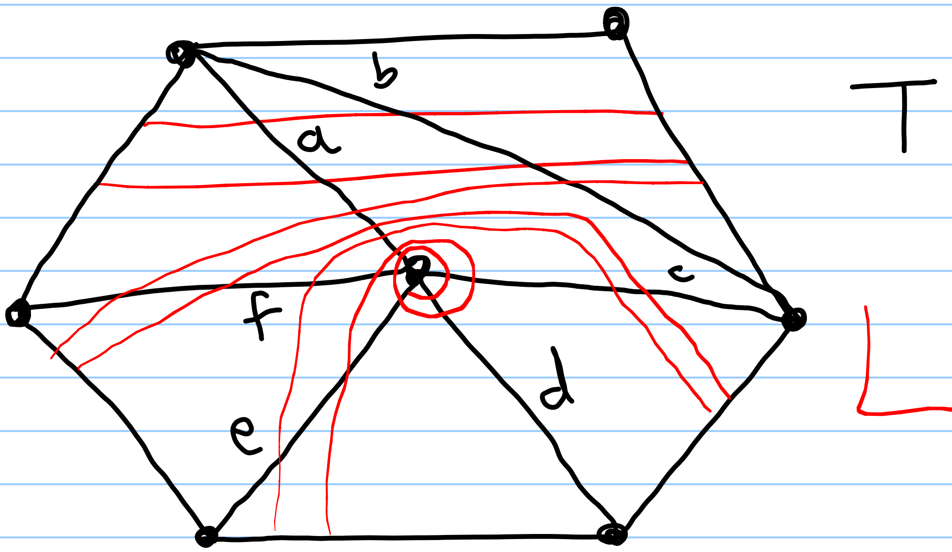
\$ Zilch   and all other crossings of adjacent sides give zero too.

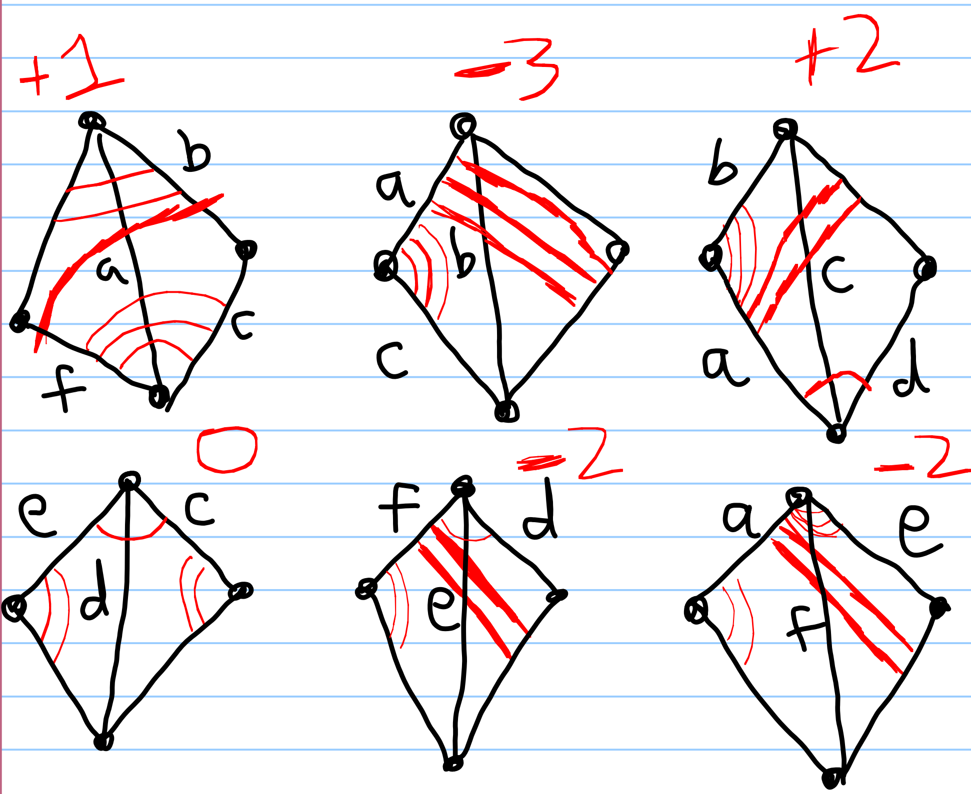
Def: A <u>multi-lamination</u> $(L_1, \ldots, L_m)$ is a collection of m laminations.

We make $\widetilde{B}$ by $\left[\frac{B}{C}\right]$, each row of

$$C = \left[b_{E_1}(T, L_j), b_{E_2}(T, L_j), \ldots, b_{E_n}(T, L_j)\right]$$

(4) Example (Fig 3 of Fomin-Thurston)





$$b_a(T,L)=1,\quad b_b(T,L)=-3,\ \ldots$$

Let us now flip $b \mapsto b'$ to get triangulation T' with L fixed:

