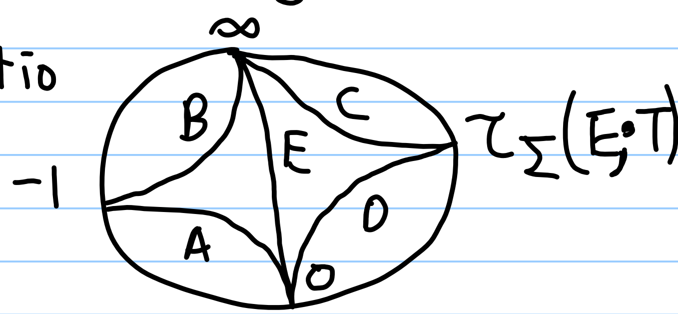
Lecture 5: Lambda Lengths & 2x2 Matrices

Note Title 4/25/2011

(1) Recall, $\mathcal{T}(S,M) =$ Teichmüller Space $= \{\text{hyperbolic metrics}\}$

Given a hyperbolic structure $\Sigma \in \mathcal{T}(S,M)$ and a triangulation $T = \{E_i\}_{i=1}^{n}$, the <u>shear coordinate</u>

$\tau_\Sigma(E;T)$ of edge $E \in T$ is the cross-ratio

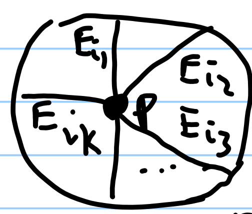


<u>Theorem</u>: The map $\mathcal{T}(S,M) \longrightarrow \mathbb{R}^n$

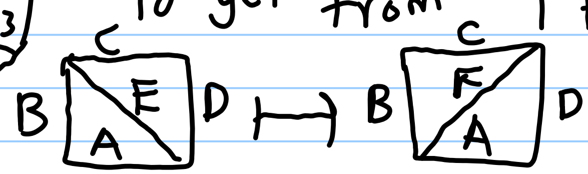
$$\Sigma \longmapsto \left(\tau_\Sigma(E_i;T)\right)_{i=1}^{n}$$

is a homeomorphism onto the subset of $\mathbb{R}^n$ where for each puncture p, and incident arcs $E_{i_1}, \ldots, E_{i_k}$, we have

$$\prod_{j=1}^{k} \tau_\Sigma(E_{i_j};T) = 1.$$



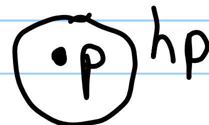
When we flip quads to get from T to T'



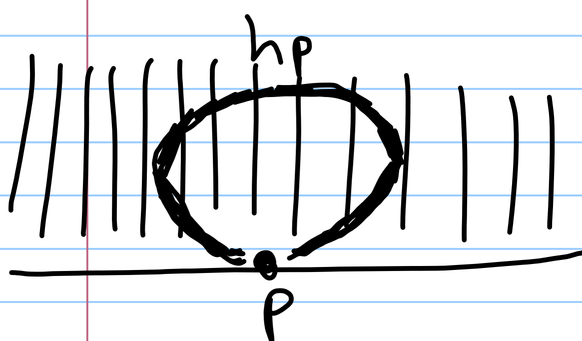
Shear coordinates change in a predictable way, see Lecture Notes 4.

2 We now switch gears and talk about Teichmüller interpretations of arcs:

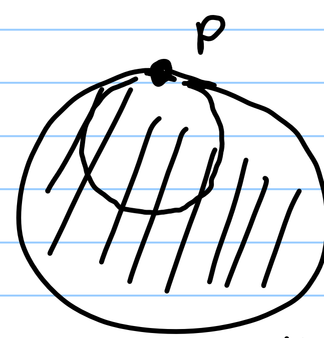
Def: A <u>horocycle</u>, at an ideal point p, is a set of points which are all equidistant to p.

Topologically: 

But in lift to hyperbolic upper half plane or the Poincaré disk,



or



Def: The <u>decorated Teichmüller space</u> $\widetilde{T}(S,M)$ is parametrized by data consisting of

- a point in $T(S,M)$,

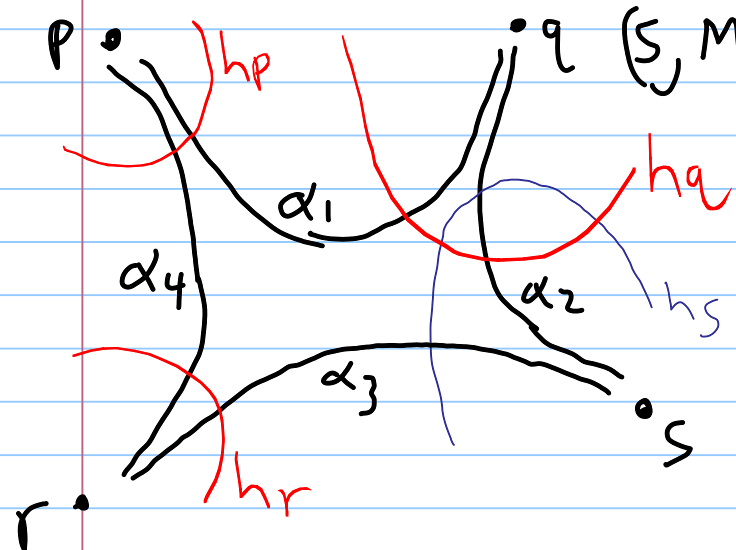
and

- a choice of horocycle around each cusp from M.

