DIRICHLET SPACE VS. TREE SPACE & CARLESON MEASURES.

$$\|f\|_{D}^2 = |f(0)|^2 + \frac{1}{\pi}\int_\Delta |f'(z)|^2 \, dx\, dy$$

$$Q_{n,j} = \left\{ re^{i\theta} : 2^{-n-1} < 1-r \le 2^{-n};\ \frac{j-1}{2^n} \le \frac{\theta}{2\pi} < \frac{j}{2^n} \right\}$$

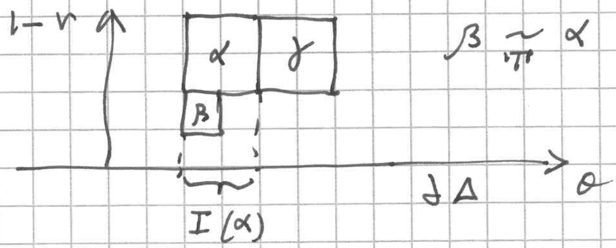
$$T = \{ \alpha = (n,j) : n \ge 0,\ 1 \le j \le 2^n \}.$$

$\alpha \underset{G}{\sim} \beta :\Longleftrightarrow \overline{Q_\alpha} \cap \overline{Q_\beta} \ne \emptyset$ (α and β are graph-related)

$\alpha \underset{T}{\sim} \beta :\Longleftrightarrow \alpha = (n,j),\ \beta = (m,k),\ |n-m| = 1$ and
$I(\alpha) \cap I(\beta) \ne \emptyset$ (α and β are tree-related)

where $I(\alpha) = \{\theta : \exists\, r \text{ s.t. } re^{i\theta} \in Q_\alpha\} \subseteq \partial\Delta$.

$\alpha \underset{G}{\sim} \beta \Longleftarrow \alpha \underset{T}{\sim} \beta$, $\quad \alpha \underset{G}{\sim} \beta \not\Longrightarrow \alpha \underset{T}{\sim} \beta$



$\beta \underset{T}{\sim} \alpha \underset{G}{\sim} \beta$

Basic estimates.

(i) $z_\alpha \in Q_\alpha$ and $z_\beta \in Q_\beta \Rightarrow |f(z_\alpha) - f(z_\beta)| \lesssim \#(\text{chain from } \alpha \text{ to } \beta) \cdot \|f\|_{D}$

where a chain $[\alpha_1, \dots, \alpha_n]$ from α to β satisfies $\alpha = \alpha_1 \underset{G}{\sim} \alpha_2 \sim \dots \underset{G}{\sim} \alpha_n = \beta$; $\#([\alpha_1, \dots, \alpha_n]) = n$.

(ii) $z_\alpha \in Q_\alpha$ and $z_\beta \in Q_\beta \Rightarrow |f(z_\alpha) - f(z_\beta)| \lesssim \dfrac{|z_\alpha - z_\beta|}{\min(|Q_\alpha|, |Q_\beta|)^{1/2}}$

Exercise. $d(z_\alpha, z_\beta) \approx \min\left\{ \#(\text{chain from } \alpha \text{ to } \beta);\ \dfrac{|z_\alpha - z_\beta|}{\min(|Q_\alpha|, |Q_\beta|)} \right\}$

where d is the hyperbolic distance, i.e. the distance w.r.t. the metric $ds^2 = |dz|^2 / (1-|z|^2)^2$.

Proof. (i) If $\alpha \underset{G}{\sim} \beta$, then

$$|f(z_\alpha) - f(z_\beta)| = \left| \int_{z_\alpha}^{z_\beta} f'(\zeta)\, d\zeta \right| \le \operatorname{diam}(Q_\alpha) \cdot |f'(\zeta_\alpha^{MAX})|$$

where $|f'(\zeta_\alpha^{MAX})| = \max\limits_{\zeta \in \overline{Q_\alpha} \cup \overline{Q_\beta}} |f'(\zeta)|$ and $\zeta \in \overline{Q_\alpha} \cup \overline{Q_\beta}$

$$= \operatorname{diam}(Q_\alpha) \cdot \left| \frac{1}{|B(\zeta_\alpha^{MAX}, \varepsilon \operatorname{diam}(Q_\alpha))|} \cdot \int_{B(\zeta_\alpha^{MAX}, \varepsilon \operatorname{diam}(Q_\alpha))} f'(\zeta)\, d\zeta_1 d\zeta_2 \right|$$

by M.V.P.

