Introduction to The Dirichlet Space

MSRI Summer Graduate Workshop

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Multipliers and Carleson Measures

Multipliers

- A function m is called a multiplier of $\mathcal{D}$ if the map M_m defined by $M_m f = mf$ maps $\mathcal{D}$ into itself. If this happens then M_m must be bounded. We denote the space of all multipliers such by $\mathcal{M}(\mathcal{D}) = \mathcal{M}$. For $m \in \mathcal{M}$ we define the multiplier norm of m, $\|m\|_{\mathcal{M}}$, to be the operator norm of M_m. With this norm $\mathcal{M}$ is a Banach space, in fact a commutative Banach algebra.
- It is easy to check that all polynomials are multipliers. The description of all multipliers is complicated.

- If $m \in \mathcal{M}$ then for each $z \in \mathbb{D}$ the function k_z is an eigenvector of M_m^* with eigenvalue $\overline{m(z)}$. To see this we compute.

$$\begin{aligned} M_m^* k_z(w) &= \langle M_m^* k_z, k_w \rangle \\ &= \langle k_z, M_m k_w \rangle \\ &= \langle k_z, m k_w \rangle \\ &= \overline{\langle m k_w, k_z \rangle} \\ &= \overline{m(z) k_w(z)} \\ &= \overline{m(z)} k_z(w). \end{aligned}$$

- One consequence of this is that multipliers are bounded; $\mathcal{M} \subset H^{\infty\cdot}$, the space of bounded holomorphic functions on the disk. More precisely, for $m \in \mathcal{M}$ we have

$$\begin{aligned} \|m\|_{H^\infty} &= \sup\left\{\left|\overline{m(z)}\right|\right\} \leq \sup\left\{|\gamma| : \gamma \text{ an eigenvalue of } M_m^*\right\} \\ &\leq \|M_m^*\| = \|M_m\| = \|m\|_{\mathcal{M}}. \end{aligned}$$

- Another consequence is this: Suppose $S = \bigvee\left\{k_{z_\alpha}\right\}$ is the closed span of a set of reproducing kernels; then S in an invariant subspace for each of the operators M_m^*, $m \in \mathcal{M}$.

- The Hardy space is a RKHS and the same reasoning shows its multiplier space is contained in H^∞. For the Hardy space that is the end of the story; $\mathcal{M}(H^2) = H^\infty$. It is now understood that many results about H^2 and H^∞ are can be constructively thought of as results relating a RKHS and it's multiplier algebra. We will see examples of this in the next lecture.
- For the Dirichlet space boundedness is not the full story. We know $1 \in \mathcal{D}$ and hence $\mathcal{M} \subset \mathcal{D}$. On the other hand we saw $H^\infty \subsetneq \mathcal{D}$. So, not every bounded function is a multiplier (although the examples are not easy to give). We now uncover the rest of the story.

Conditions on Multipliers

- We make the inessential, but technically convenient, restriction to functions that vanish at the origin.
- To show $m \in \mathcal{M}$ we must, for $f \in \mathcal{D}$, bound $\mathcal{D}(mf)$. We have

$$\begin{aligned}
\mathcal{D}(mf) &= \int \left|(mf)'\right|^2 \lesssim \int \left|f'\right|^2 |m|^2 + \int |f|^2 \left|m'\right|^2 \\
&\lesssim \|m\|_{H^\infty}^2 \mathcal{D}(f) + \int |f|^2 \left|m'\right|^2 \\
&= I + II.
\end{aligned}$$

If I and II are under control we have a multiplier. Conversely, if $m \in \mathcal{M}$ then m is bounded and hence I is under control. Once we know that, a bit of further manipulation will show that II is also bounded. Thus m is a multiplier if and only if two conditions hold; first, m is bounded and, second, there is a positive constant $C(m)$ so that for all $f \in \mathcal{D}$

$$\int |f(z)|^2 \left|m'(z)\right|^2 dxdy \leq C(m)^2 \|f\|^2. \qquad \text{(MCM)}$$

Two Extremal Problems

- For $z, w \in \mathbb{D}$ we consider two extremal problems. First, consider

$$\sup\left\{\operatorname{Re} f(z) : f \in \mathcal{D},\ \ \|f\| \leq 1,\ f(w) = 0\right\}.$$

Elementary Hilbert space theory insures that there is a unique function for which the supremum is attained; we will denote that function by $k_{w,z}$. Second, we consider the analogous quantity for multipliers;

$$M = \sup\left\{\operatorname{Re} m(z) : m \in \mathcal{M},\ \ \|m\|_{\mathcal{M}} \leq 1,\ m(w) = 0\right\}.$$

$\boxed{\textbf{If}}$ there is a unique function for which that supremum is attained we will denote it by $m_{w,z}$.

