Introduction to The Dirichlet Space

MSRI Summer Graduate Workshop

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Overview

Study of the Dirichlet space is an area where many different viewpoints and techniques come together. I will discuss many of the themes and tools. The discussion will generally be informal and sketchy.

The choice of particular topics is influenced by my attempt to build a few coherent narratives and, of course, by my own interests.

I will give five talks.

1. Introduction to the Dirichlet space
2. Multipliers and Carleson Measures
3. Interpolation and the Pick Property
4. Zero Sets
5. Discrete Models

I will be here for the entire program and would be glad to talk to any of you about the material any time.

The Dirichlet Space

Geometric Definition:

- For $f(z) = \sum a_n z^n \in \mathrm{Hol}(\mathbb{D})$ set

$$\begin{aligned} \mathcal{D}(f) &= \frac{1}{\pi} \int \int_{\mathbb{D}} \left| f'(z) \right|^2 dxdy = \sum_1^\infty n \left| a_n \right|^2 \\ &= \left(\text{area } f(\mathbb{D}) \text{ with multiplicity} \right)^2 \end{aligned}$$

- The Dirichlet space is

$$\mathcal{D} = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}(f) < \infty \right\}$$

Using $\|f\|^2 = |a_0|^2 + \mathcal{D}(f)$ and the associated inner product, $\langle f, g \rangle$, $\mathcal{D}$ is a Hilbert space.

- The monomials are orthogonal and the polynomials are dense in $\mathcal{D}$. The set $\{1\} \cup \{nz^n\}_1^\infty$ is an orthonormal basis.
- The definition of $\mathcal{D}$ is, almost, but not quite, strong enough to force $\mathcal{D} \subset A(\mathbb{D})$ the algebra of functions in $\mathrm{Hol}(\mathbb{D})$ which extend continuously to the boundary. Functions with $\sum_1^\infty n \log n |a_n|^2 < \infty$ are in $A(\mathbb{D})$. The fact that $\mathcal{D}$ contains the Riemann map onto any simply connected domain of finite area shows the general $f \in \mathcal{D}$ need not extend continuously to the boundary and it also shows that $\mathcal{D}$ contains unbounded functions. The inclusion $A(\mathbb{D}) \subset \mathcal{D}$ also fails; if that inclusion held it would be continuous, but the monomials are a bounded set in $A(\mathbb{D})$ and an unbounded set in $\mathcal{D}$.

Reproducing Kernels

- A Hilbert spaces with reproducing kernels ($=$ reproducing kernel Hilbert space $=$ RKHS) is a Hilbert space whose vectors are functions on a set X and with the evaluation at points of X being continuous. Those continuous point evaluation functionals are realized by the reproducing kernels: $f \longrightarrow f(x) = \langle f, k_x \rangle$.
- In an RKHS it is possible to define pointwise products of Hilbert space vectors. This additional structure leads to the distinctive aspects of the theory of RKHS. Hilbert spaces of holomrophic functions generally have this additional structure. $L^2(0,1)$ does not.
- The general formula for the reproducing kernel is $k_z(w) = \sum \overline{e_n(z)} e_n(w)$ with $\{e_n\}$ any orthonormal basis. For the Dirichlet space we get $k_z(w) = 1 + \log \{1/(1 - \bar{z}w)\}$.
- We have

$$\|k_z\|^2 = \langle k_z, k_z \rangle = k_z(z) = 1 + \log \frac{1}{1 - |z|^2}.$$

We denote the normalized kernel by $\hat{k}_z$; $\hat{k}_z = k_z / \|k_z\|$.