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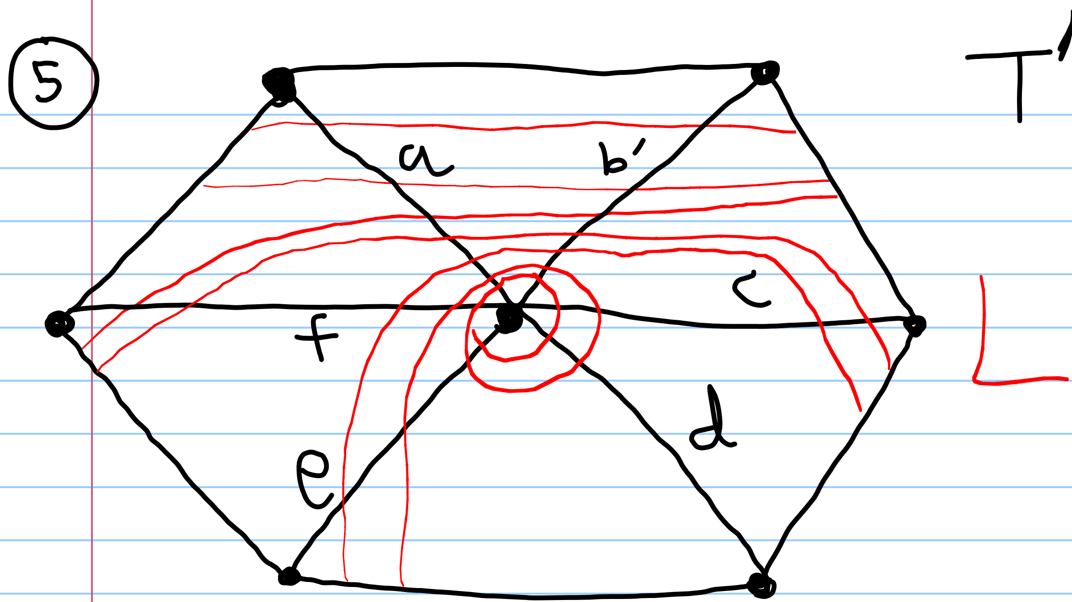
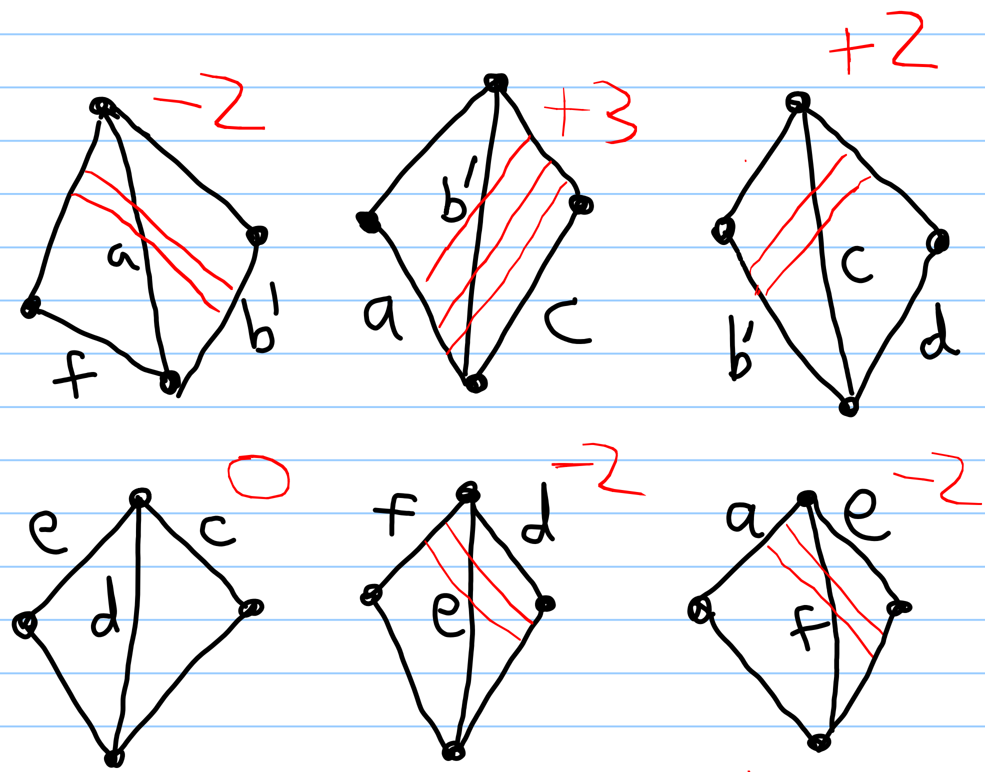




<u>Notice:</u> $\overset{+3}{b_{b'}(T',L)} \overset{=}{=} \overset{-(-3)}{-b_b(T,L)}$

$$\overset{-2}{b_a(T',L)} \overset{=}{=} \overset{1}{b_a(T,L)} \overset{-}{-} \overset{\max(-(-3),0)}{\max(-b_b(T,L),0)}$$

$$\overset{+2}{b_c(T',L)} \overset{=}{=} \overset{2}{b_c(T,L)} \overset{+}{+} \overset{\max(-3,0)}{\max(b_b(T,L),0)}$$

$$b_\tau(T',L) = b_\tau(T,L) \quad \text{o.w.}$$

⑥ $\widetilde{B}(T) = \begin{array}{c|c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \hline L \end{array} & \left[\begin{array}{cccccc} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & -3 & 2 & 0 & -2 & -2 \end{array}\right] \end{array}$

$\Big\downarrow \mu_b$

$\widetilde{B}(T') = \begin{array}{c|c} & \begin{array}{cccccc} a & b' & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b' \\ c \\ d \\ e \\ f \\ \hline L \end{array} & \left[\begin{array}{cccccc} 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ \hline -2 & 3 & 2 & 0 & -2 & -2 \end{array}\right] \end{array}$

observe:

Agrees with how tropical shear coordinates change under mutating T to T'.