$$\leq \operatorname{diam}(Q_\alpha) \cdot \left(\frac{1}{|B(\zeta_\alpha^{MAX}, \varepsilon \operatorname{diam}(Q_\alpha))|} \int_{B(\zeta_\alpha^{MAX}, \varepsilon \operatorname{diam}(Q_\alpha))} |f'(\zeta)|^2\, d\zeta_1 d\zeta_2 \right)^{1/2}$$

by Jensen

$$\asymp \left(\int_{\bigcup Q_\gamma :\, d_G(\gamma, \alpha) \leq 2} |f'(\zeta)|^2\, d\zeta_1 d\zeta_2 \right)^{1/2} \qquad \text{provided } \varepsilon \leq 1/8.$$

Iterating along a chain:

$$|f(z_\alpha) - f(z_\beta)| \lesssim \sum_{\gamma \text{ in the chain}} \left(\int_{\bigcup Q_\eta :\, d_G(\eta, \gamma) \leq 2} |f'(\zeta)|^2\, d\zeta_1 d\zeta_2 \right)^{1/2}$$

$$\leq \left(\sum_{\gamma \in \text{chain}} 1 \right)^{1/2} \cdot \left(\sum_{\gamma \in \text{chain}} \int_{\bigcup Q_\eta :\, d_G(\eta, \gamma) \leq 2} |f'(\zeta)|^2\, d\zeta_1 d\zeta_2 \right)^{1/2}$$

Cauchy-Schwarz

$\leq \#(\gamma \text{ in the chain})^{1/2} \cdot \|f\|_{\mathcal{D}}^{*}$ since the overlapping of the boxes over which we integrate is bounded.

Here, $\|f\|_{\mathcal{D}}^{*\,2} = \frac{1}{\pi} \int_{\Delta} |f'|^2\, dx dy$

is the conformally invariant seminorm of $\mathcal{D}$.

(ii) The quantity on the R.H.S. of (ii) dominates that on the R.H.S. of (i) if α and β are not G-related (Exercise), then we can assume $\alpha \underset{G}{\sim} \beta$, in which case $|f(z_\alpha) - f(z_\beta)| = \left| \int_{z_\alpha}^{z_\beta} f'(\zeta) d\zeta \right|$

$$\leq |z_\alpha - z_\beta| \cdot |f'(\zeta_\alpha^{MAX})| \lesssim \frac{|z_\alpha - z_\beta|}{|Q_\alpha|^{1/2}} \cdot \|f\|_{\mathcal{D}}$$

$\asymp \dfrac{|z_\alpha - z_\beta|}{\min(|Q_\alpha|, |Q_\beta|)^{1/2}} \cdot \|f\|_{\mathcal{D}}$ by the calculation

we used in proving (ii) ∎

<u>Corollary</u>. $|f(z) - f(w)| \lesssim d(z,w)^{1/2} \cdot \|f\|_{\mathcal{D}}^{*}$

i.e. $f \in \mathcal{D} \Rightarrow f : (\Delta, hyp) \to (\mathbb{C}, Euc)$ is ~~Lipschitz~~ Hölder-1/2.

Obs. $\#(\text{chain from } \alpha \text{ to } \beta) \approx d(\alpha,\beta)+1$

$$\approx \log \frac{1+\left|\frac{z-w}{1-z\bar{w}}\right|}{1-\left|\frac{z-w}{1-z\bar{w}}\right|} + 1$$

The tree T. Vertices: $\alpha \in T$.

Edges: (α,β) with $\alpha \underset{T}{\sim} \beta$

Distance: $d_T(\alpha,\beta) = \min\{\#(T\text{-chain from } \alpha \text{ to } \beta)\} - 1$

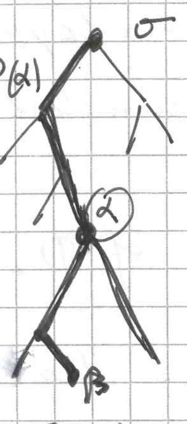
Root: $\sigma = (0,1)$

Geodesics: $[\alpha,\beta]$ = minimal chain between α and β.

