Introduction to The Dirichlet Space

MSRI Summer Graduate Workshop

Richard Rochberg
Washington University
St, Louis MO, USA

June 16, 2011

Interpolation and the Pick Property

In this lecture I will:

1. Introduce three questions about interpolation of values; that is, questions of whether there is a function in a particular function space which takes specified target values at particular points. The questions were originally raised and answered for the Hardy space H^2 and its multiplier algebra H^∞. Here I will formulate the problems for $\mathcal{D}$ and $\mathcal{M}(\mathcal{D})$.
2. I will collect the easy necessary conditions for the problem to have solutions. They are the same for the Hardy and Dirichlet space.
3. I will tell what is known about the solutions.
4. There are striking similarities between the Hardy space and Dirichlet space results. It is now understood that there is a deep unification of the two through the Pick property. I will introduce that and discuss it briefly.
5. I will say a few words about invariant subspaces, multipliers and the Pick property.

Questions

In 1916 Georg Pick published a solution to the following problem in classical function theory: what are the necessary and sufficient conditions on sets $\{z_i\}_{i=1}^n, \{w_i\}_{i=1}^n \subset \mathbb{D}$ for there to be $f \in \mathrm{Hol}(\mathbb{D})$ with $\sup \{|f(\zeta)|\} \leq 1$ which performs the interpolation $f(z_i) = w_i$; $i = 1, ...n$?

In line with the earlier comment about the value of viewing H^∞ as a multiplier algebra, this suggests the following question for $\mathcal{M} = \mathcal{M}(\mathcal{D})$:

Pick Interpolation Question: Given $\{z_i\}_{i=1}^n, \{w_i\}_{i=1}^n \subset \mathbb{D}$, is there $m \in \mathcal{M}$ with $\|m\| \leq 1$ and which performs the interpolation $m(z_i) = w_i$; $i = 1, ...n$.

There are two related questions about interpolation of values that we also want to consider.

- **Hilbert Space Interpolation Question:** Characterize the Hilbert space interpolation sequences, HSIS. That is, for which $Z = \{z_i\}_1^\infty \subset \mathbb{D}$ is it true that the scaled restriction map S, from $\mathcal{D}$ to sequences defined on Z, given by $Sf = \left\{ \left\| k_{z_i} \right\|^{-1} f(z_i) \right\}$, maps $\mathcal{D}$ into and onto $\ell^2(Z)$?
- Note that $|f(z_i)| = |\langle f, k_{z_i} \rangle| \leq \|f\| \, \|k_{z_i}\|$ and thus S automatically maps into $\ell^\infty(Z)$.
- **Multiplier Interpolation Question:** Characterize the multiplier interpolation sequences, MIS. That is, for which $Z = \{z_i\}_1^\infty \subset \mathbb{D}$ is it true that the restriction map R, from $\mathcal{D}$ to sequences defined on Z, given by $Rf = \{f(z_i)\}$, maps $\mathcal{M}(\mathcal{D})$ into and onto $\ell^\infty(Z)$?

Necessary Conditions

- **Necessary Condition for Pick Interpolation::** In order for the Pick interpolation problem to have a positive solution it is necessary that the matrix $\mathrm{Mx}(T) = \left[\left(1 - w_j \bar{w}_i\right) k_j\left(z_i\right) \right]_{i,j=1}^{n}$ be positive semidefinite.
- **Proof:** Set $K = \bigvee \left\{ k_{z_i} \right\}_{i=1}^{n}$. If there is an m which does the interpolation then, as we noted earlier, for each i, $M_m^* k_i = \overline{m(z_i)} k_i$ and M_m^* maps K to itself. Let T be the operator M_m^* restricted to K. We have $\|T\| \leq \|M_m^*\| = \|M_m\| = \|m\|_{\mathcal{M}} \leq 1$. Hence for any scalars $\{a_j\}$ we must have

$$\left\| \sum a_j \bar{w}_j k_j \right\|^2 = \left\| T\left(\sum a_j k_j\right) \right\|^2 \leq \left\| \sum a_j k_j \right\|^2.$$

When we compute both norms explicitly, recall that $\langle k_j, k_i \rangle = k_j\left(z_i\right)$, and rearrange terms we find that

$$\sum \left(1 - w_j \bar{w}_i\right) k_j\left(z_i\right) a_j \bar{a}_i \geq 0.$$

The scalars $\{a_j\}$ were arbitrary and thus this is the condition that $\mathrm{Mx}(T)$ is positive semidefinite.