Two Other Hilbert Spaces: The Hardy Space

- The Hardy space. $H^2 = H^2(\mathbb{D})$

$$
\begin{aligned}
H^2 &= \left\{ f \in \mathrm{Hol}(\mathbb{D}) : f = \sum a_n z^n, \|f\|^2 = \sum |a_n|^2 < \infty \right\}, \\
&= \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|^2 = \limsup_{r \nearrow 1} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 d\theta/2\pi < \infty \right\}, \\
&= \left\{ f \in L^2(\mathbb{T}) : f \sim \sum_0^\infty a_n e^{in\theta}, \ \ \|f\|^2 = \sum_0^\infty |a_n|^2 < \infty \right\}.
\end{aligned}
$$

- The reproducing kernel (Cauchy kernel, Szegö Kernel) is $k_z(w) = (1 - \bar{z}w)^{-1}$. H^2 is a closed subspace of $L^2(\mathbb{T}, d\theta/2\pi)$. The map

$$F \longrightarrow \langle F, k_z \rangle_{L^2(\mathbb{T})}$$

gives the orthogonal projection (Cauchy-Szegö projection) of $L^2(\mathbb{T})$ onto H^2.

- The Bergman space. $A^2 = A^2(\mathbb{D})$

$$\begin{aligned}\mathcal{A}^2 &= \left\{ f \in \mathrm{Hol}(\mathbb{D}) \cap L^2(\mathbb{D}, \frac{dxdy}{\pi}), \|f\|^2 = \int_{\mathbb{D}} |f|^2 \frac{dxdy}{\pi} < \infty \right\} \\ &= \left\{ f \in \mathrm{Hol}(\mathbb{D}),\ f = \sum a_n z^n, \|f\|^2 = \sum \frac{|a_n|^2}{n+1} < \infty \right\}.\end{aligned}$$

- For $\mathcal{A}^2$ the reproducing kernel (Bergman kernel) is $k_z(w) = (1 - \bar{z}w)^{-2}$. $\mathcal{A}^2$ is a closed subspace of $L^2(\mathbb{D}, dxdy/\pi)$. The map

$$F \to \langle F, k_z \rangle_{L^2(\mathbb{D})}$$

gives the orthogonal projection (Bergman projection) of $L^2(\mathbb{D})$ onto $\mathcal{A}^2$.

The Dirichlet Space is Different

- $\mathcal{D}$ is not a subspace of any $L^2(X, d\mu)$. That leads to a less important role for measure theory in the study of $\mathcal{D}$ and a more important role for potential theory.
- By comparing the formulas for the norms we see

$$\begin{aligned} \mathcal{D} &\subset H^2 \subset \mathcal{A}^2, \\ \mathcal{D} &= \left\{ f \in \mathrm{Hol}\,(\mathbb{D}) : f' \in \mathcal{A}^2 \right\}. \end{aligned}$$

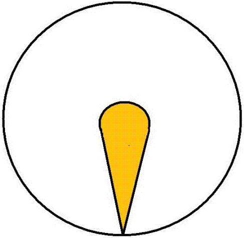
- One can introduce a notion of half-order differentiation and integration and produce versions of the last statement which also involve H^2. Potential operators carry functions back and forth between these three spaces.

The Dirichlet Space, Alternative Definition

- An equivalent norm for $\mathcal{D}$ is given by $\|f\|^2 = \sum_0^\infty (n+1)|a_n|^2$. This norm does not have a clean geometric interpretation.
- On the other hand, the associated kernel function, $-(\bar{z}w)^{-1}\log(1-\bar{z}w)$, has an important and striking algebraic/analytic property, the complete Pick property, which we will discuss later.
- Most results can be presented comfortably using either norm, however the discussion of Carleson's formula uses the first norm, the discussion of Pick interpolation the second; hopefully......

Boundary Value Theory

- Because $\mathcal{D} \subset H^2$ we have access to the Hardy space boundary value theory:
- Any $f = \sum a_n z^n \in H^2$ has radial (in fact, nontangential) boundary values a.e. The boundary value function is in $L^2(\mathbb{T})$ and its Fourier coefficients are the $\{a_n\}$. We will identify the function and its boundary values whenever convenient and without further comment.