Preorder: $\alpha \le \beta \iff \alpha \in [\sigma,\beta]$

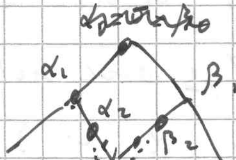
Predecessor set: $P(\alpha) = \{\gamma : \sigma \le \gamma \le \alpha\} = [\sigma,\alpha]$

Successor set: $S(\alpha) = \{\beta : \beta \ge \alpha\} = \{\beta : \alpha \in P(\beta)\}$



- The tree is simpler than the graph.
- The tree does not faithfully represent the geometry of the disc: $\exists\ \alpha_n, \beta_n$ s.t.

$$d_T(\alpha_n,\beta_n) / d_G(\alpha_n,\beta_n) \longrightarrow \infty \quad n \to \infty$$



$d_T(\alpha_n,\beta_n) = 2n$

$d_G(\alpha_n,\beta_n) = 1$

Hardy operator. If $\varphi : T \to \mathbb{C}$,

$$I\varphi(\alpha) = \sum_{\beta \in P(\alpha)} \varphi(\beta) \quad ; \quad I\varphi : T \to \mathbb{C}.$$

$$I\varphi(\alpha) = \sum_{\beta \in T} \varphi(\beta)\, \chi_{P(\alpha)}(\beta) \quad \text{and} \quad (\alpha,\beta) \mapsto \chi_{P(\alpha)}(\beta)$$

can be seen as a kernel.

Its formal ℓ^2-adjoint I^* is

$$I^*\psi(\beta) = \sum_\alpha \chi_{P(\alpha)}(\beta)\,\psi(\alpha) = \sum_\alpha \chi_{S(\beta)}(\alpha)\,\psi(\alpha) ;$$

$$\langle I\varphi, \psi\rangle_{\ell^2} = \sum_\alpha I\varphi(\alpha)\,\overline{\psi(\alpha)} = \sum_\alpha \sum_{\beta \in P(\alpha)} \varphi(\beta)\,\overline{\psi(\alpha)} = \langle \varphi, I^*

Carleson measures for $\mathcal{D}$ – Part 1

$\mu \geq 0$ a Borel measure on Δ.

$\mu \in CM(\mathcal{D})$ is a Carleson measure for $\mathcal{D}$

if $\int_\Delta |f|^2 d\mu \leq [\mu]_{CM(\mathcal{D})} \cdot \|f\|_{\mathcal{D}}^2$

(then $[\mu]_{CM(\mathcal{D})} = |||Id|||^2_{\mathcal{B}(\mathcal{D}, L^2(\mu))}$).

Motivation. (1) Multipliers for $\mathcal{D}$;
(2) Interpolating sequences for $\mathcal{D}$ and its multiplier space; (3) Hankel-type forms on $\mathcal{D}$; (4) Boundary values.

Multiplier space. $g \in H(\Delta)$ is a multiplier for $\mathcal{D}$ if $f \longmapsto M_g f := gf \in \mathcal{B}(\mathcal{D})$.
We let $\|g\|_{M(\mathcal{D})} := \|M_g\|_{\mathcal{B}(\mathcal{D})}$.

Proposition. $g \in M(\mathcal{D}) \iff g \in H^\infty$ and $|g'|^2 dx\,dy \in CM(\mathcal{D})$

Also: $\|g\|_{M(\mathcal{D})} \approx [|g'|^2 dx\,dy]^{1/2}_{CM(\mathcal{D})} + \|g\|_{H^\infty}$.

Proof. ($\Leftarrow$). $\|M_g f\|^2_{\mathcal{D}} = |f(0)|^2|g(0)|^2 + \frac{1}{\pi}\int_\Delta |(fg)'(z)|^2 dx\,dy$

$\lesssim \|g\|^2_\infty |f(0)|^2 + \int_\Delta |f|^2 \cdot |g'|^2 dx\,dy + \int_\Delta |g|^2 \cdot |f'|^2 dx\,dy$

$\leq \|g\|^2_\infty \cdot \|f\|^2_{\mathcal{D}} + [|g'|^2 dx\,dy]_{CM(\mathcal{D})} \cdot \|f\|^2_{\mathcal{D}}$

($\Rightarrow$) Lemma: $M_g^* K_z = \overline{g(z)} \cdot K_z$.

Pf. $M_g^* K_z(w) = \langle M_g^* K_z, K_w\rangle_{\mathcal{D}} = \langle K_z, M_g K_w\rangle_{\mathcal{D}}$

$= \overline{\langle M_g K_w, K_z\rangle} = \overline{M_g K_w(z)} = \overline{g(z)} \cdot \overline{K_w(z)}$

$= \overline{g(z)} \cdot K_z(w)$

Then, $|\overline{g(z)}| \cdot \|K_z\|^2_{\mathcal{D}} = |\overline{g(z)}\, K_z(z)| = |\langle M_g^* K_z, K_z\rangle_{\mathcal{D}}|$

$\leq \|M_g^*\|_{\mathcal{B}(\mathcal{D})} \cdot \|K_z\|^2_{\mathcal{D}} \Longrightarrow \|g\|_{H^\infty} \leq \|M_g^*\|_{\mathcal{B}(\mathcal{D})} = \|g\|_{M(\mathcal{D})}$.