Thm (Fomin-Thurston)

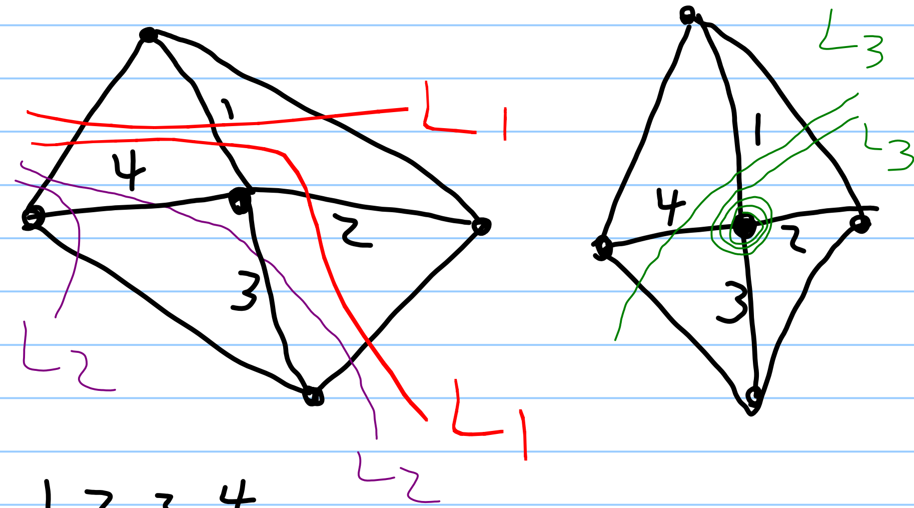
[Based on work by Fock-Goncharov and W. Thurston]

If we define $(m+n)\times n$ $\widetilde{B}$-matrix by ideal triangulation $T=(\tau_1,\dots,\tau_n)$ and multi-lamination $(L_1,\dots,L_m)$, and we mutate $T \xrightarrow[\mu_k]{} T'$,

then $\mu_k(\widetilde{B}(T,L)) = \widetilde{B}(T',L)$.

Rem: Also an extension to tagged triangulations, see Fomin-Thurston.

⑦ Another Example:



$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & -1 & 0 & 1 \\
2 & 1 & 0 & -1 & 0 \\
3 & 0 & 1 & 0 & -1 \\
4 & -1 & 0 & 1 & 0 \\
\hline
L_1 & -1 & 1 & 0 & 0 \\
L_2 & 0 & 0 & -1 & 1 \\
L_3 & 2 & 0 & 0 & -1
\end{array}$$

Recall, the Theorem

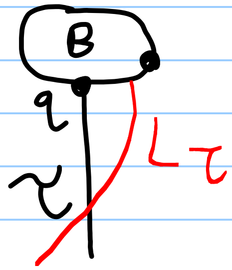
any row in $\mathbb{Z}^n$ can be achieved by the choice of some lamination.

A special case: Principal coeffs

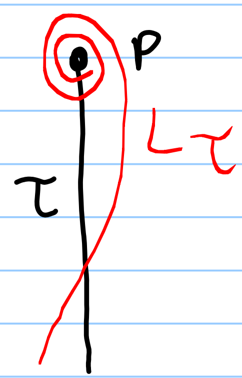
$$\left[\begin{array}{c} B \\ \hline I_n \end{array}\right]$$

For any tagged arc τ in a triangulation, we define the <u>elementary lamination</u> L_τ associated to τ to be the single curve obtained from τ by changing τ's endpoints in one of the following 3 ways:

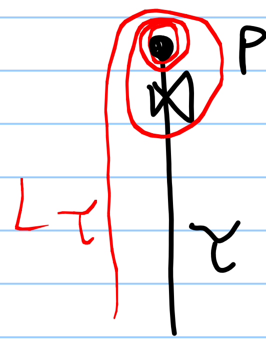
⑧ If τ's endpoint q lies on a boundary B, we move q slightly "counter-clockwise" around B so that it misses marked points:



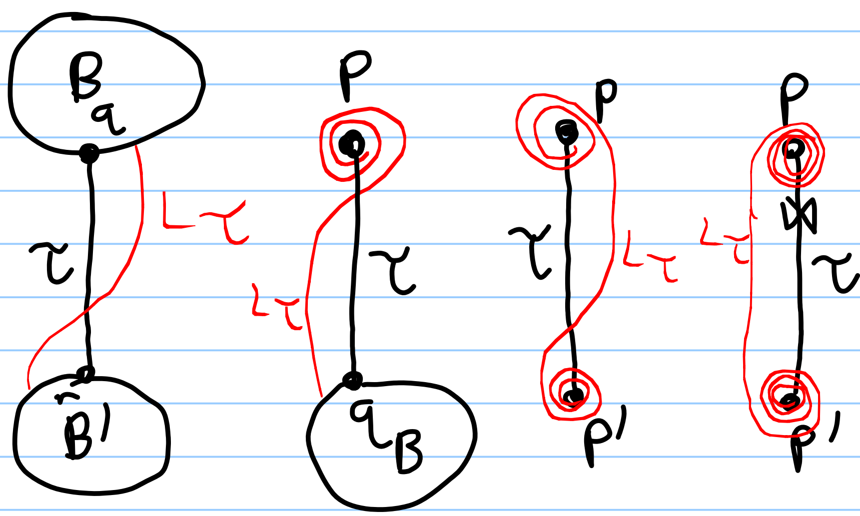
If τ's endpoint p lies at a puncture & τ is tagged plain, we rotate L_τ counter-clockwise around p:



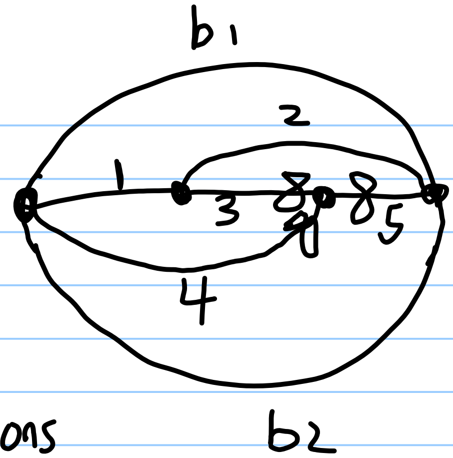
Finally, if τ is notched at p, then we rotate L_τ clockwise around p instead:



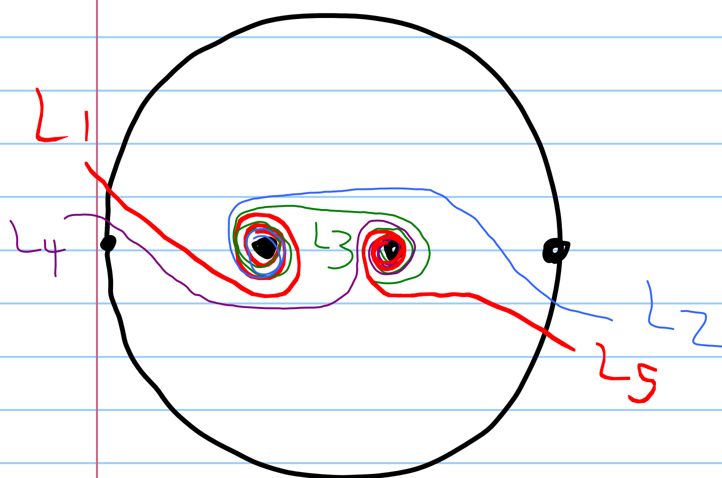
Putting these rules together:

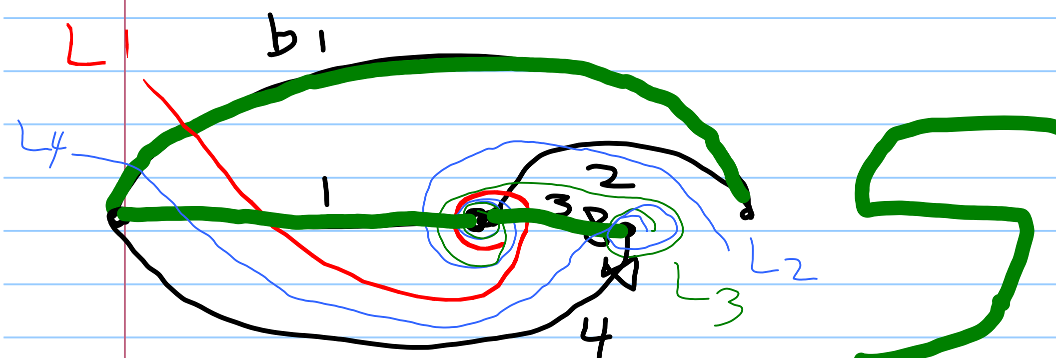


9 Example:

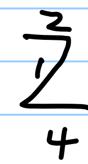


gives laminations





No lamination crosses τ_4 so crossing quadrilateral in negative formation impossible.



only L_1 crosses b_1, τ_1, and τ_3

so column corresponding to τ_1 is indeed

$$\begin{array}{c} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The other 4 columns give the 5x5 identity matrix on the bottom.

(10) We now switch gears and talk about Teichmüller theory and how to use hyperbolic geometry to interpret cluster variables and coefficients another way.

Given a marked surface (S,M) $\mathcal{T}(S,M)$ <u>Teichmüller Space</u> is defined to be the space of metrics on (S,M) satisfying the following properties:

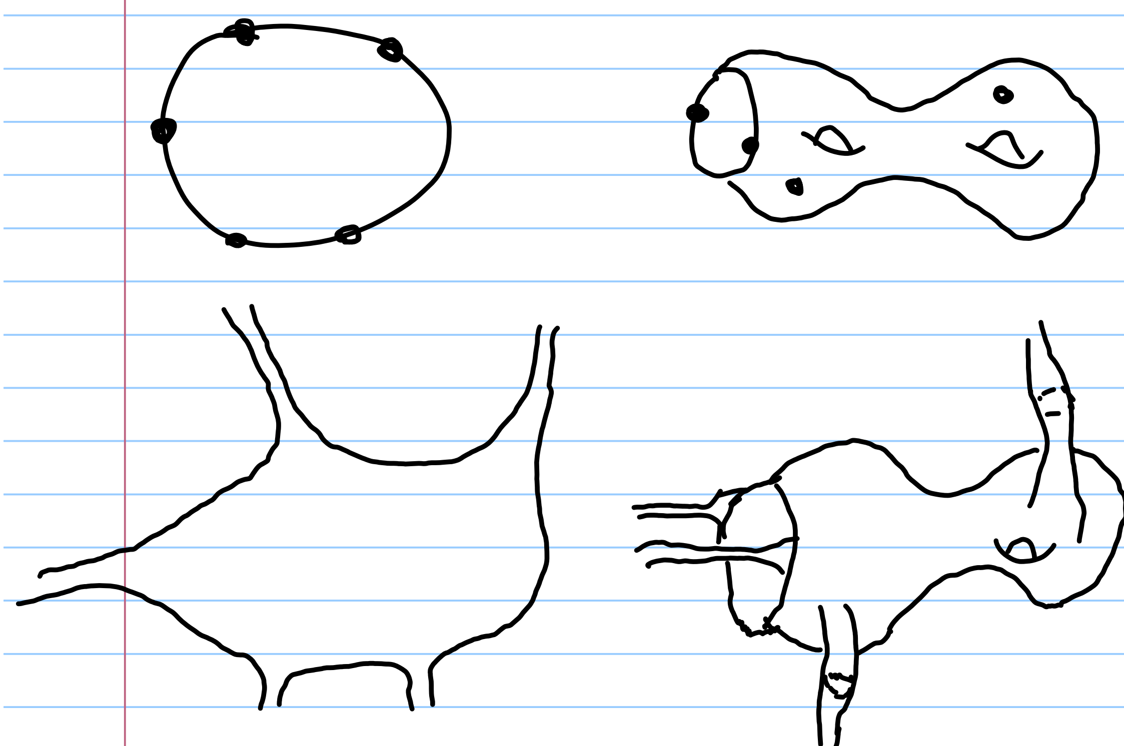
- <u>hyperbolic</u> (constant curvature -1)
- has <u>geodesic boundary</u> on boundary of S

  i.e. nbhd of $p \in \partial S$ looks like nbhd of one side of a geodesic in $\mathbb{H}^2$.