Def (Penner): For an arc E on (S,M) and a choice $\Sigma \in \widetilde{T}(S,M)$, the <u>length</u> $\ell_\Sigma(E) :=$ length of the geodesic rep. of E between intersections with horocycles.

Note: If horocycles chosen largely enough so that they intersect, $\ell_\Sigma(\gamma)$ is negative instead.

③ In topological viewpoint



(S, M) $M = \{p, q, r, s\}$

In this e.g., $l_\Sigma(\alpha_2)$ is negative.

Def: The λ-length of E is defined as $\boxed{\lambda_\Sigma(E) = e^{l_\Sigma(E)/2} \in \mathbb{R}_{>0}}$.

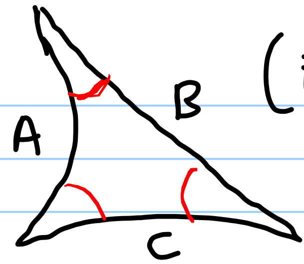
Penner coordinates for decorated Teichmüller space:

Theorem (Penner): For any triangulation $T = \{E_i\}_{i=1}^n$ without self-folded triangles, the map

$$\prod_{\gamma \in T \cup \{\text{Boundary Arcs}\}} \lambda(\gamma) : \widetilde{\mathcal{T}}(S, M) \to \mathbb{R}_{>0}^{n+c}$$

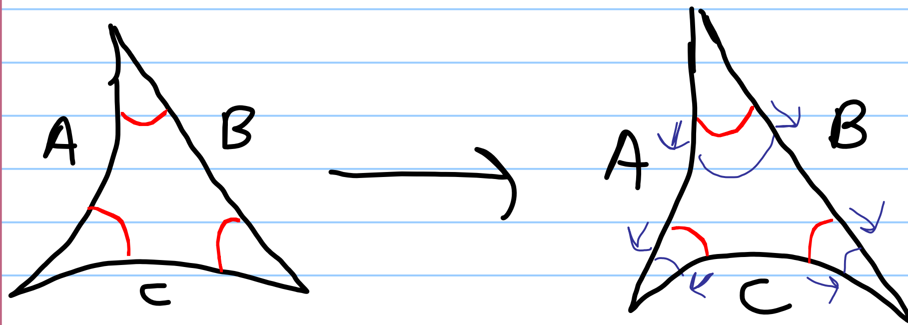
is a homeomorphism.

Sketch of Proof: If our surface was a single triangle with fixed vertices, then choosing $l_\Sigma(A), l_\Sigma(B), l_\Sigma(c)$ uniquely determines the decoration with horocycles.

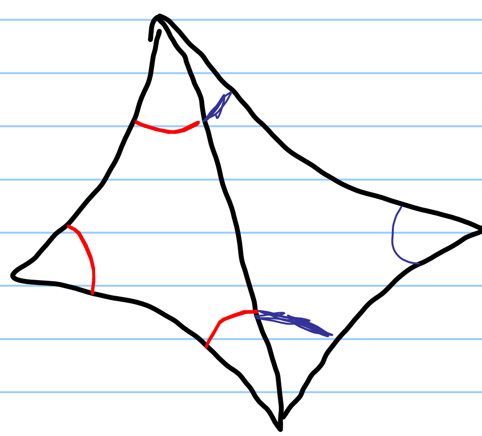


(Each $l_\Sigma(\cdot)\in(-\infty,\infty)$, so $\lambda_\Sigma(\cdot)\in\mathbb{R}_{>0}$)

Example: if $l_\Sigma(A)$ $l_\Sigma(B)$ held fixed, but $l_\Sigma(C)$ were increased, would change all horocycles accordingly;



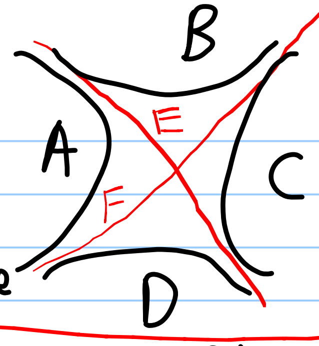
$\Longrightarrow$ Lengths on all triangles determines decorated triangles, and a unique way to glue adjacent triangles.



Also from λ-lengths (or $l_\Sigma(E)$) specified for each arc of a triangulation (and the boundary segments), we can obtain the lengths of any arc in the surface.

We now use the above specified λ-lengths to obtain λ-length corresponding to an arc obtained from flipping an arc:

⑤ Ptolemy Relation



For any ideal quadrilateral and $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, we have

$$\boxed{\lambda_\Sigma(E)\lambda_\Sigma(F) = \lambda_\Sigma(A)\lambda_\Sigma(C) + \lambda_\Sigma(B)\lambda_\Sigma(D).}$$

Not just algebraic statement, but statement about exponentials of these hyperbolic lengths.

Notice that this is a "tropical"-like statement about lengths

$$e^{l_\Sigma(E)/2} e^{l_\Sigma(F)/2} = e^{l_\Sigma(A)/2} e^{l_\Sigma(C)/2} + e^{l_\Sigma(B)/2} e^{l_\Sigma(D)/2}$$

$$\Rightarrow l_\Sigma(E) + l_\Sigma(F) = \log\left(e^{l_\Sigma(A)+l_\Sigma(C)} + e^{l_\Sigma(B)+l_\Sigma(D)}\right)$$

<u>Moral</u>: Let $x_{E_i} := \lambda_\Sigma(E_i)$ for each $E_i \in \widetilde{T}$, a triangulation with no self-folded triangles plus boundary arcs.