We can now compute:

$$\int_\Delta |f|^2 |g'|^2 dx\,dy \lesssim \int_\Delta |(fg)'|^2 dx\,dy + \int_\Delta |g|^2 |f'|^2 dx\,dy$$

$$\lesssim \|M_g f\|_{\mathcal{D}}^2 + \|g\|_\infty^2 \cdot \|f\|_{\mathcal{D}}^2$$

$$\lesssim \|g\|_{M(\mathcal{D})}^2 \cdot \|f\|_{\mathcal{D}}^2 + \|g\|_\infty^2 \cdot \|f\|_{\mathcal{D}}^2 \leq 2\|g\|_{M(\mathcal{D})}^2 \cdot \|f\|_{\mathcal{D}}^2$$

i.e. $[|g'|^2 dx\,dy]_{CM(\mathcal{D})} \lesssim \|g\|_{M(\mathcal{D})}^2$ ∎

Let $\mu \geq 0$ be a Borel measure on Δ and define $\tilde{\mu} : T \to [0, +\infty)$,

$\tilde{\mu}(\alpha) := \mu(Q_\alpha)$.

<u>Theorem</u>. $\mu \in CM(\mathcal{D}) \iff \ell^2(T) \xrightarrow{I} L^2(T, \tilde{\mu})$.

Message: the problem of characterizing Carleson measures for $\mathcal{D}$ can be completely discretized.

<u>Proof</u>. $(\Leftarrow)$ Let $f \in \mathcal{D}$. Using the basic estimates, we have, for $z \in \Delta$, $z \in Q_\alpha$,

$$|f(z)| \leq |f(0)| + |f(z) - f(0)|$$

$$\lesssim |f(0)| + \sum_{\beta \in P(\alpha)} |f(z_\beta) - f(z_{\beta-1})|, \text{ with } z = z_\alpha.$$

$$\lesssim |f(0)| + \sum_{\beta \in P(\alpha)} \left(\int_{\cup Q_\eta : d_G(\eta, \beta) \leq 2} |f'(w)|^2 du\,dv \right)^{1/2}$$

$$= |f(0)| + I\varphi(\alpha), \text{ with}$$

$$\varphi(\beta) = \left(\int_{\cup Q_\eta : d_G(\eta, \beta) \leq 2} |f'(w)|^2 du\,dv \right)^{1/2}$$

Observe that $I \in \mathcal{B}(\ell^2(T), L^2(T, \tilde{\mu})) \Rightarrow \tilde{\mu}(T) < \infty$

(Test the inequality on $\varphi = \delta_0$; $I\varphi \equiv 1$).

Thus, $\int_\Delta |f(z)|^2 d\mu \lesssim \mu(\Delta)\,|f(0)|^2 + \sum_\alpha \mu(Q_\alpha)\, \mathcal{I}\varphi(\alpha)^2$

$\leq \tilde{\mu}(T)\,|f(0)|^2 + \sum_\alpha \tilde{\mu}(\alpha)\cdot \mathcal{I}\varphi(\alpha)^2 \lesssim$

$\leq |||\mathcal{I}|||^2_{\mathcal{B}(\ell^2(T),\, L^2(\tilde{\mu}))} \quad \|\varphi\|^2_{\ell^2(T)}$ by hypothesis.

where we have let $|f(0)| = \varphi(0)$.

Now, $\|\varphi\|^2_{\ell^2(T)} \approx |f(0)|^2 + \int_\Delta |f'|^2 dx\,dy = \|f\|^2_{\mathcal{D}}$,

because $\forall \eta \in T$ there are at most $C>0$ β s.t. $d_G(\eta, \beta) \leq 2$. Overall,

$$\|\mu\|_{CM(\mathcal{D})} \lesssim |||\mathcal{I}|||^2_{\mathcal{B}(\ell^2,\, L^2(\tilde{\mu}))}.$$

($\Rightarrow$) We dualize the inequality:

$\mu \in CM(\mathcal{D}) \Leftrightarrow \mathcal{D} \xrightarrow{Id} L^2(\mu)$ is bnded

$\Leftrightarrow \mathcal{D} \xleftarrow{\Theta = Id^*} L^2(\mu)$ is bnded.

and similarly $\ell^2 \xrightarrow{\mathcal{I}} L^2(\tilde{\mu})$ is bnded

$\Leftrightarrow \ell^2 \xleftarrow{\mathcal{I}^*_\mu} L^2(\tilde{\mu})$ is bnded

We compute the adjoints.