- has <u>cusps</u> at marked points M.

  (go off to ∞ while area stays bounded. (S,M) is complete.)

<u>Examples:</u>



11 These metrics are considered up to diffeomorphisms homotopic to identity.

Facts: ① $\mathcal{T}(S,M)$ is a $\mathbb{R}$-manifold of dimension $6g-6+2p+3b+c$

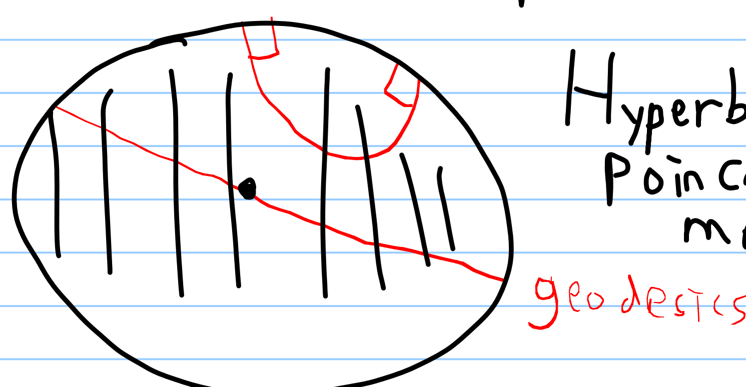
② $\mathcal{T}(S,M)$ is contractible.

Coordinates on $\mathcal{T}(S,M)$ by Shear coordinates (tropical versions defined above)

Thm (W. Thurston) Given (S,M), and a triangulation $T=\{\tau_1,\dots,\tau_n\}$, then

$$L \to \left(b_\tau(T,L)\right)_{\tau\in T}$$ is a bijection between the set of (integral unbounded measured) laminations and $\mathbb{Z}^n$.

Need hyperbolic geometry to define (non-tropicalized) shear coords.



Hyperbolic $\mathbb{D}$
Poincaré disk model

metric $= ds$ satisfying $ds^2 = \dfrac{dx^2+dy^2}{(1-r^2)^2}$

$r=\sqrt{x^2+y^2}$

Points on Boundary are infinitely far away from the center.

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Example: Distance from $(1,0)$ to $(0,0)$ is
$$\int_0^1 \frac{dx}{1-x^2} = \operatorname{arctanh}(x)\Big|_0^1 = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\Big|_0^1 \to \infty.$$

Related Model (Upper Halfplane $\mathbb{H}^2$)

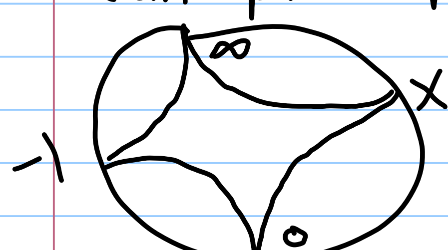
Metric $ds^2 = \dfrac{dx^2+dy^2}{y^2}$

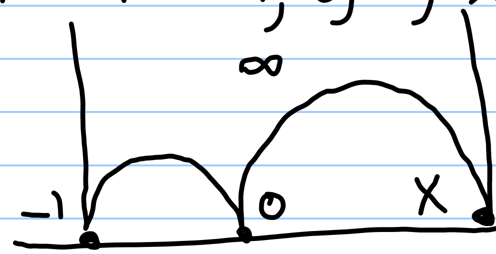
points on real line are infinitely far away.

Symmetries by $PSL_2(\mathbb{R})$ acting by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \quad \text{(Fractional linear transf.)}$$

Action of $PSL_2(\mathbb{R})$ fixes 3 pts on the boundary and allows freedom to choose others.

e.g. For quadrilateral on $\mathbb{D}$ or $\mathbb{H}^2$ can place points at $-1, 0, \infty, X$



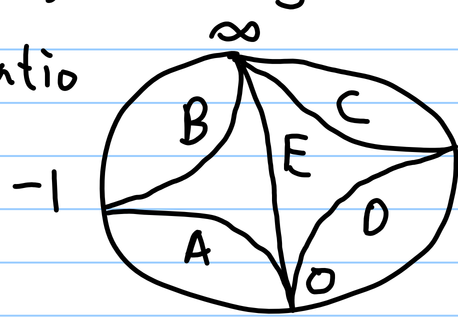


(13) Remaining corner of a quadrilateral is free $x \in \mathbb{R}_{>0}$ but is an invariant of the quadrilateral, cross-ratio
$\parallel$
shear coordinates.

Given a hyperbolic structure $\Sigma \in \mathcal{T}(S,M)$ and a triangulation $T = \{E_i\}_{i=1}^n$, the shear coord.