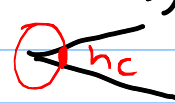
Then, choice of $\{x_{E_i}\}$'s (as in $\mathbb{R}_{>0}^{n+c}$) uniquely determines <u>data</u> $\Sigma \in \widetilde{\mathcal{T}}(S,M)$

$\Rightarrow$ all other $\lambda_\Sigma(\gamma)$ for γ another arc of (S,M) determined.

6. By iterations of Ptolemy Relations, each $\lambda_\Sigma(\gamma)$ is a Laurent polynomial in the X_{E_i}'s and can be thought of as a function acting on points of $\tilde{\mathcal{T}}(S,M)$.

$\boxed{\text{These are the cluster variables.}}$

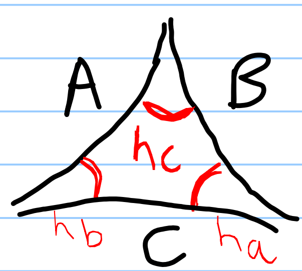
We now prove above Ptolemy Relation hyperbolically. First, a Lemma:

Note: Since horocycles lie away from cusps, their hyperbolic lengths, $L(h)$, or sector lengths, $L(h_c)$ 

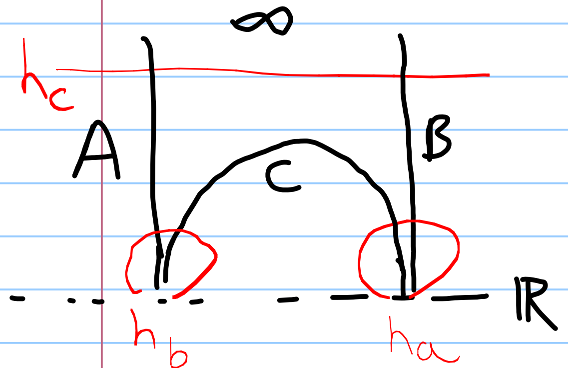
, are well-defined.

Lemma If A, B, C are sides of an ideal triangle, then for $\Sigma \in \tilde{\mathcal{T}}(S,M)$

$$\boxed{L(h_c) = \lambda_\Sigma(C) / \lambda_\Sigma(A)\lambda_\Sigma(B)}.$$



Pf: We consider the upper-half plane model where $ds = \dfrac{\sqrt{dx^2+dy^2}}{y}$



Assume h_a & h_b both have diameter 1 and h_c is the line $y = y_0$.
Assume A, B are vertical lines $x=0, x=x_0$, resp.

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$$\ell(A)=\ell(B)=\int_{y=1}^{y_0}\frac{\sqrt{dx^2+dy^2}}{y}=\int_1^{y_0}\frac{dy}{y}=\ln(y_0)$$

$$\Rightarrow \lambda(A)=\lambda(B)=\sqrt{y_0}\quad\left(\text{since } \lambda(\gamma):=e^{\frac{\ell(\gamma)}{2}}\right)$$

$$L(h_c)=\int_{x=0}^{x_0}\frac{dx}{y_0}=\frac{x_0}{y_0}.$$ So consider the expression

$\boxed{L(h_c)\lambda(A)\lambda(B)=x_0}$. Thus, this value does not depend on the height of h_c.

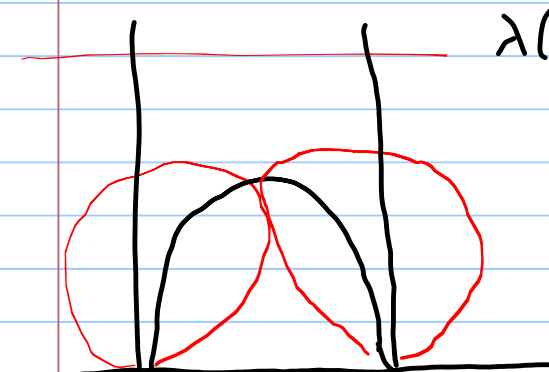
By symmetry, does not depend on radii of h_a, h_b either.

$\Rightarrow$ invariant must be $\boxed{\lambda(C)/\lambda(A)\lambda(B)}$.

We now let $x_0=1$ and then if diameters of h_a and h_b are both chosen to be one, then they are tangent $\Rightarrow \ell(C)=0$.

$$L(h_c)\,\lambda(A)\,\lambda(B)=1=e^{\ell(C)/2}$$

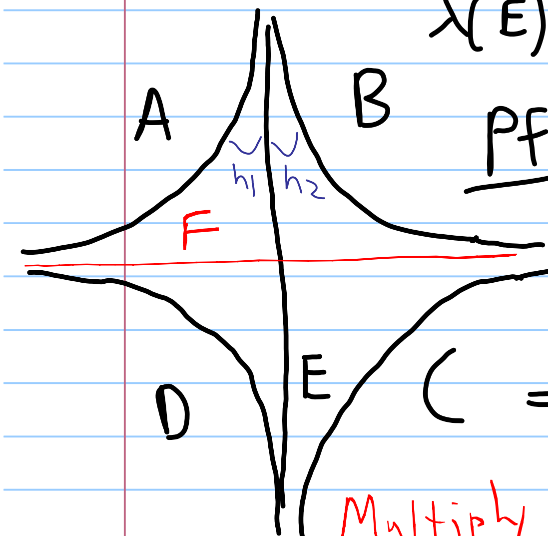
$$\Rightarrow L(h_c)=\frac{\lambda(C)}{\lambda(A)\lambda(B)}.$$ [i.e., no other constant needed]