$g \in L^2(\mu) \Rightarrow \Theta g(z) = \langle \Theta g, K_z\rangle_{\mathcal{D}} = \langle g, K_z\rangle_{L^2(\mu)}$

$= \int_\Delta g(w)\,\overline{K_z(w)}\, d\mu(w)$

$\psi \in L^2(\tilde{\mu}) \Rightarrow \sum_\beta \mathcal{I}^*_\mu \psi(\beta)\,\overline{\varphi(\beta)} = \langle \mathcal{I}^*_\mu \psi, \varphi\rangle_{\ell^2} = \langle \psi, \mathcal{I}\varphi\rangle_{L^2(\tilde{\mu})}$

$= \sum_\alpha \psi(\alpha) \overline{\sum_{\beta \in P(\alpha)} \varphi(\beta)}\ \tilde{\mu}(\alpha) = \sum_\beta \left(\sum_{\alpha \in S(\beta)} \psi(\alpha)\tilde{\mu}(\alpha)\right) \cdot \overline{\varphi(\beta)}$

$\Rightarrow \mathcal{I}^*_\mu \psi(\beta) = \sum_{\alpha \in S(\beta)} \psi(\alpha)\,\tilde{\mu}(\alpha).$

Now, if $g \in L^2(\mu)$, $(\Theta g)'(z) = \frac{d}{dz}\left[\int_\Delta g(w)\left(1 + \log\frac{1}{1 - z\overline{w}}\right) d\mu(w)\right]$

$= \int_\Delta \frac{g(w)\cdot \overline{w}}{1 - z\overline{w}}\, d\mu(w)$

Pick $\Psi \in L^2(\tilde{\mu})$, $\Psi(o)=0$, and define $g : \Delta \to \mathbb{C}$:

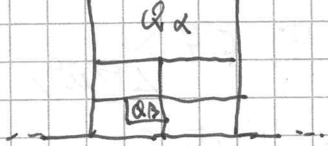
$g(w)\,\bar{w} = \Psi(\alpha)$ for $w \in Q_\alpha$. Suppose $\Psi \geq 0$.

Then,
$$\int_\Delta |g|^2 \, d\mu = \sum_\alpha \int_{Q_\alpha} |g(w)|^2 \, d\mu(w)$$
$$\approx \sum_\alpha \Psi(\alpha)^2 \mu(Q_\alpha) = \|\Psi\|^2_{L^2(\tilde{\mu})}.$$

Also observe that, if $z = re^{i\theta}$ and $w = \rho e^{i\eta}$,
$$\mathrm{Re}\left(\frac{1}{1-z\bar{w}}\right) = \mathrm{Re}\left(\frac{1-z\bar{w}}{|1-z\bar{w}|^2}\right) = \mathrm{Re}\left(\frac{1-r\rho\, e^{i(\theta-\eta)}}{|1-r\rho\, e^{i(\theta-\eta)}|^2}\right)$$
$$= \frac{1-r\rho\cos(\theta-\eta)}{1-2r\rho\cos(\theta-\eta)+r^2\rho^2} = \frac{1-r\rho\cos(\theta-\eta)}{(1-r\rho)^2+2r\rho(1-\cos(\theta-\eta))}$$

so that $\mathrm{Re}\,\dfrac{1}{1-z\bar{w}} \geq 0 \quad \forall z, w \in \Delta$

and
$$\mathrm{Re}\,\frac{1}{1-z\bar{w}} \approx \frac{1-r\rho + r\rho(1-\cos(\theta-\eta))}{(1-r\rho)^2+2r\rho(1-\cos(\theta-\eta))}$$
$$\approx \frac{1-r+r(1-\rho)+2r\rho\sin^2((\theta-\eta)/2)}{[1-r+r(1-\rho)]^2+4r\rho\cdot\sin^2((\theta-\eta)/2)}$$
$$\approx \frac{1-r+1-\rho+(\theta-\eta)^2}{(1-r)^2+(1-\rho)^2+(\theta-\eta)^2} \gtrsim \frac{1}{1-r} \approx |Q_\alpha|^{-1/2}$$
if $z \in Q_\alpha$ and $w \in Q_\beta$, $\beta \in S(\alpha)$