- It will be convenient to formulate the results using the distance function $\delta(z, w)$ given by

$$\delta(z, w) = \left(1 - \left|\left\langle \hat{k}_z, \hat{k}_w \right\rangle\right|^2\right)^{1/2}$$

(The triangle inequality can be obtained starting from the fact that $\left|\left\langle \hat{k}_z, \hat{k}_w \right\rangle\right|^2 = \cos^2(\text{angle between } k_z \text{ and } k_w.))$

Results

- Proposition 1:
- 1) Given $z, w \in \mathbb{D}$ the unique solution to the Hilbert space extremal problem is
$$k_{w,z} = \delta(z,w)^{-1} \left(\hat{k}_z - \left\langle \hat{k}_z, \hat{k}_w \right\rangle \hat{k}_w \right).$$
It produces the extremal value $k_{w,z}(z) = \|k_z\| \, \delta(z,w)$.
- 2) $M \leq \delta(z,w)$. If there is an $m \in \mathcal{M}$, $\|m\|_{\mathcal{M}} \leq 1$, $m(w) = 0$ and $m(z) = \delta(z,w)$ then
$$M = \delta(z,w) \text{ and } m_{w,z} = m$$
and the solutions to the two problems are related by
$$m_{w,z} \hat{k}_z = k_w. \qquad \text{(Relation)}$$

- Proof: 1) First we consider the Hilbert space problem. Note that projecting a purported extremal onto $\operatorname{span}\{k_z, k_w\}$ would produce an improved competitor. Thus the solution must be in this two dimensional space. In that space the supremum will be attained at a unit vector orthogonal to k_w. A Gram-Schmidt computation finishes the argument.
- 2) Now we consider the multipliers. If $t \in \mathcal{M}, \|t\|_{\mathcal{M}} \leq 1,\ t(w) = 0$ then $t\hat{k}_z$ is a competitor for the Hilbert space extremal problem we just solved; that insures $M \leq \delta(z, w)$. If there is an m as described in the statement then $m\hat{k}_z$ would attain the supremum for that extremal problem and hence, by the uniqueness of $k_{w,z}$ we would have $m\hat{k}_z = k_{w,z}$. That establishes that if there is such an m it is unique and (Relation) holds.

Carleson Measures

- A positive measure μ supported on $\mathbb{D}$ is called a Carleson measure (for $\mathcal{D}$) if $\exists C(\mu) > 0$ so that $\forall f \in \mathcal{D}$,

$$\int_{\mathbb{D}} |f|^2 \, d\mu \leq C(\mu)^2 \, \|f\|^2 .$$

The Carleson measure norm of μ, $\|\mu\|_{\mathrm{CM}}$, is defined to be the infimum of possible choices for $C(\mu)$ in this inequality. Equivalently, μ is a Carleson measure if the natural inclusion, J_μ, of $A(\mathbb{D})$ into $L^2(d\mu)$ extends to a bounded map of $\mathcal{D}$ to $L^2(d\mu)$ with norm $\|\mu\|_{\mathrm{CM}}$. We denote the set of all such measures by $\mathcal{CM}(\mathcal{D})$. In this language our second requirement for multipliers, (MCM), is that, $|m|^2 \, dxdy \in \mathcal{CM}(\mathcal{D})$.

- These measures play an important, but quite technical, role in the study of the Dirichlet space and it is important to understand them. We will offer two characterizations of them but will not discuss the proofs in these talks.