Completes the pf of the Lemma. ▨

⑧ Lemma (Ptolemy Theorem)

$$\lambda(E)\lambda(F) = \lambda(A)\lambda(C) + \lambda(B)\lambda(D)$$



Pf: $L(h_1) + L(h_2)$

$$= \frac{\lambda(D)}{\lambda(A)\lambda(E)} + \frac{\lambda(C)}{\lambda(B)\lambda(E)}$$

$$\subset = \lambda(F)/\lambda(A)\lambda(B).$$

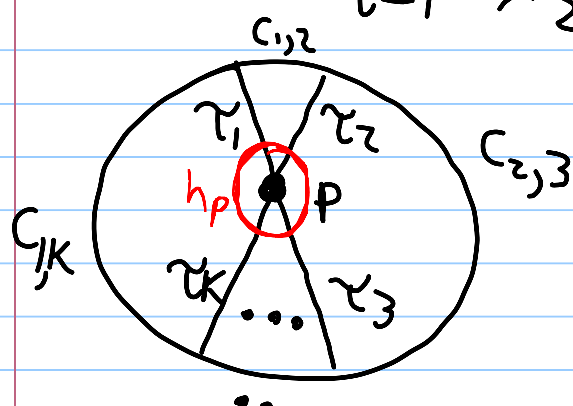
Multiply through by $\lambda(A)\lambda(B)\lambda(E)$:

$$\Rightarrow \lambda(E)\lambda(F) = \lambda(A)\lambda(B)\lambda(E)\left[\frac{\lambda(D)}{\lambda(A)\lambda(E)} + \frac{\lambda(C)}{\lambda(B)\lambda(E)}\right]$$

$$\Rightarrow \lambda(E)\lambda(F) = \lambda(B)\lambda(D) + \lambda(A)\lambda(C) \quad \blacksquare$$

Another Corollary: Around a puncture p incident to $\tau_1, \dots, \tau_K$ with opposite edges labeled as below, we have

$$L(h_p) = \sum_{i=1}^{K} \frac{\lambda_\Sigma(c_{i,i+1})}{\lambda_\Sigma(\tau_i)\lambda_\Sigma(\tau_{i+1})}$$



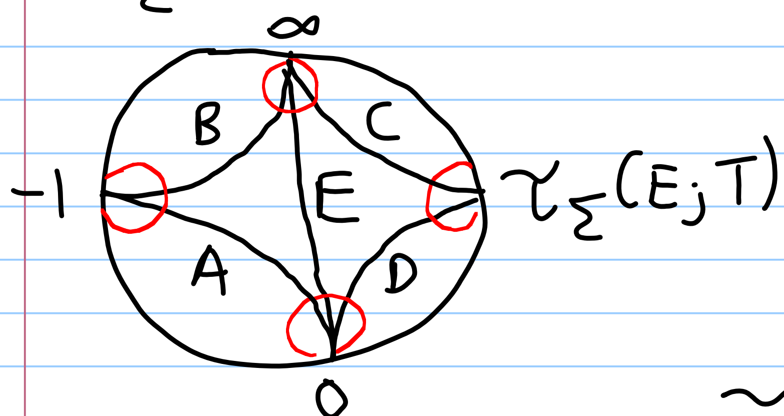
h_p = full circle

Pf: Sum together $L(h_p) = \sum_{i=1}^{K} L(h_{i,i+1})$ where $h_{i,i+1}$ is the arc segment between τ_i and τ_{i+1}.

9 Relation to shear coordinates

Given a hyperbolic structure (undecorated) $\Sigma \in \mathcal{T}(S,M)$ and triangulation $T = \{E_i\}$

$\tau_\Sigma(E;T) =$ cross ratio

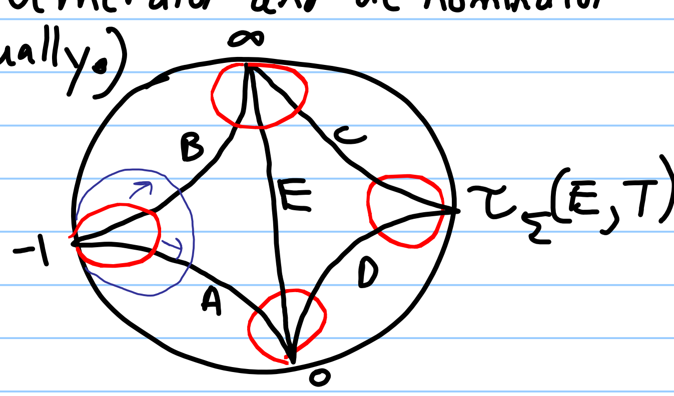


We can lift to an element $\tilde{\Sigma} \in \tilde{\mathcal{T}}(S,M)$ by choosing horocycles and then

$$\tau_\Sigma(E;T) = \frac{\lambda_{\tilde{\Sigma}}(A)\,\lambda_{\tilde{\Sigma}}(C)}{\lambda_{\tilde{\Sigma}}(B)\,\lambda_{\tilde{\Sigma}}(D)}$$