Then,
$$\int_\Delta |(\Theta g)'(z)|^2 \, dx\,dy =$$
$$= \sum_\alpha \int_{Q_\alpha} \left| \int_\Delta \frac{g(w)\,\bar{w}}{1-z\bar{w}} \, d\mu(w) \right|^2 dx\,dy$$
$$\geq \sum_\alpha \int_{Q_\alpha} \left| \int_{\substack{\cup Q_\beta \\ \beta \geq \alpha}} \mathrm{Re}\left(\frac{1}{1-z\bar{w}}\right) \Psi(\beta)\, d\mu(w) \right|^2 dx\,dy$$
$$\gtrsim \sum_\alpha \int_{Q_\alpha} \left(\sum_{\beta \in S(\alpha)} \frac{\Psi(\beta)\,\tilde{\mu}(\beta)}{|Q_\alpha|^{1/2}} \right)^2 dx\,dy = \sum_\alpha \left(\sum_{\beta \in S(\alpha)} \Psi(\beta)\,\tilde{\mu}(\beta) \right)^2$$

$$= \| \mathcal{I}_\mu^* \varphi \|_{\ell^2}^2$$

We can also allow $\varphi(o) \neq 0$, letting $g(w)\overline{w} = \begin{cases} 0 & \text{if } |w| \leq \varepsilon \\ \varphi(o) & \text{if } |w| \geq \varepsilon, \quad w \in Q_o \end{cases}$

~~[illegible]~~ Then, if $\mu \in CM(\mathcal{D})$,

$$\|\varphi\|^2_{L^2(\tilde{\mu})} \approx \int_\Delta |g|^2 d\mu \gtrsim_{\cancel{[illegible]}\,CM(\mathcal{D})} \|\Theta g\|^2_{\mathcal{D}}$$

$$\gtrsim \| \mathcal{I}^*_\mu \varphi \|^2_{\ell^2} \qquad \forall \varphi \geq 0, \ \varphi \in L^2(\tilde{\mu})$$

But this implies that $\|\mathcal{I}^*_\mu \varphi\|^2_{\ell^2} \lesssim \|\varphi\|^2_{L^2(\tilde{\mu})}$ $\forall \varphi \in L^2(\tilde{\mu})$

(the kernel of $\mathcal{I}^*_\mu$ is positive),

hence that $\mathcal{I}^*_\mu : L^2(\tilde{\mu}) \longrightarrow \ell^2$ is bndd,

with $\quad ||| \mathcal{I}^*_\mu |||^2_{\mathcal{B}(L^2(\tilde{\mu}), \ell^2)} \lesssim [\mu]_{CM(\mathcal{D})}$.

Corollary of the proof:

$$||| \mathcal{I}^*_\mu |||^2_{\mathcal{B}(L^2(\tilde{\mu}), \ell^2)} \approx [\mu]_{CM(\mathcal{D})}.$$

Theorem 2. Let μ be a Borel measure on $\overline{T}$ and consider $\mathcal{I}\varphi(\zeta) = \sum_{\alpha \in P(\zeta)} \varphi(\alpha)$ for $\alpha \in \overline{T}$.

Then, $\mathcal{I} : \ell^2(\cancel{[illegible]}) \longrightarrow L^2(\mu)$ is bndd iff $\mu(\overline{T}) < \infty$ and $\mathcal{I}^*(\mathcal{I}^*\mu)^2 \leq C \cdot \mathcal{I}^*\mu$; i.e. if $\forall \alpha \in \overline{T}$:

$$\sum_{\beta \in S(\alpha)} \mu(\overline{S(\beta)})^2 \leq C \cdot \mu(\overline{S(\alpha)}). \qquad (TC)$$

Here, $\overline{S(\alpha)}$ is the completion of $S(\alpha)$ in $\overline{T}$ and for $\zeta \in \partial T$, $P(\zeta)$ is the half-geodesic starting at o ~~and~~ with end-point ζ.

Obs. Coupled with Theorem 1, Theorem 2 provides a geometric characterization of $\mu \in CM(\mathcal{D})$.

