GEOMETRY OF THE TREE AND ITS BOUNDARY.

T: the dyadic tree; $o \in T$ its root; d_T: the edge-counting distance (a model for hyperbolic geometry).

A *path* in T is a sequence of adjacent edges and a *geodesic* is a path in which no edge appears more than once. $\forall \alpha, \beta \in T \; \exists!$ geodesic Γ joining α and β. $d_T(\alpha, \beta) = \#$ edges in Γ.

$\partial T = \{\Gamma : \Gamma$ is a half-infinite geodesic starting at $o\}$. If $\zeta \in \partial T$, $\zeta = (\zeta_n)_{n=0}^{\infty}$ can be seen as a sequence of points $\zeta_n \in T$.

Let $d_T(\alpha) = \#(P(\alpha)) = d_T(\alpha, o) + 1$.

For $\zeta \in \partial T$, $P(\zeta) = \{\zeta_n\}_{n=0}^{\infty}$ is the set of its vertices. $\overline{T} = T \cup \partial T$.

Let $\zeta, \xi \in T$. $\zeta \wedge \xi := \min(P(\zeta) \cap P(\xi))$; i.e. $P(\zeta \wedge \xi) = P(\zeta) \cap P(\xi)$.

Assign to each edge from x s.t. $d(x) = n$ to any of its children $x_\pm$, $d(x_\pm) = n+1$, the length 2^{-n}. Then, we can define a distance $\delta_T : \overline{T} \times \overline{T} \to [0, \infty)$

$\delta_T(\zeta, \xi) =$ length of the geodesic between ζ and ξ $= 2\left\{\left(\frac{1}{2}\right)^{d(\zeta\wedge\xi, o)} - \frac{1}{2}\cdot\left[\left(\frac{1}{2}\right)^{d(\zeta, o)} + \left(\frac{1}{2}\right)^{d(\xi, o)}\right]\right\}$

$= 4 \cdot \left\{\left(\frac{1}{2}\right)^{d(\zeta\wedge\xi)} - \frac{1}{2}\left[\left(\frac{1}{2}\right)^{d(\zeta)} + \left(\frac{1}{2}\right)^{d(\xi)}\right]\right\}$

Obs. that $\delta_T(\zeta, \xi) = 4 \cdot \left(\frac{1}{2}\right)^{d(\zeta \wedge \xi)}$ if $\zeta, \xi \in \partial T$.

- $(\overline{T}, \delta_T)$ is compact, complete.
- ∂T is compact; ∂T is the accumulation set of T.
- ∂T is perfect; it has Hausdorff dimension 1.

Examples with log-capacity on trees.

(i) $\operatorname{Cap}(\{\alpha\}) = d(\alpha)^{-1} \quad \forall \alpha \in T$

Among f's s.t. $If(\alpha) \geq 1$, $\|f\|_{\ell^2}^2$ is minimal if $\operatorname{supp}(f) \subseteq P(\alpha)$. For f supported in $P(\alpha)$ s.t. $If(\alpha) \geq 1$, $\|f\|_{\ell^2}^2$ is reduced if we replace f by $\operatorname{MAX}(f,0) / (I\operatorname{MAX}(f,0))(\alpha)$. If $P(\alpha) = \{\alpha_j\}_{j=0}^{n}$, and $If(\alpha_n) = \sum_{j=0}^{n} f(\alpha_j) = 1$, $f \geq 0$, then

$$\|f\|_{\ell^2}^2 = \sum_{j=0}^{n} f(\alpha_j)^2 \geq \left(\sum_{j=0}^{n} f(\alpha_j)\right)^2 / (n+1) \quad \text{by Cauchy-Schwarz}$$

$$= 1/(n+1) \quad \text{with equality if } f(\alpha_j) = \frac{1}{n+1} \ \forall j$$

(ii) Lemma the inf in the def. of log-Cap on T can be taken among f's s.t. $f \geq 0$ and $\operatorname{supp}(f) \subseteq \bigvee_{\zeta \in E} P(\zeta)$.

(iii) Let $\tilde{E}$ be the set of the minimal points in E: $\zeta \in \tilde{E} \iff \zeta \in E$ and for no $\alpha \in E$ we have $\zeta \in S(\alpha) / \{\alpha\}$. Then, $\operatorname{Cap}(E) = \operatorname{Cap}(\tilde{E})$. and let $E(\text{hence } \tilde{E}) \subseteq T$

(iv) Let $T_E = T \setminus \bigcup_{\alpha \in \tilde{E} \cap T} (S(\alpha)/\{\alpha\})$. The inf might be taken among f's s.t. $F = If$ is harmonic on $T_E \setminus (E \cup \{o\})$; i.e. s.t.

$$F(x) = \frac{1}{3} \sum_{y \underset{T}{\sim} x} F(y) \qquad \forall x \in T \setminus (E \cup \{o\}).$$

[In fact, the harmonicity is on $T \setminus (E \cup \{o\})$.