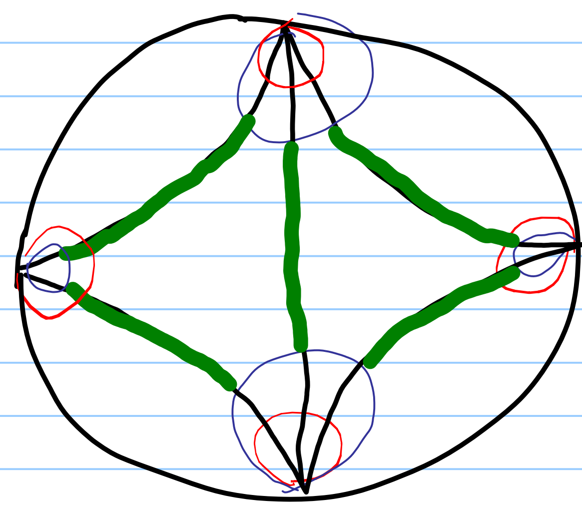
Note: Does not depend on lift $\tilde{\Sigma}$, i.e. choice of horocycles.

(If we make a horocycle bigger or smaller it affects consecutive sides of quadrilateral equally ⇒ numerator and denominator affected equally.)



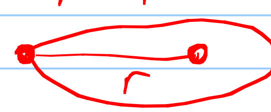
Remark: The hyperbolic lengths $\ell_{\tilde{\Sigma}}(A), \dots, \ell_{\tilde{\Sigma}}(D)$ do not determine $\ell_{\tilde{\Sigma}}(E)$:

Shrinking horocycles on left and right while expanding horocycles on top and bottom can keep $\ell_{\tilde{\Sigma}}(A), \dots, \ell_{\tilde{\Sigma}}(D)$ fixed while shrinking $\ell_{\tilde{\Sigma}}(E)$.



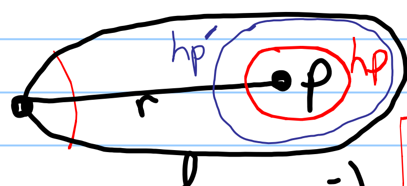
Next topic: Tagging interpreted hyperbolically. (we want $x_r x_{r^{(D)}} = x_\ell$)

for



Def: Define two horocycles h and h' around p to be conjugate if $L(h') = 1/L(h)$.

Cor: In a once-punctured monogon



with conjugate horocycles h_p and h_p', then

i) $\boxed{L(h_p) = \dfrac{\lambda_{\Sigma}(\ell)}{\lambda_{\Sigma}(r)^2}}$

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ii) Letting $\Sigma^{(p)} \in \widetilde{\mathcal{T}}(S,M)$ be the metric + horocyclic decoration with the same metric as $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, and all horocycles except h_p' instead of h_p.

$$\boxed{\lambda_{\Sigma^{(p)}}(\ell) = \lambda_{\Sigma}(\ell) = \lambda_{\Sigma}(r)\cdot\lambda_{\Sigma^{(p)}}(r)}.$$

Pf: First statement follows from earlier Lemma.

For second, notice $\lambda_{\Sigma^{(p)}}(\ell) = \lambda_{\Sigma}(\ell)$ as ℓ does not intersect h_p nor h_p'.

Since h_p' defined such that

$L(h_p') = 1/L(h_p)$, we have

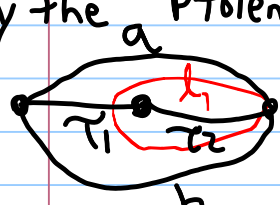
$L(h_p') = \lambda_{\Sigma^{(p)}}(\ell)/\lambda_{\Sigma^{(p)}}(r)^2$ &

$L(h_p') = \lambda_{\Sigma}(r)^2/\lambda_{\Sigma}(\ell)$, $\Rightarrow$

$$\lambda_{\Sigma^{(p)}}(\ell)\,\lambda_{\Sigma}(\ell) = \lambda_{\Sigma}(r)^2\lambda_{\Sigma^{(p)}}(r)^2$$
$$= \lambda_{\Sigma}(\ell)^2 = \lambda_{\Sigma^{(p)}}(\ell)^2.$$

Taking square-roots of both sides finishes the proof.

Moral: Combinatorially, if we have a 2-gon, by the Ptolemy Relation, we have

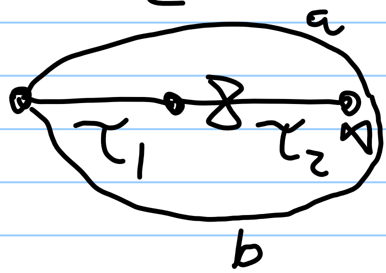


$$\boxed{\ell_{\Sigma}(\tau_1)\,\ell_{\Sigma}(\ell_1) = \big(\ell_{\Sigma}(a) + \ell_{\Sigma}(b)\big)\,\ell_{\Sigma}(\tau_2)}.$$

(12) Dividing both sides by $\ell_\Sigma(\tau_2)$, we get

$$\boxed{\ell_\Sigma(\tau_1)\,\ell_{\Sigma^{(p)}}(\tau_2) = \ell_\Sigma(a) + \ell_\Sigma(b)}$$

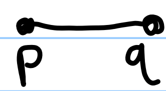
$\Rightarrow \ell_{\Sigma^{(p)}}(\tau_2)$ behaves like tagged arc.