Proof. We can write $\| \mathcal{J}^*_\mu \Psi \|^2_{\ell^2} =$ $(\Psi \geq 0)$

$$= \sum_{\beta \in \mathcal{T}} \left(\int_{\overline{S(\beta)}} \Psi \, d\mu \right)^2 = \sum_{\beta \in \mathcal{T}} \left(\frac{1}{\mu(\overline{S(\beta)})} \int_{\overline{S(\beta)}} \Psi \, d\mu \right)^2 \mu(\overline{S(\beta)})^2$$

$$\leq \sum_{\beta \in \mathcal{T}} \left[\mathcal{M}_\mu \Psi(\beta) \right]^2 \rho(\beta) \quad \text{where} \quad \rho(\beta) = \mu(\overline{S(\beta)})^2$$

and $\mathcal{M}_\mu \Psi(\beta) = \max\limits_{\alpha \in P(\beta)} \left[\frac{1}{\mu(\overline{S(\alpha)})} \int_{\overline{S(\alpha)}} \Psi \, d\mu \right]$

is a maximal function.

We have $|\mathcal{M}_\mu \Psi(\beta)| \leq \|\Psi\|_\infty$, $\forall \beta \in \mathcal{T}$.

For $\lambda > 0$, let $E_\lambda = \{\beta : \mathcal{M}_\mu \Psi(\beta) > \lambda\}$.

By definition of $\mathcal{M}_\mu$, $\exists \{\beta_n\}$ in $\mathcal{T}$ s.t.

$E_\lambda = \coprod_n S(\beta_n)$. Then, if $\rho(S(\beta)) \leq C_1 \cdot \mu(\overline{S(\beta)})$,

$$\lambda \cdot \rho(E_\lambda) = \sum_n \lambda \, \rho(S(\beta_n)) \leq C_1 \sum_n \lambda \, \mu(\overline{S(\beta_n)})$$

$$\leq C_1 \sum_n \int_{\overline{S(\beta_n)}} \Psi \, d\mu \quad \left\{ \begin{array}{l} \text{because } \beta_n \text{ is minimal with} \\ \mathcal{M}_\mu \Psi(\beta) > \lambda, \text{ hence} \\ \frac{1}{\mu(\overline{S(\beta_n)})} \int_{\overline{S(\beta_n)}} \Psi \, d\mu > \lambda \end{array} \right.$$

$$\leq C_1 \|\Psi\|_{L^1(\mu)}$$

by disjointness of the $\overline{S(\beta_n)}$'s.

If $\rho(S(\beta)) \leq C_1 \cdot \mu(\overline{S(\beta)})$ $\forall \beta$, then,

$\mathcal{M}_\mu$ is strongly bounded in L^∞ and weakly bounded in L^1, hence it is bounded in L^2-norm, as wished.

Now, $\rho(S(\beta)) \leq C_1 \mu(\overline{S(\beta)})$ is just another way to write $\mathcal{J}^*(\mathcal{J}^* \mu)^2 \leq C_1 \mathcal{J}^* \mu$.

On the other hand, the inequality $\|\mathcal{J}^*_\mu \Psi\|^2_{\ell^2} \leq C_1 \|\Psi\|^2_{L^2(\mu)}$ tested on $\Psi = \chi_{\overline{S(\beta)}}$ ~~implies~~ gives (TC) ∎

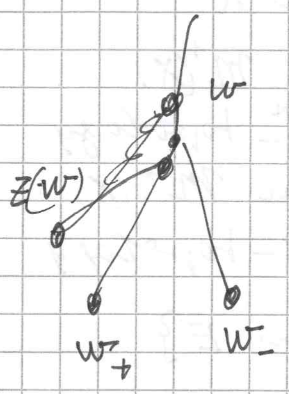
Remarks. (i) Testing $\|J\varphi\|^2_{L^2(\mu)} \le C_1 \cdot \|\varphi\|^2_{\ell^2}$

on $\varphi = \chi_{P(\alpha)}$ we obtain

(SC) $\mu(\overline{S(\alpha)}) \le C_1 \cdot (d(\alpha,o)+1)^{-1}$,

i.e. $\mu(S(z)) \le C_1 \cdot \left(1+\log\frac{1}{1-|z|^2}\right)^{-1}$ (on Δ),

which is necessary in order that $\mu \in CM(\mathcal{D})$.
The following example shows that is not sufficient.
Let $W \hookrightarrow T$ be a subtree of T ($w_1, w_2 \in W \Rightarrow w_1 \wedge w_2 \in W$),
W dyadic, W having 2^n points w s.t. $d(w) = e_n$.



To each $w \in W$ associate $z(w) \in T$,
$d(z(w)) = d_n$ if $d(w) = e_n$, with
$z(w) > w$ and $z(w) \wedge w^+$ having
distance 1 from w.