$\tau_\Sigma(E;T)$ of edge $E \in T$ is

the cross-ratio



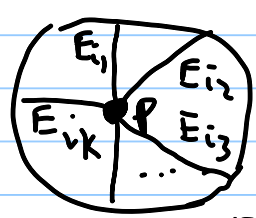
$\tau_\Sigma(E;T)$

Theorem: The map $\mathcal{T}(S,M) \longrightarrow \mathbb{R}^n$

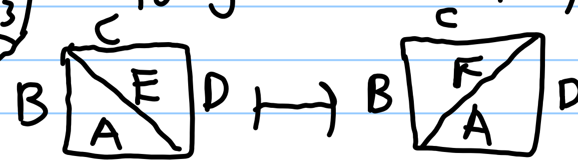
$$\Sigma \longmapsto \left(\tau_\Sigma(E_i;T)\right)_{i=1}^n$$

is a homeomorphism onto the subset of $\mathbb{R}^n$ where for each puncture p, and incident arcs $E_{i_1}, \dots, E_{i_k}$, we have

$$\prod_{j=1}^{k} \tau_\Sigma(E_{i_j};T) = 1.$$



When we flip quads to get from $T \to T'$,



(14)

shear coordinates change in a predictable way:

$$\tau_\Sigma(F;T') = \tau_\Sigma(E;T)^{-1}$$

$$\tau_\Sigma(A;T') = \tau_\Sigma(A;T)\left(1+\tau_\Sigma(E;T)^{-1}\right)^{-1}$$

$$\tau_\Sigma(B;T') = \tau_\Sigma(B;T)\left(1+\tau_\Sigma(E;T)\right)$$

$$\tau_\Sigma(C;T') = \tau_\Sigma(C;T)\left(1+\tau_\Sigma(E;T)^{-1}\right)^{-1}$$

$$\tau_\Sigma(D;T') = \tau_\Sigma(D;T)\left(1+\tau_\Sigma(E;T)\right).$$

As we will see shortly, these mutations also appear in cluster alg's!

Recall, the <u>original definition</u>

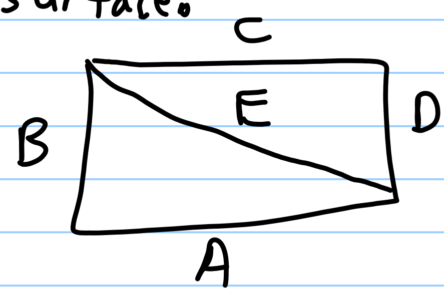
$(\mathbb{X}, \mathbb{Y}, B)$ $\mathbb{X} = \{x_1, \dots, x_n\}$ initial cluster,

$\mathbb{Y} = \{y_1, \dots, y_n\}$ initial coeffs, $B =$ $n \times n$ exchange matrix

$$x_k' x_k = \frac{y_k \prod\limits_{b_{ik}>0} x_i^{b_{ik}} + \prod\limits_{b_{ik}<0} x_i^{-b_{ik}}}{(y_k \oplus 1)}.$$

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j\left(y_k/(y_k \oplus 1)\right)^{b_{kj}} & \text{if } j \neq k, \ b_{kj} > 0, \\ y_j\,(y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \ \& \ b_{kj} \leq 0. \end{cases}$$

15. Let us focus on case of tropical semifield and cluster algebra from a surface.

Then 

gives rise to

portion of exchange matrix:

$$\begin{array}{c|c} & \begin{matrix} E & A & B & C & D \end{matrix} \\ \begin{matrix} E \\ A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & & & \\ 1 & & 0 & & * \\ -1 & * & & 0 & \\ 1 & & & & 0 \end{bmatrix} \end{array}$$

Thus, if we mutate μ_E, $x_{E'} = x_F$ and

$y_F = y_E' = y_E^{-1}$,

$y_A' = y_A\left(\frac{y_E}{y_E \oplus 1}\right) = y_A \frac{y_E}{y_E}\left(\frac{1}{1 \oplus 1/y_E}\right)$

$y_B' = y_B(y_E \oplus 1)$

$y_C' = y_C\left(\frac{y_E}{y_E \oplus 1}\right) = y_C \frac{y_E}{y_E}\left(\frac{1}{1 \oplus 1/y_E}\right)$

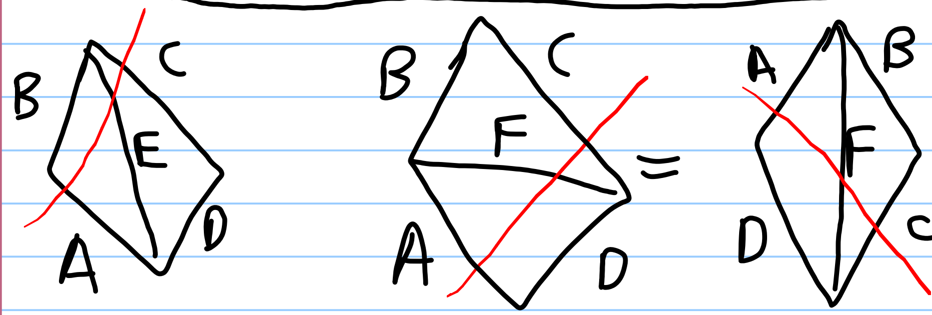
$y_D' = y_D(y_E \oplus 1)$

Moral: Coeff Dynamics behave like Tropical shear coordinate dynamics.

Exactly the Tropical Shear Coordinates by (T, L)'s defined above.