This allowed Fomin-Thurston to think of cluster vars corresponding to tagged arcs as

"Take $\Sigma \in \tilde{\mathcal{T}}(S,M)$,

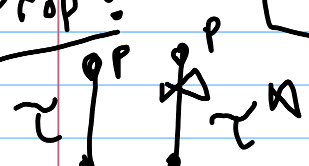
if τ notched at p or q,



change decoration Σ by changing horocycle at p and/or to its conjugate, i.e. $\Sigma^{(p)}$, $\Sigma^{(q)}$, or $\Sigma^{(pq)}$.

Then $\boxed{X_\tau = \lambda_{\Sigma^{(p)}}(\tau)}$

Prop:



$$\boxed{X_{\tau^{\bowtie}} = L(h_p)\cdot\lambda_\Sigma(\tau)}$$

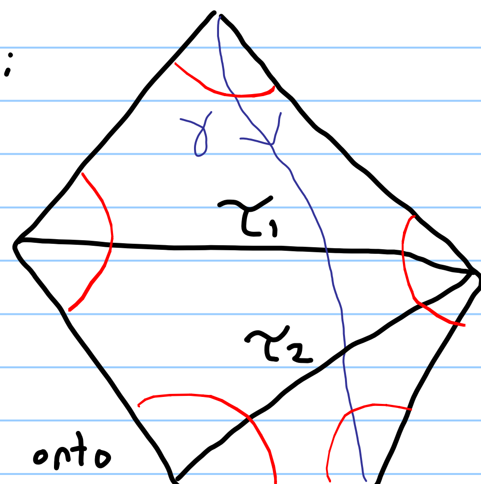
13 2x2 Matrix Formulas (Coeff. free)

While it is beyond the scope of this course, we can use the hyperbolic interpretation to obtain cluster variable formulas in terms of $PSL_2(\mathbb{R})$ matrices:

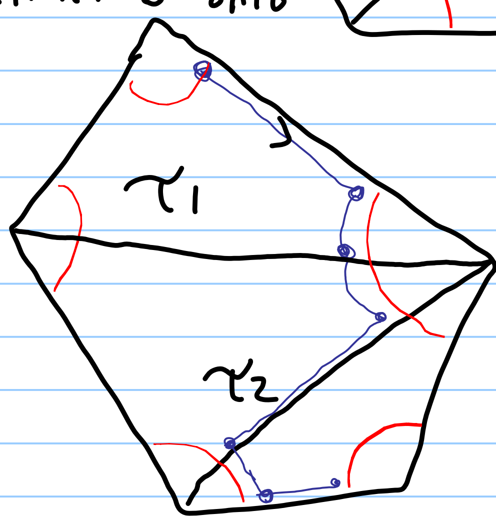
Construction (Fock-Goncharov):

Draw a circle around each $m \in M$. Any arc γ can be retracted onto a path travelling along an arc $\tau \in T$ or along an arc segment around m.

Example:

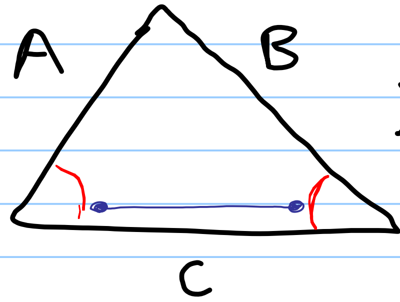


Retract γ onto



(14)

For a step of the type



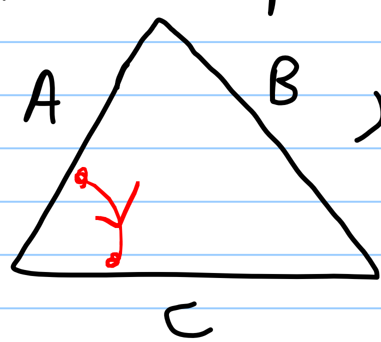
, we use the matrix $\begin{bmatrix} 0 & C \\ -\frac{1}{C} & 0 \end{bmatrix}$

Note: $\begin{bmatrix} 0 & C \\ -\frac{1}{C} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -C \\ 1/C & 0 \end{bmatrix}$ so

$\begin{bmatrix} 0 & C \\ -1/C & 0 \end{bmatrix}$ is an involution in $PSL_2(\mathbb{R})$.

Hence, the orientation of the step is irrelevant.

For a step of the type



, we use the matrix $\begin{bmatrix} 1 & 0 \\ \frac{B}{AC} & 1 \end{bmatrix}$.

This time orientation <u>does</u> matter.