- **Necessary Condition for HSIS:** In order for Z to be a HSIS it is necessary that the points of the sequence be uniformly separated: $\exists \varepsilon > 0$ so that $\forall i, j, i \neq j$,

$$\delta(z_i, z_j) > \varepsilon. \tag{SEP}$$

Also the associated measure, $\mu_Z = \sum_{i=1}^{\infty} \|k_{z_i}\|^{-2} \delta_{z_i}$, must be a Carleson measure:

$$\mu_Z \in CM(\mathcal{D}). \tag{CM}$$

- **Proof:** If S maps into ℓ^2 it must be bounded. Hence there is a $C > 0$ so that for all i, j $\exists$ $f_{i.j} \in \mathcal{D}$ with $\left(Sf_{i,j}\right)_i = 0$, $\left(Sf_{i,j}\right)_j = 1$ and $\|f_{i,j}\| \leq C$. Using the definition of S and the estimate from Proposition 1 we find

$$\begin{aligned} 1 &= \left(Sf_{i,j}\right)_j = \left\|k_{z_j}\right\|^{-1} f_{i,j}(z_j) \\ &\leq \left\|k_{z_j}\right\|^{-1} \left\|f_{i,j}\right\| \left\|k_{z_j}\right\| \delta(z_i, z_j) \\ &\leq C\delta(z_i, z_j) \end{aligned}$$

  which insures separation.
- The fact that S is bounded means that for any f

$$\sum_{i=1}^{\infty} \left\|k_{z_i}\right\|^{-2} \left|f(z_i)\right|^2 \leq C \left\|f\right\|^2 .$$

  The left side is $\int |f|^2 \, d\mu_Z$ and hence this is the condition (CM).

- **Necessary Condition for MIS:** In order for Z to be a MIS it is necessary that it satisfy (SEP) and (CM).
- **Proof:** If R maps onto ℓ^∞ then one can find $g_{i.j} \in \mathcal{M}$ so that $g_{i,j}(z_i) = 0$, $g_{i,j}(z_j) = 1$ and $\|g_{i.j}\|_{\mathcal{M}} \leq C$. On the other hand, from the estimate in the second part of Proposition 1 we conclude that $\|g_{i.j}\|_{\mathcal{M}}^{-1} \leq \delta(z_i, z_j)$. Combining these two estimates gives (SEP).
- The analysis for (CM) is more subtle. One can give a functional analysis argument showing that such a Z must also be an HSIS and then use the previous proposition. We will not discuss that further.

- An aside: There is another question that is historically and technically intertwined with the questions just discussed; it is the Corona Question for the multiplier algebra. Suppose you are given $\{f_i\}_1^n$ in the multiplier algebra. Are there $\{g_i\}$ in the algebra so that $\sum f_i g_i = 1$. Because multipliers must be bounded we have a necessary condition

$$\inf \sum |f_i| \geq \delta > 0.$$

Carleson's '58 work on the MIS sufficed to show that for H^∞, the multiplier algebra of H^2, in a critical special case, this necessary condition is also sufficient. In '62 he proved that the condition is always sufficient (the "Corona Theorem"). The lines of inquiry triggered by those results are still very active and include many of the ideas discussed here.

What is known

- The Hardy space: The questions also make sense, *mutatis mutandis,* for the Hardy space, H^2, and its multiplier algebra, H^∞. In fact that is where these questions were first considered. In each case the answer is that the necessary condition we gave is, in fact, also sufficient– either to insure that there is an interpolating function of to insure that Z in an interpolation sequence. For Pick interpolation this was established by Pick in 1916 , The characterization of the MIS was done by Carleson in '58 , the HSIS were characterized by Shapiro and Shields in '61 .
- The Dirichlet space: It is the same story: the answer in each case is that the necessary condition we gave is, in fact, also sufficient. The Pick type theorem was established by Agler in '88 ; the MIS and HSIS were characterized in '94 by Bishop and, independently, in '94 by Marshall and Sundberg . Unfortunately their papers were not published.

Two natural questions

:

- What are the proofs?
- Why are the results so similar?
- On the first question: the proofs are substantial and I don't want to give a rushed overview of them.
- The answer to the second question leads into a large, and relatively new, research area and I will discuss after a brief digression.

Digression: Onto Interpolation

- I am emphasizing the similarities between the Hilbert space results and the Dirichlet space results. I will just mention here a fundamental difference between the two which will be discussed later. In fact, the HSIS question we formulated asks when a certain normalized restriction map takes the function space into and onto an ℓ^2. The question and answer are very similar for $\mathcal{D}$ and H^2. However, for H^2 the assumption that the restriction map be "into", that is, bounded, can be dropped. In that case if the map is onto ℓ^2 it can be shown to be bounded. (And because that is true the boundedness is sometimes omitted when formulating the question.)
- This is not true for $\mathcal{D}$ and hence there is a question for $\mathcal{D}$ which has no real analog for H^2 : How do you characterize the sequences for which the restriction map is onto ℓ^2 even if it is perhaps not boundedly so. This is an interesting and difficult question and we will hear more about it later.

The Pick Property

- Suppose H is a RKHS of holomorphic functions on $\mathbb{D}$. Suppose further that the reproducing kernel has the form $k_z(w) = \sum_0^\infty a_n(\bar{z}w)^n$ with $a_0 > 0$. Define $\{c_n\}$ by the formula

$$\left(\sum_0^\infty a_n t^n\right)\left(\sum_0^\infty c_n t^n\right) = 1 \text{ for } t \text{ near } 0.$$

  We say such an H has the *complete Pick property* if $c_n \leq 0 \quad \forall n \geq 1$.
- The Hardy space clearly has this property, the Bergman space clearly does not. The Dirichlet space has the property but showing that requires some clever work with power series.