Nontangential approach

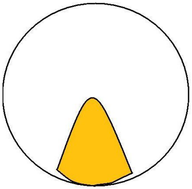
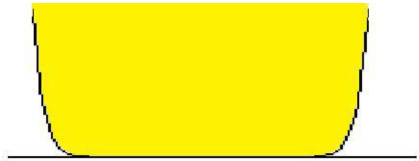
- For $f \in \mathcal{D}$ much more is true. Such f have radial limits off much smaller exceptional set (capacity zero, of which more later). Alternatively, if we are willing to accept a much large exceptional set, a set of measure zero, we can have a much larger approach regions. For $e^{i\theta} \in \mathbb{T}$ set

$$\mathrm{NRS}\left(e^{i\theta}\right) = \left\{z \in \mathbb{D} : \left|z - e^{i\theta}\right| \leq \left|\log(1 - |z|)\right|^{-1}\right\}$$

- Theorem: (Nagel, Rudin, Shapiro 1982): Given $f \in \mathcal{D}$ there is an exceptional set $E \subset \mathbb{T}$, $|E| = 0$ so that for each $e^{i\theta} \in \mathbb{T} \setminus E$

$$\lim_{\substack{z \to e^{i\theta} \\ z \in \mathrm{NRS}(e^{i\theta})}} f(z) \text{ exists.}$$

There are two proofs, both are hard.

- The boundaries of the regions $\mathrm{NRS}\left(e^{i\theta}\right)$ have infinite order contact with the circle.
-  

  NRS(1) Close-up; $y = \exp(-1/\left|x\right|)$

Inner-Outer Factorization

Because $\mathcal{D} \subset H^2$ we also have access to the Hardy space factorization theory:

- If $f \in H^2$ then its zero set, $Z = \{z_j\}_{j=1}^{\infty}$, satisfies the *Blaschke condition*
$$\sum (1 - |z_i|) < \infty. \tag{B}$$
The associated Blaschke product is
$$B_f(z) = B_Z(z) = \prod_{i=1}^{\infty} \frac{\bar{z}_i}{|z_i|} \frac{z_i - z}{1 - \bar{z}_i z}.$$
(B) insures that the product converges. Its zero set is exactly Z.

- Set
$$O_f(z) = \exp\left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log\left| f\left(e^{it}\right)\right| dt \right\}. \qquad \text{(out)}$$
- There is a unimodular constant c and a finite positive Borel measure μ_f on $\mathbb{T}$, $\mu_f \perp d\theta$, so that, if we set
$$S_f(z) = \exp\left\{ -\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} d\mu_f(t) \right\},$$
then we can form a factorization of f :
$$f = cB_f S_f O_f.$$
- The functions B_f, S_f, and O_f, are called, respectively, the Blaschke, singular, and outer factors of f. The function $I_f = cB_f S_f$ is called the inner factor of f and the representation $f = I_f O_f$ is called the inner-outer factorization of f.

- At almost every boundary point $|B_f| = |S_f| = 1$, and $|f| = |O_f|$. In particular each factor is in H^2 and $\|B_f\|_{H^2} = \|S_f\|_{H^2} = 1$, $\|O_f\|_{H^2} = \|f\|_{H^2}$. Thus O_f is a zero-free function which carries the size of f; B_f carries the zeros, and S_f, which tends to zero strongly near the support of μ_f, carries information about how f approaches zero near the boundary of the disk.
- Conversely, if B_f and S_f are as described, $\left|f(e^{i\theta})\right|$ is specified on the boundary subject to

$$\begin{aligned} \int_{\mathbb{T}} |f|^2 &< \infty, \\ -\int_{\mathbb{T}} 0 \wedge \log|f| &< \infty \end{aligned} \tag{log}$$

and O_f is defined by (out); then $f = B_f S_f O_f$ is in H^2 and that product is its canonical factorization.

- A great deal of Hardy space theory is based on this factorization. There is no truly satisfactory substitute for functions in the Dirichlet space.
- The Hardy space norm of f is the same as that of its outer factor. Surprisingly, the Dirichlet norm of a function can also be read off from the data in the factorization. We now present that.