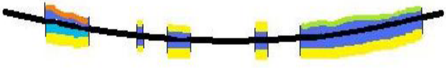
Stegenga's Characterization

In 1980 Stegenga gave a characterization of measures in $\mathcal{CM}(\mathcal{D})$ that used a capacity function to measure the size or sets in $\mathbb{T}$. For $Y \subset \mathbb{T}$ set

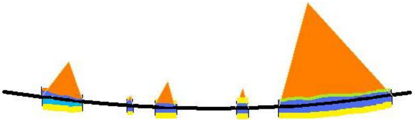
$$\operatorname{cap}(Y) = \inf \left\{ \|f\|^2 : f \in \mathcal{D}, \quad \operatorname{Re} f \geq 1 \text{ on } Y \right\}.$$

- Every countable set has capacity zero; every set of capacity zero has Lebesgue measure zero. The Cantor middle-thirds set has positive capacity and Lebesgue measure zero.
- This is a "Bessel capacity". There is a closely related notion of "logarithmic capacity". The two capacities have the same null sets but there is no simple general comparison of their values. The two are sometimes confounded in the literature. Another quantity, with a similar name, "analytic capacity", is quite different.

- Given $Y \subset \mathbb{T}$, a union of intervals, we define $T(Y)$ the "union of tents over Y" by pictures:



Y



$T(Y)$

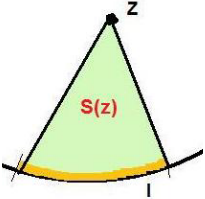
- Theorem : μ is a Carleson measure for $\mathcal{D}$ if and only if there is a constant $C(\mu)$ so that given any $Y \subset \mathbb{T}$, a union of intervals, we have

$$\mu(T(Y)) \leq C(\mu) \operatorname{cap}(Y). \tag{ST}$$

- The same condition, but considered only for the special cases of Y a single interval, is necessary but not sufficient. That condition is sometimes called a "single box" condition. (In some presentations of these ideas our triangles are replaced by boxes.)
- Proof comment: Starting with the definition of capacity is is straightforward to find functions that can be used to show that the condition is necessary. To establish sufficiency requires technical tools ("capacitary strong type inequalities").

A Testing Condition Characterization

- For $z \in \mathbb{D}$ we define the shadow under z, $S(z)$, by a picture. It is the same as the tent over associated interval



$$S(z) = T(I)$$

- Theorem: (Arcozzi, R, Sawyer, '02): μ is a Carleson measure for $\mathcal{D}$ if and only if there is a constant $C(\mu)$ so that $\forall \zeta \in \mathbb{D}$

$$\int_{S(\zeta)} \mu(\overline{S(z)})^2 \frac{dA(z)}{(1-|z|^2)} \leq C(\mu)\mu(\overline{S(\zeta)}). \tag{TC}$$

- A natural approach for trying to characterize the boundedness of the inclusion J_μ is to study the condition obtained by assuming J_μ is uniformly bounded when applied to the reproducing kernels. This type of testing gives a necessary condition. For some questions on a RKHS the necessary condition obtained this way is also sufficient (a "success of the Reproducing Kernel Thesis"). That doesn't happen here. In fact the necessary condition obtained is equivalent to the "one box" condition.
- One can extend this approach by studying the estimates obtained by testing the boundedness of J_μ^* on a set of fundamental functions in $L^2(d\mu)$. The uniform boundedness of J_μ^* on characteristic functions of the sets $S(\zeta)$ yields (TC) and hence that condition must be necessarily. The condition is also sufficient. That is best understood by first looking at an analogous question in a discrete model. That will be discussed in later talks.
- A consequence of these two results is an indirect proof that the conditions (ST) and (TC) are equivalent. This equivalence will also be discussed later.

The Simple Case

- If μ is supported on $[.5, 1]$ then the one box condition is necessary and sufficient. Furthermore is is possible to estimate the relevant capacities.
- μ is supported on $[.5, 1]$ is a Carleson measure if and only if for $.5 < x < 1$, $\mu([x, 1]) \leq C \left|(\log(1 - x))^{-1}\right|$. For instance

$$d\mu(x) = \frac{\chi_{(.5,1)}(x)}{(1 - x)\left(\log(1 - x)\right)^2} dx$$

is a Carleson measure.

- If $X = \{x_n\} \subset (0,1)$ then $\mu_X = \sum_{i=1}^{\infty} \left\| k_{x_n} \right\|^{-2} \delta_{x_n}$ will be a Carleson measure (a question of interest in the next lecture) if and only if

$$\sum_{x_n > x} \log \frac{1}{1 - x_n^2} \leq C \log \frac{1}{1 - x^2}.$$

In particular:

$$\begin{aligned} \text{For } x_n &= 1 - 2^{-n},\ \mu_X \text{ is not a Carleson measure;} \\ \text{For } x_n &= 1 - 2^{-n^2},\ \mu_X \text{ is a Carleson measure.} \end{aligned}$$