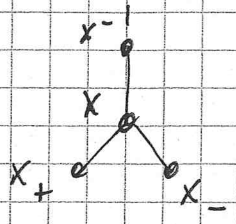
(v) Let $E \subseteq T$; $x, x_+, x_- \in T_E$. Let Cap_x be capacity measured on the tree $S(x)$ from the root x.

Then,
$$\operatorname{Cap}_x(E_x) = \frac{\operatorname{Cap}_{x_+}(E_{x_+}) + \operatorname{Cap}_{x_-}(E_{x_-})}{1 + \operatorname{Cap}_{x_+}(E_{x_+}) + \operatorname{Cap}_{x_-}(E_{x_-})},$$

where $E_x = E \cap S(x)$.

Formula (v) is one of the main reasons for us to care about tree-capacity.

Proof of (iv). Given that $F(x^{-1})$, $F(x_+)$, $F(x_-)$ remain constant, we want to minimize $f(x)^2 + f(x_+)^2 + f(x_-)^2 =$



$$= f(x)^2 + (c_+ - f(x))^2 + (c_- - f(x))^2$$

(with $c_\pm$ constant). This happens if

$$0 = f(x) - (c_+ - f(x)) - (c_- - f(x)) = f(x) - f(x_+) - f(x_-),$$

which is equivalent to (iv).

Proof of (v). Observe that $F = \mathcal{I}f$ is harmonic at x iff $f(x_+) + f(x_-) = f(x)$. Let $H_\pm = \mathcal{I}h_\pm$ be the extremal functions for $\mathrm{Cap}_{x_\pm}(E_{x_\pm})$ and let

$h = \lambda_\pm h_\pm$ on $S(x_\pm)$. Since $\mathcal{I}_{x_\bullet} h = 1$ on E_x, then

$$h(x) = \mathcal{I}_x h(y) - \mathcal{I}_{x_+}(h(y)) = 1 - \lambda_+ \quad (y \in E_{x_+})$$

and $h(x) = 1 - \lambda_-$, so that $(\lambda = \lambda_+ = \lambda_-)$

$$\|h\|^2_{\ell^2(S(x))} = (1-\lambda)^2 + \lambda^2 \|h_+\|^2_{\ell^2(S(x_+))} + \lambda^2 \|h_-\|^2_{\ell^2(S(x_-))}$$

$$= (1-\lambda)^2 + \lambda^2 \left(\mathrm{Cap}_{x_+}(E_{x_+}) + \mathrm{Cap}_{x_-}(E_{x_-})\right)$$ by extremality of $h_\pm$.

Minimizing w.r.t. λ, $\lambda = \dfrac{1}{1 + \mathrm{Cap}_{x_+}(E_{x_+}) + \mathrm{Cap}_{x_-}(E_{x_-})}$,

which gives $\mathrm{Cap}_x(E_x) = \|h\|^2_{\ell^2(S(x))} = \dfrac{\mathrm{Cap}_{x_+}(E_{x_+}) + \mathrm{Cap}_{x_-}(E_{x_-})}{1 + \mathrm{Cap}_{x_+}(E_{x_+}) + \mathrm{Cap}_{x_-}(E_{x_-})}$

as wished.

We have implicitly used:

<u>Lemma</u>. If $E \subseteq \mathbb{T}$, an extremal h exists (<u>Proof</u>. Exercise).

We have a corollary.

(vi) If $H = \mathcal{I}h$ is the extremal for $E \subseteq \mathbb{T}$, then $\mathrm{Cap}(E) = h(o)$.

<u>Pf</u>. $h(o) = 1 - \lambda = \dfrac{\mathrm{Cap}_{o_+}(E) + \mathrm{Cap}_{o_-}(E)}{\dots} = \mathrm{Cap}(E)$

MORE ON ABSTRACT POTENTIAL THEORY.

Let $E \subseteq X$ and $\Omega_E = \{ f \geq 0 : \kappa f \geq 1 \text{ q.e. on } E \}$,
where "q.e." means "but for a set of null capacity".
Then
$$\mathrm{Cap}_k(E) = \inf \{ \|f\|^2_{L^2(M)} : f \in \Omega_E \}.$$

Theorem: $\mathrm{Cap}(E) < \infty \Rightarrow \exists!\ f^E \in L^2_+(M)$ s.t.
$\kappa f^E \geq 1$ q.e. on E and $\mathrm{Cap}(E) = \|f\|^2_{L^2(M)}$.