Carleson's Formula

Suppose we have $f \in \mathcal{D}$, $f = cB_f S_f O_f$. For any $z \in \mathbb{D}$ let $P_z(e^{i\theta})$ be the Poisson kernel for evaluation at z :

$$P_z(e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}$$

Set $u = \log|f| = \log|O_f|$ on $\mathbb{T}$. Then (Carleson, 1960)

$$\begin{aligned}
\pi\mathcal{D}(f) &= \int_T \left(\sum P_{z_i}(e^{i\theta})\right) \left|f(e^{i\theta})\right|^2 d\theta \\
&+ \int_T \int_T \frac{2}{|e^{i\theta} - e^{i\phi}|^2} d\mu_f(e^{i\phi}) \left|f(e^{i\theta})\right|^2 d\theta \\
&+ \int_T \int_T \frac{\left(e^{2u(e^{i\theta})} - e^{2u(e^{i\phi})}\right)\left(u(e^{i\theta}) - u(e^{i\phi})\right)}{|e^{i\theta} - e^{i\phi}|^2} d\theta d\phi
\end{aligned} \tag{CF}$$

- Corollary: The only inner functions in $\mathcal{D}$ are the finite Blaschke products. (Set $|f| = 1$ and use Fubini's theorem.)
- The formula shows that removing factors from the Blaschke product or replacing $d\mu_f$ by a smaller measure will reduce the size of $\mathcal{D}(f)$. In particular if $f \in \mathcal{D}$ then its outer factor $O_f \in \mathcal{D}$ and $\mathcal{D}(O_f) \leq \mathcal{D}(f)$.
- The formula is valuable tool for constructing and studying functions in the Dirichlet space; we will look at an example later.
- A far reaching generalization of this approach, a formula for the "local Dirichlet integral", has been developed and used very effectively by Richter and Sundberg .

Friends of the Dirichlet space I: The Quasinorm

- $\mathcal{D}(f)$ is not a norm on $\mathcal{D}$, is is a quasinorm. (Or, if you like, it is a norm on $\mathcal{D}_0 = \mathcal{D} \ominus \mathbb{C} = \{f \in \mathcal{D} : f(0) = 0\}$.) and some properties are clearer when working with $\mathcal{D}_0$ rather than $\mathcal{D}$.
- "The Dirichlet space is conformally invariant" $=$ For Φ is a conformal automorphism of the disk (a Mobius transformation) define the renormalized composition operator by $f(z) \to f(\Phi(z)) - f(\Phi(0))$. This mapping is unitary on $\mathcal{D}_0$. (Recall the geometric interpretation of $\mathcal{D}(\cdot)$.)
- Theorem (Arazy-Fisher '86). If K is a Hilbert space of holomorphic functions on the disk which has a quasinorm which is invariant under the full group of Mobius transforms then that quasinorm is a constant multiple of $\mathcal{D}$.

Friends of the Dirichlet space II: The Besov Spaces

- The Hardy space H^2 is an element in a one parameter family of Banach spaces of holomorphic functions, the Hardy spaces, H^p, $1 \leq p \leq \infty$. The Dirichlet space also lives in a natural family of Banach spaces of holomorphic functions, the Besov spaces.
- For $1 < p < \infty$ let B^p be the space of holomrophic functions on the disk for which

$$\int_{\mathbb{D}} \left|(1-|z|^2)f'(z)\right|^p \frac{dxdy}{(1-|z|^2)^2} < \infty$$

  - For $p = 2$ this is the Dirichlet space.
  - The scale can be extended to $p = 1$ and $p = \infty$ but the definitions must be adjusted.
  - The reason for the particular way of writing the integrand is that both the differential operator $(1-|z|^2)\frac{d}{dz}$ and the measure $(1-|z|^2)^{-2}dxdy$ behave well under changes of variables by Mobius transformations.