Let $\mu = \sum_{w \in W} d(z(w))^{-1} \cdot \delta_{z(w)}$

(SC) $\Longleftrightarrow \forall w \in W : \sum_{\substack{s \in W \\ s \ge w}} d(z(s))^{-1} \lesssim d(w)^{-1}$

i.e. $\sum_{n \ge m} 2^{n-m} d_n^{-1} \lesssim e_m^{-1}$ i.e. $\sum_{n\ge m} 2^n/d_n \lesssim 2^m/e_m$

(TC) $\sum_{n \ge m} 2^{n-m}(e_{n+1}-e_n) \cdot \left(\sum_{k \ge n} 2^{k-n} d_k^{-1}\right)^2 \lesssim \sum_{n \ge m} 2^{n-m} d_n^{-1}$

i.e. $\sum_{n \ge m} \frac{e_{n+1}-e_n}{2^n} \cdot \left(\sum_{k \ge n} 2^k/d_k\right)^2 \lesssim \sum_{n \ge m} 2^n/d_n$

Let $d_n = 2^n \cdot n^p$ $(p>1)$ and $e_n = 2^n \cdot n^{p-1}$,
so that $e_{n+1} - e_n \approx e_n$. Then:

(SC) $\Longleftrightarrow \sum_{n \ge m} 1/n^p \lesssim 1/m^{p-1}$, which is correct,

(TC) $\Longleftrightarrow \sum_{n \ge m} n^{p-1} \cdot \left(\sum_{k \ge n} 1/k^p\right)^2 \lesssim \sum_{n \ge m} 1/n^p$, which is not.

POTENTIAL THEORY.

Abstract setting (linear). (X, d) metric, loc. compact; (M, m) measure

$$k : X \times M \to [0, +\infty]$$

$x \mapsto k(x,y)$ lower semicontinuous $\liminf\limits_{x_n \to x_0} k(x_n, y) \ge k(x_0, y)$

$y \mapsto k(x,y)$ measurable

$kf(x) := \int_M k(x,y) f(y)\, dm(y)$ if $f \ge 0$ is measurable

$\check{k}\mu(y) := \int_X k(x,y)\, d\mu(x)$ if $\mu \ge 0$ is a Borel measure

$$\mathcal{E}(\mu, f) := \int_M \check{k}\mu(y)\, dm(y) = \int_X kf(x)\, d\mu(x)$$

Proposition. (a) $x \mapsto kf(x)$ is l.s.c. on X

(b) $\mu \mapsto \check{k}\mu(y)$ is l.s.c. on $\mathcal{M}^+(X)$ w.r.t. weak*-topology

(c) $\mu \mapsto \mathcal{E}(\mu, f)$ is l.s.c. on $\mathcal{M}^+(X)$ w.r.t. weak*-topology

$E \subseteq X$. $\Omega_E = \{ f \ge 0 \text{ in } L^2(M) : kf \ge 1 \text{ on } E \}$

$\mathrm{Cap}_k(E) = \inf\{ \|f\|^2_{L^2(M)} : f \in \Omega_E \}$ is the capacity of E.

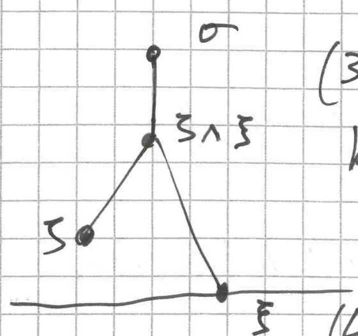
Relevant examples.

(1) Bessel $(2, 1/2)$-capacity on $\partial\Delta$.

$k : \partial\Delta \times \partial\Delta \to [0, +\infty]$; $k(e^{is}, e^{it}) = |e^{is} - e^{it}|^{-1/2}$

(2) Bessel $(2, 1/2)$-capacity on $\partial\overline{T}$

$k : \partial\overline{T} \times \partial\overline{T} \to [0, +\infty]$; $k(\zeta, \xi) = \delta_T(\zeta, \xi)^{-1/2}$



(3) Log-Tree capacity on $\overline{T}$.

$k : \overline{T} \times T \to [0, +\infty)$

$k(\zeta, \alpha) = \chi(\zeta \ge \alpha) = \chi_{P(\zeta)}(\alpha) = \chi_{S(\alpha)}(\zeta)$

(4) Log-capacity on $\overline{\Delta}$:

$k(z, w) = \log \dfrac{1}{|z - w|}$. (Complication due to sign).