Proposition. (1) $\mathrm{Cap}(\emptyset) = 0$
(2) $E \subseteq F \Rightarrow \mathrm{Cap}(E) \leq \mathrm{Cap}(F)$
(3) $K_i \downarrow K$ compact $\Rightarrow \mathrm{Cap}(K_i) \downarrow \mathrm{Cap}(K)$
(4) $E_i \nearrow E \Rightarrow \mathrm{Cap}(E_i) \nearrow \mathrm{Cap}(E)$

$E \subseteq X$ is capacitable if $\mathrm{Cap}(E) = \sup\{ \mathrm{Cap}(K) : K \subseteq E \text{ compact}\}$

Theorem. All Suslin sets (hence, Borel sets) are capacitable.

Theorem. $E_i \nearrow E$ and $\mathrm{Cap}(E) < \infty \Rightarrow f^{E_i} \xrightarrow[i]{} f^E$ in $L^2(M)$

Dual definition of capacity:

Theorem. K compact in $X \Rightarrow \mathrm{Cap}^{1/2}(K) = \sup\{ \mu(K) : \mathrm{supp}(\mu) \subseteq K$
and $\|\check{\kappa}\mu\|^2_{L^2(M)} \leq 1 \}$

The Theorem extends to Suslin sets.

Thm. K compact in $X \Rightarrow \exists\, \mu^K \geq 0$, $\mathrm{supp}(\mu^K) \subseteq K$ s.t.
$f^K = \check{\kappa}\mu^K$ and
$$\mathrm{Cap}(K) = \mu^K(K) = \int_M (\check{\kappa}\mu^K)^2\, dm = \int_X \kappa f^K \cdot d\mu^K$$

The potential of $\mu \geq 0$ is:
$$V^\mu(x) = \kappa(\check{\kappa}\mu)(x) = \int_M \kappa(x,y) \int_X g(z,y)\, d\mu(z) \cdot dm(y)$$

and the energy is $\mathcal{E}(\mu) = \int_X V^\mu\, d\mu = \int_M (\check{\kappa}\mu)^2\, dm$

__Thm.__ K compact in $X \Rightarrow Kf^K = V^{\mu^K} \leq 1$ on $\operatorname{supp}(\mu_K)$ and $\operatorname{cap}(K) = \max\{\mu(K) : \mu \in \mathcal{M}_+(K)$ and $V^\mu \leq 1$ on $\operatorname{supp}(\mu)\}$

__Thm.__ Supp. $\operatorname{cap}(E) < \infty$ and that M is loc. compact, with the property that $\varphi \in C_c(M) \Rightarrow K\varphi$ is continuous and $\lim_{x\to\infty} K\varphi(x) = 0$.

Then, $\exists\, \mu^E$ in $\mathcal{M}_+(\bar{E})$ s.t.

(i) $f^E = \check{K}\mu^E$; (ii) $Kf^E \geq 1$ q.e. on E;

(iii) $Kf^E \leq 1$ q.e. on $\operatorname{supp}(\mu^E)$

(iv) $\operatorname{cap}(E) = \mu^E(\bar{E}) = \int_M (\check{K}\mu^E)^2 \, dm = \int_X Kf^E \cdot d\mu^E$

For all of this, see Adams–Hedberg.

__Exercise__ (iv) – (vi) extend to $E \subseteq T \cup \partial T = \overline{T}$

(In (vi), the extremal H satisfies $H \geq 1$ q.e. on E).

It is also useful to know that

__Lemma.__ If $E = \bigcup_j \partial S(\alpha_j)$ and $\tilde{E} = \{\alpha_j\}_j \subseteq T$,

Then $\operatorname{cap}(E) \leq \operatorname{cap}(\tilde{E}) \leq 4 \cdot \operatorname{cap}(E)$.

__Pf.__ The 1^{st} inequality is trivial.

Let φ be the extremal function for $\operatorname{cap}(E)$:

$I\varphi \geq 1$ q.e. on $\bigcup \partial S(\alpha_j)$ [Hence, $I\varphi \geq 1$ on $\bigcup \partial S(\alpha_j)$] and $\operatorname{cap}(E) = \|\varphi\|^2_{\ell^2}$. Then $\varphi = I^*\mu$, where μ is the equilibrium measure and μ is constant (exercise!) on $\partial S(\alpha_j)$: $\exists\, \Gamma_j$ s.t.

$\varphi = \Gamma_j \, 2^{-d(\beta)} \quad \forall \beta \in S(\alpha_j)$. Then,

$1 - I\varphi(\alpha_j) = I\varphi(\zeta) - I\varphi(\alpha_j) = \Gamma_j \, 2^{-d(\alpha_j)} = \varphi(\alpha_j)$

$\Rightarrow I\varphi(\alpha_j) = 1 - \varphi(\alpha_j) \geq 1 - 1/d(\alpha_j) \geq 1/2 \Rightarrow \varphi/2$ is good for $\operatorname{cap}(\tilde{E})$.