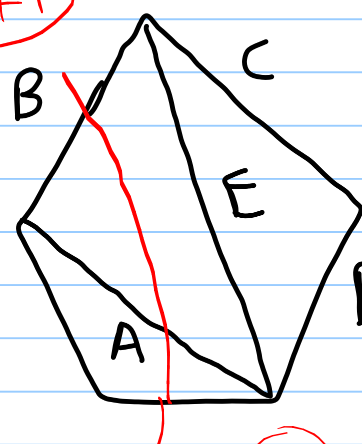
1b

More examples comparing how $b_\tau(T,L) \to b_\tau(T',L)$ changes



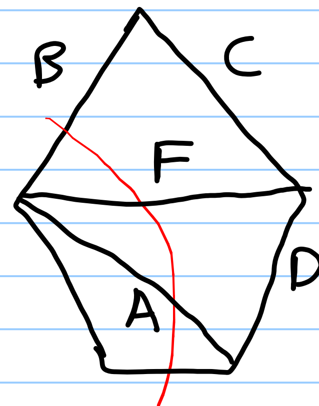
sign reversal

+1



+1 still

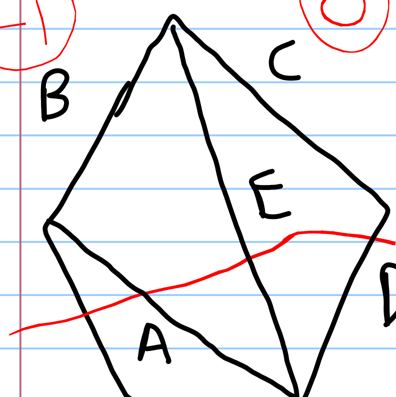
versus



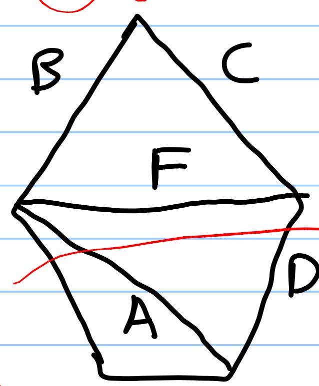
0 w.r.t. E

-1 still

-1



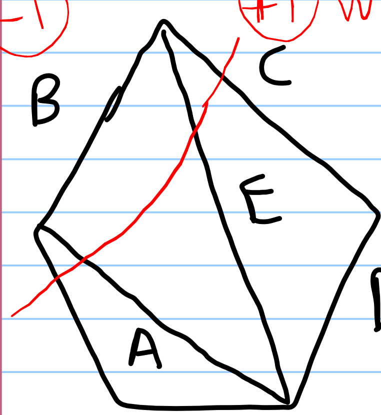
versus



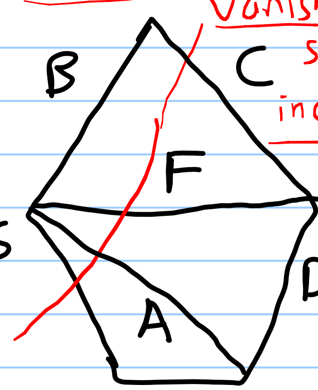
-1

+1 w.r.t. E

0 contribution vanishes so increased



versus



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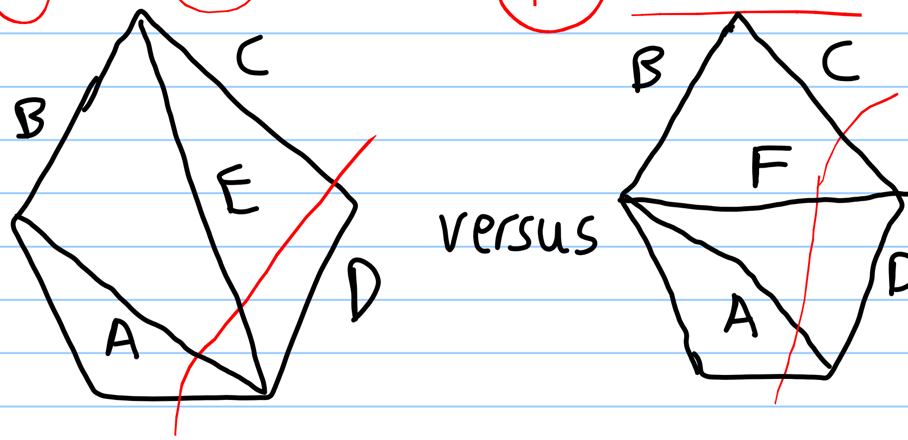
$\textcircled{0}$ $\textcircled{+1}$ w.r.t. E $\textcircled{+1}$ increased



$$b_A(T', L) = b_A(T, L) + \underbrace{b_E(T, L)}_{\text{if } > 0}$$

$$= b_A(T, L) + \max(b_E(T, L), 0).$$

Moral: if we let $\oplus = \max(-,-)$, coefficient dynamics agree with computing trop. shear coordinates with laminations.

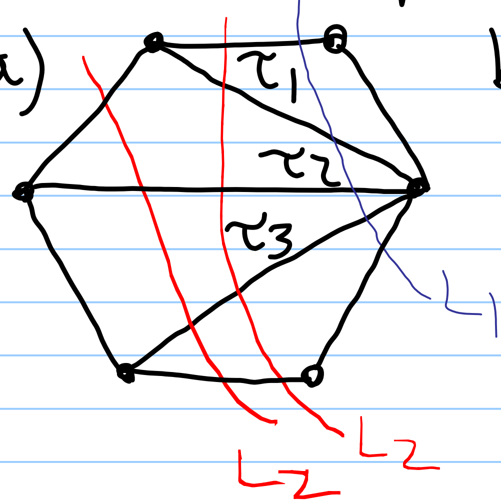
Next time: More on Hyperbolic geometry and Lambda Lengths.

Will also discuss related 2x2 matrix formulas (time permitting).