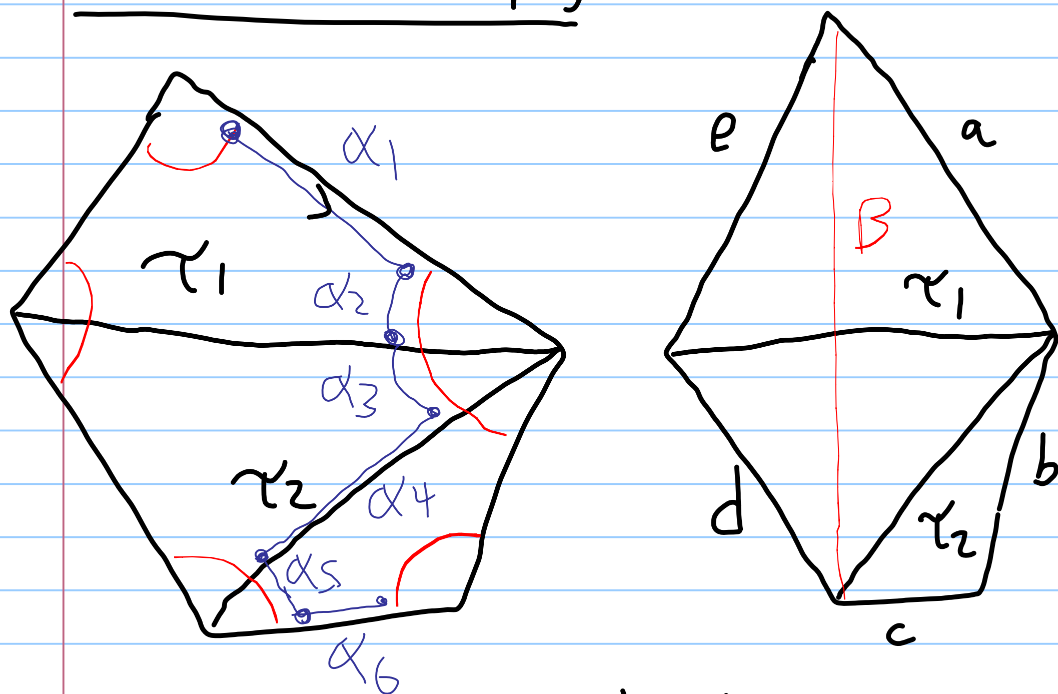
If counter-clockwise, we use

$\begin{bmatrix} 1 & 0 \\ -B/AC & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ B/AC & 1 \end{bmatrix}^{-1}$ instead.

We concatenate the matrices corresponding to these steps

$M(\alpha_K) \circ \cdots \circ M(\alpha_1)$ for $\overleftarrow{\gamma = \alpha_K \circ \cdots \circ \alpha_1}$

In above example,



$$M(\gamma)=M(\alpha_6)M(\alpha_5)M(\alpha_4)M(\alpha_3)M(\alpha_2)M(\alpha_1)$$

$$=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}1&0\\\frac{1}{x_2}&1\end{bmatrix}\begin{bmatrix}0&x_2\\-\frac{1}{x_2}&0\end{bmatrix}\begin{bmatrix}1&0\\-\frac{1}{x_1x_2}&1\end{bmatrix}\begin{bmatrix}1&0\\-\frac{1}{x_1}&1\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

$$=\begin{bmatrix}-1&-(x_1+x_2+1)/x_1x_2\\x_2&(x_2+1)/x_1\end{bmatrix}$$

Thm (Fock–Goncharov)

$|$upper right entry of $M(\gamma)|$ is $\lambda_\Sigma(\gamma)$.

Second Example:

Also, consider B that only crosses τ_1.

$$M(B)=M(\alpha_4)M(\alpha_3)M(\alpha_2)M(\alpha_1)=\begin{bmatrix}-x_2&*\\0&-\frac{1}{x_2}\end{bmatrix}$$

where $* = -(x_2+1)/x_1$.

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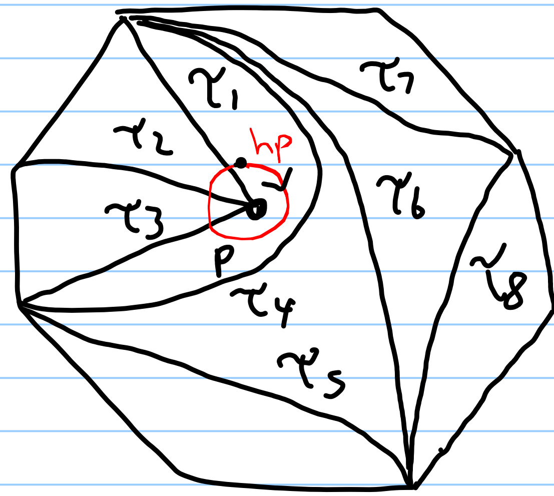
Another example, Type D_8

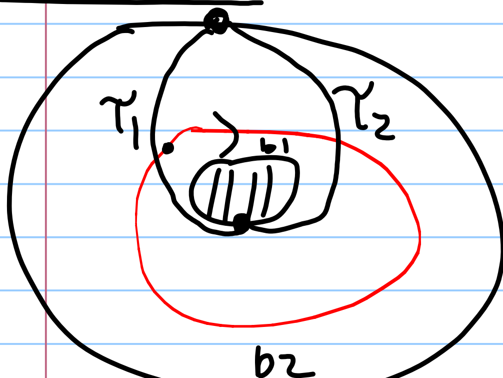


Thm (FG) For a closed curve, $|\mathrm{trace}(M(\gamma))|$ is $\lambda_\Sigma(\gamma)$.

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{x_1x_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{x_2x_3} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{x_4}{x_1x_3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ h_p & 1 \end{bmatrix}$$

so trace $= 2$.