COMPARISON OF VARIOUS CAPACITIES.

Theorem. Let $E \subseteq \partial T$ and let

$\mathrm{Cap}_T(E)$ be the log-Tree capacity and

$\mathrm{Cap}^B_T(E)$ be the Bessel $(2, 1/2)$-Tree capacity.

Then,
$$\mathrm{Cap}_T(E) \approx \mathrm{Cap}^B_T(E).$$

Proof. It suffices to show that the energies of a measure $\mu \geq 0$, $\mathrm{supp}(\mu) \subseteq \partial T$, are comparable.

$$\mathcal{E}_T(\mu) = \sum_\alpha I^*\mu(\alpha)^2 = \sum_\alpha \left(\int_{\partial T} \chi(\zeta \geq \alpha)\, d\mu(\zeta) \right)^2$$

$$= \int_{\partial T} d\mu(\zeta) \int_{\partial T} d\mu(\xi)\; K_T(\zeta, \xi), \text{ where}$$

$$K_T(\zeta, \xi) = \sum_{\zeta, \xi \geq \alpha} 1 = d(\zeta \wedge \xi).$$

$$\mathcal{E}^B_T(\mu) = \int_{\partial T} \left(\int_{\partial T} \delta_T(\eta, \zeta)^{-1/2}\, d\mu(\zeta) \right)^2 d\eta \quad (d\eta : \text{linear measure on } \partial T).$$

$$= \int_{\partial T} d\mu(\zeta) \int_{\partial T} d\mu(\xi) \cdot K^B_T(\zeta, \xi),$$

where
$$K^B_T(\zeta, \xi) = \int_{\partial T} \delta_T(\eta, \zeta)^{-1/2} \cdot \delta_T(\eta, \xi)^{-1/2}\, d\eta$$

$$= \sum_{\alpha = o}^{\zeta \wedge \xi} 2^{d(\alpha)/2} \cdot 2^{d(\alpha)/2} \cdot 2^{-d(\alpha)} + \sum_{\substack{\alpha \geq \zeta \wedge \xi \\ \alpha \neq \zeta \wedge \xi}}^{\zeta} 2^{-d(\alpha)} \cdot 2^{d(\alpha)/2} \cdot 2^{d(\zeta \wedge \xi)/2} + (\text{same with } \xi \text{ inst. of } \zeta)$$

$$= d(\zeta \wedge \xi) + 2 \cdot C, \text{ with } C = \frac{1}{\sqrt{2} - 1}, \text{ I believe.} \blacksquare$$

Consider the map $\partial T' \xrightarrow{\Lambda} [0,1]$.

$$\zeta = (\zeta_n)_{n=1}^{\infty} \longmapsto \left(\bigcap_{n\geq 1} \overline{I(\zeta_n)}\right)/2\pi$$

Exercise: Λ is Lipschitz from $(\partial T, \bar{d}_{T'})$ to $[0,1]$.

Let ~~$\mu \in \mathcal{M}([0,1])$~~, $\nu \in \mathcal{M}_+(\partial T')$

$\Lambda_{\#}\nu(F) := \nu(\Lambda^{-1}(F))$ defines a measure on $\partial T'$. In the inverse direction, if $\mu \in \mathcal{M}([0,1])$ and $E \subseteq \partial T'$ is a Borel set,

$$\Lambda^{\#}\mu(E) = \int_{[0,1]} \frac{\#(\Lambda^{-1}(\{x\}) \cap E)}{\#(\Lambda^{-1}(\{x\}))}\, d\mu(x).$$

i.e. $\Lambda^{\#}\mu$ distributes the weight $\mu(\{x\})$ among the (at most 2) preimages of x in $\partial T'$.

Exercise. (i) If E is Borel measurable in ~~$[0,1]$, then~~ $\partial T'$, then $x \longmapsto \dfrac{\#(\Lambda^{-1}(\{x\})\cap E)}{\#(\Lambda^{-1}(\{x\}))}$ is Borel (m) on $[0,1]$.

(ii) If μ ~~is non-atomic~~ has no atoms, then $\Lambda^{\#}\mu$ has no atoms and

$$\Lambda^{\#}\mu(\partial S_{T'}(\alpha)) = \mu(I(\alpha)) \quad \forall \alpha \in T'.$$

Theorem. Let $E \subseteq \partial T'$ be closed. Then,

(1) $\mathrm{Cap}^B_{T'}(E) \approx \mathrm{Cap}^B(\Lambda(E))$ ($\Lambda(E)$ is closed in $[0,1]$).

Let $F \subseteq [0,1]$ closed. Then,

(2) $\mathrm{Cap}^B(F) \approx \mathrm{Cap}^B_{T'}(\Lambda^{-1}(F))$.

~~Obs. It is clear that (2) $\Rightarrow$ (1):~~

~~Cap~~