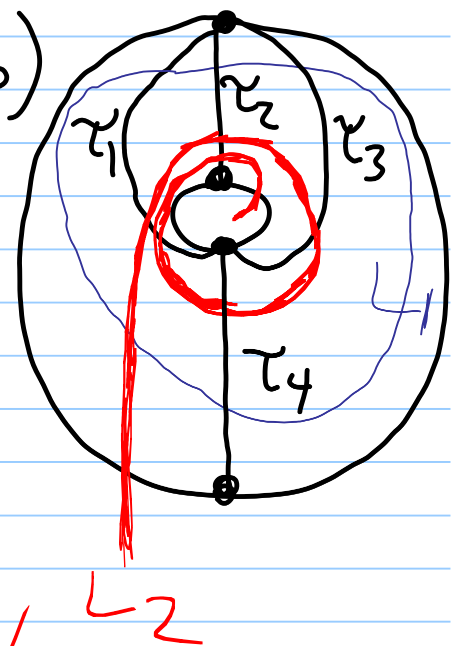
Lecture 4 Exercises

4–1) Compute $\widetilde{B}$ for some of the following triangulation/multi-lamination pairs:

a)

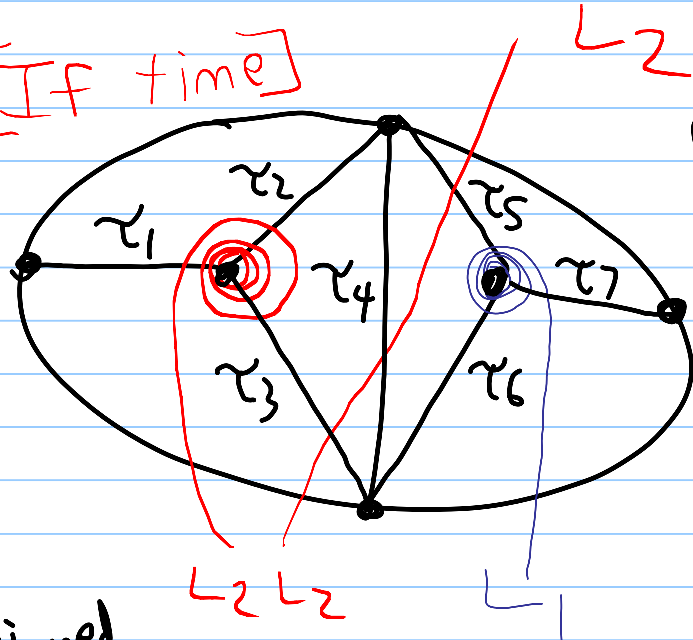


b)



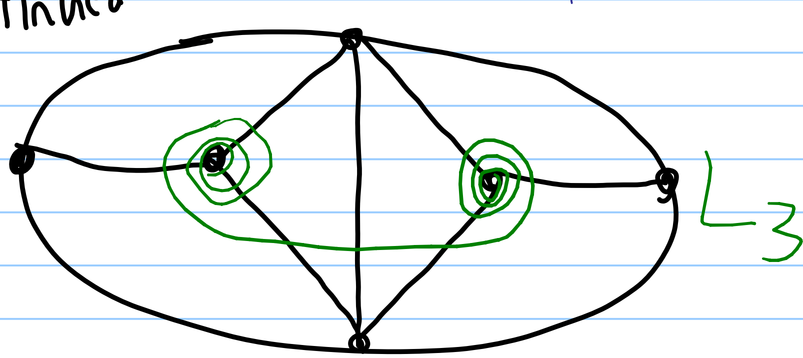
(*) [If time]

c)



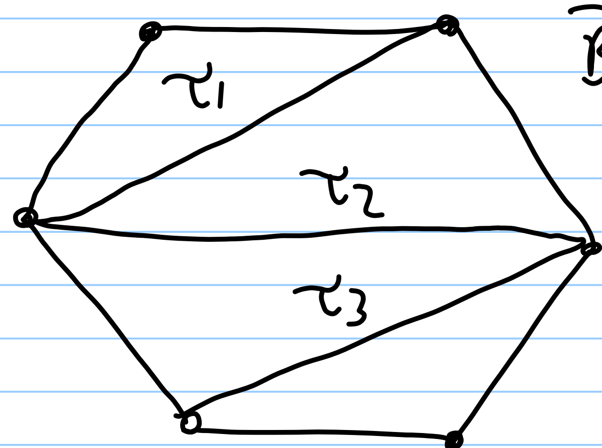
compute bottom of $\widetilde{B}$ only for this example

c) continued



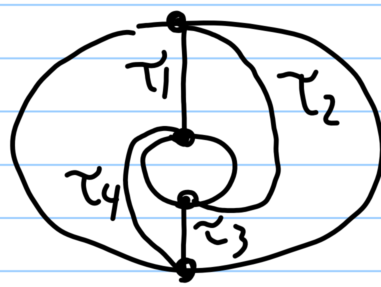
4-2) For the following triangulation T and $\tilde{B}$ matrix, compute a corresponding multi-lamination:

a)



$$\tilde{B} = \begin{bmatrix} & * & \\ \hline 0 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

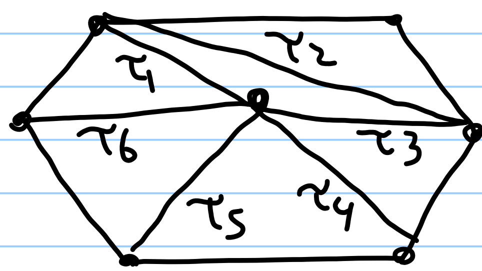
b)



$$\tilde{B} = \begin{bmatrix} & & * & \\ & & & \\ \hline 1 & -1 & 1 & -1 \end{bmatrix}$$

c)

* if time



$$\tilde{B} = \begin{bmatrix} & & * & & \\ & & & & \\ \hline 1 & 0 & 0 & -1 & -1 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

4-3) Prove that when we flip triangulation T to T', and let $b'_\gamma(T,L) := b_\gamma(T',L)$, ($\gamma \in T$ or T') we obtain: Exchange relations for $b_\gamma(T,L)$'s are tropical version of exchange relations for shear coordinates $\tau_\Sigma(\gamma,T)$.

Hint: Look at calculations on pages 14-17 of today's lecture notes.