Example: annulus



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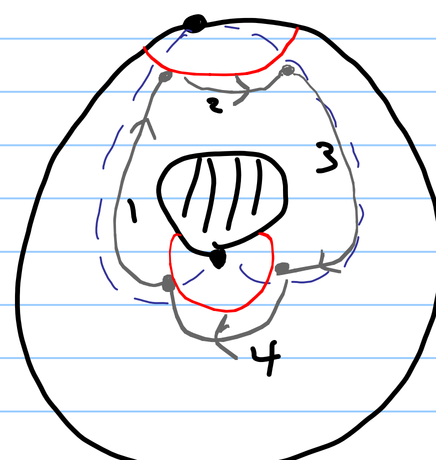


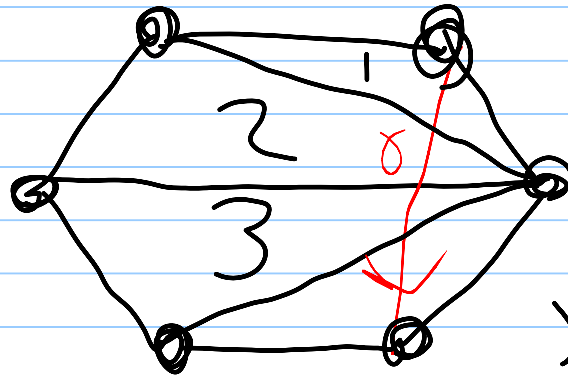
$$\begin{bmatrix} 1 & 0 \\ \frac{b_2}{x_1x_2} & 1 \end{bmatrix}\begin{bmatrix} 0 & x_2 \\ \frac{-1}{x_2} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{-b_1}{x_1x_2} & 1 \end{bmatrix}\begin{bmatrix} 0 & x_1 \\ -1/x_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -x_2/x_1 & -b_1 \\ -b_2/x_1^2 & -(x_1^2+b_1b_2)/x_1x_2 \end{bmatrix}$$

$$|\mathrm{Trace}| = \frac{x_2^2+x_1^2+b_1b_2}{x_1x_2} \quad (\text{let } b_1=b_2=1)$$

Lecture 5 Exercises

5-1) Compute $M(\gamma)$ for γ in

a) 

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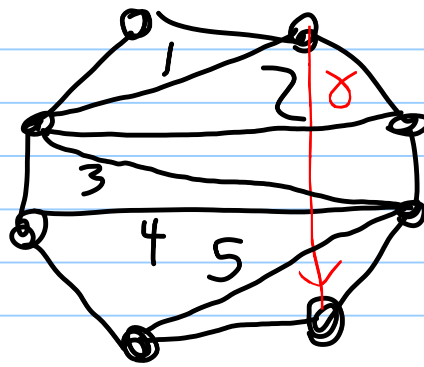
b) 

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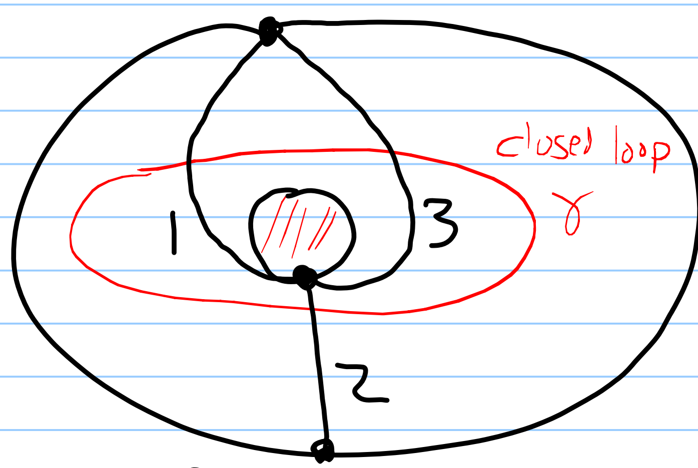
c) 

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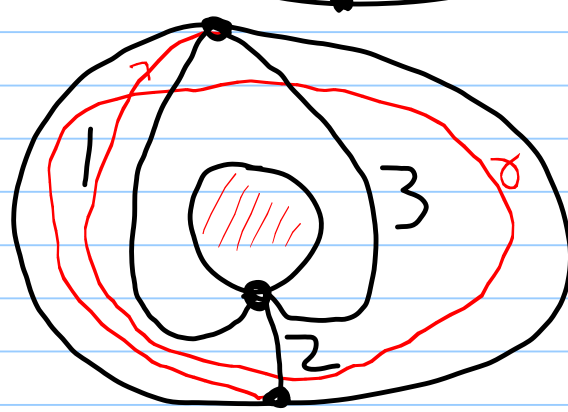
d) 

Hint:

[Compute $M(\gamma)$ as usual.]

e) How do the answers for (c) & (d) compare? Why is that?

5-2) Let $x_n x_{n-2} = x_{n-1}^2 + 1$ for $n \in \mathbb{Z}$.

Find a "conserved quantity" γ such that if $x_\gamma = \dfrac{P(x_1, x_2)}{x_1^{d_1} x_2^{d_2}}$ in $\{x_1, x_2\}$,

then for any other cluster $\{x_{n-1}, x_n\}$, we have $x_\gamma = \dfrac{P(x_{n-1}, x_n)}{x_{n-1}^{d_1} x_n^{d_2}}$ for the same polynomial $P(\cdot,\cdot)$ and integers d_1, d_2.

5-3) Let T be any ideal triangulation, γ be an ordinary arc (plain on both ends), $\gamma^{(p)}$ notched at punc p and $\gamma^{(pq)}$ notched at both ends, punctures p, q.

a) Prove that in the coefficient-free case,

$$X_{\gamma^{(p)}} = X_\gamma \cdot L(h_p),$$

and

$$X_{\gamma^{(pq)}} = X_\gamma \cdot L(h_p) L(h_q).$$

b) Look back at exercises from Lecture 3 and factor Laurent polynomials showing up there (in the coefficient-free case).