In this note I complete the argument that I did not complete during the lecture.

Set-up: (R, m) is a Noetherian local ring, M is a finitely generated R-module, $\operatorname{ann} M = 0$,

$$C_\bullet : 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

is a complex of finitely generated free R-modules such that $C_\bullet \otimes_R M$ is exact, $I(\varphi_n)$ is contained in m, and there exists $x \in I_k(\varphi_k)$ for all k that is a non-zerodivisor on M and on R.

In the lecture I wrote: let $C_k = \ker(\varphi_k)$.

Instead, define $C_k = \ker(\varphi_k \otimes_R M)$. Since C_k is a subset of $F_k \otimes M$, which is a direct sum of copies of M, it follows that x is non-zerodivisor on C_k. The exact sequence $C_\bullet \otimes M$ can be spliced up into:

$$\begin{aligned}
&0 \to F_n \otimes M \to F_{n-1} \otimes M \to C_{n-2} \to 0 \\
&0 \to C_{n-2} \to F_{n-2} \otimes M \to C_{n-3} \to 0 \\
&\ \vdots \\
&0 \to C_2 \to F_2 \otimes M \to F_1 \otimes M \to 0.
\end{aligned}$$

Since x is a non-zerodivisor on R, we get that tensoring the short exact sequences above with R/xR yields short exact sequences, which then in turn splice together into the long exact sequence

$$0 \to F_n \otimes \frac{M}{xM} \to F_{n-1} \otimes \frac{M}{xM} \to F_{n-2} \otimes \frac{M}{xM} \to \cdots \to C_2 \to F_2 \otimes \frac{M}{xM} \to F_1 \otimes \frac{M}{xM}.$$

By induction on n, since this complex arises from tensoring a truncation of $C_\bullet$, we conclude that $\operatorname{depth}(I(\varphi_k, M/xM) \geq k-1$ for all $k \geq 2$, so that $\operatorname{depth}(I(\varphi_k, M) \geq k$ for all $k \geq 2$. But Lemma 4 also proved that $\operatorname{depth}(I(\varphi_1, M) \geq 1$, which proves that $\operatorname{depth}(I(\varphi_k, M) \geq k$ for all $k \geq 1$. This finishes the proof of (b).