Consequences of the Pick Property:

1. For any H as above one can again formulate the Pick interpolation question. The necessary condition we obtained earlier still applies. If H has the complete Pick property then this necessary condition is sufficient.
2. There are matricial versions of the Pick interpolation question and an analogous matricial necessary condition for having a solution. Again, if H has the complete Pick property then this necessary condition is sufficient.
3. If H has the complete Pick property its HSIS and its MIS are the same. (It is conjectured that, in this case, the conditions (CM) and (SEP) are, together, necessary and sufficient.)
4. If H has the complete Pick property then a great deal of structural information about multipliers and invariant subspaces follows. We will see a simple example in a moment.

- The workhorse idea is statement 2. The definition we gave is a devolved version that is simple, restricted, and hides a lot.

An Application of Pick's Theorem.

In the second talk we considered an extremal problem for multipliers of the Dirichlet space and we obtained a conditional result: if we could find $m \in \mathcal{M}$ with certain properties then the extremal problem had a unique solution and we had a formula for it. We now complete the analysis by using Pick's theorem to show there is such an m.

- **Proposition:** There is an $m \in \mathcal{M}$, with $\|m\|_{\mathcal{M}} \leq 1$, $m(w) = 0$, and $\operatorname{Re} m(z) = \delta(z, w)$.
- **Proof:** Pick's theorem for the Dirichlet space tells us that the necessary and sufficient condition for the existence of an m which satisfies the first conditions and has $\operatorname{Re} m(z) = M$ is that the 2×2 matrix $T = \left[(1 - w_j \bar{w}_i)\, k_j(z_i)\right]_{i,j=1}^{2}$ built using the data

$$z_1 = z,\ z_2 = w,\ w_1 = 0,\ w_2 = M$$

must be positive definite. We have

$$T = \begin{pmatrix} k_1(z) & k_2(z) \\ k_1(w) & (1 - M^2)\, k_2(w) \end{pmatrix}.$$

The upper left entry is positive so T will be positive definite if the determinant is positive

$$\det T = k_1(z)\left(1 - M^2\right) k_2(w) - |k_2(z)|^2$$

and this is positive exactly if $M \leq \delta(z, w)$.

Pick's Theorem, Multipliers, and Invariant Subspaces

The simplest nonconstant multiplier on H^2 is z. Multiplication by z acting on H^2 shifts each normalized monomial one place and the operator is called the Shift. Our understanding of the operator theory associated with the shift is extremely rich. The modern theory begins with Beurling's description of the invariant subspaces using the inner-outer factorization Multiplication by z acting on $\mathcal{D}$, M_z, is called the Dirichlet Shift. Some of the most basic questions about its operator theory are open: What are the invariant subspaces of M_z? What are the invariant subspaces of M_z^*? How do you characterize the $f \in \mathcal{D}$ that are not contained in any nontrivial M_z invariant subspace (that is, the *cyclic vectors*).

Some of the theory now used to study the Dirichlet Shift uses the interrelationship between Hilbert space extremal problems, multiplier extremal problems, and Pick's theorem. The proposition we just proved is a model example of that. That proposition establishes a link between the invariant subspace of functions that vanish at w, a Hilbert space extremal problem associated with w (and an auxiliary base point z) and the solution to a multiplier extremal problem.

Some of the theory now in place for invariant subspaces of the Dirichlet Shift uses the interrelationship between Hilbert space extremal problems, multiplier extremal problems, and Pick's theorem. I want to take a moment to state some results of that sort. I will state them in minimal generality. (In that context some of them are almost trivial.) However these are all results that mirror results for the Hardy Shift, they are all results that extend to more general invariant subspaces of the Dirichlet Shift, and, in fact, because the primary tools are the tools we have been discussing, they have variations for general RKHS with the complete Pick property.

For $\lambda \in \mathbb{D}$, $\lambda \neq 0$ let V_λ be the subspace $\mathcal{D}$ consisting of all functions which vanish at λ. Set $zV_\lambda = \{zf : f \in V_\lambda\}$ then

1. V_λ is an invariant subspace
2. $\dim(V_\lambda \ominus zV_\lambda) = 1$
3. If h_λ is the unit vector in $V_\lambda \ominus zV_\lambda$ which is positive at the origin then
$$h_\lambda = k_{\lambda,0} = m_{\lambda,0}.$$
In particular h_λ is a contractive multiplier.
4. V_λ is generated by h_λ; that is, V_λ is is the smallest closed invariant

Comments:

1. The most easily described invariant subspaces are the subspaces of functions which vanish on a specified set Z. However, as we will see in the next lecture, it is not at all clear which Z give a nontrivial subspace
2. This is called the "codimension one" property. It is elementary in this case. The subtle fact is that every invariant subspace of the Dirichlet Shift has it.
3. It is particularly interesting to know that V contains a multiplier, see 5. below.
4. Again, this case is elementary. The subtle fact is that the h_λ of 3. will always be a generator.