Interpolating sequences and bilinear Hankel forms for the classical Dirichlet space

The classical Dirichlet space

Eric T.Sawyer

McMaster University

June 20, 2011

Part 1

Overview

Overview

- In these lectures we discuss two theorems regarding the function theory of the classical Dirichlet space $\mathcal{D}$:

Overview

- In these lectures we discuss two theorems regarding the function theory of the classical Dirichlet space $\mathcal{D}$:

1. The characterization of interpolating sequences $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ for $\mathcal{D}$ and its multiplier algebra $M_{\mathcal{D}}$ in terms of separation of the points z_j and embedding of the Dirichlet space in a Lebesgue space $L^2(\mu_Z)$, where $\mu_Z = \sum_{j=1}^{\infty} \frac{1}{1+\beta(0,z_j)} \delta_{z_j}$;

Overview

- In these lectures we discuss two theorems regarding the function theory of the classical Dirichlet space $\mathcal{D}$:

1. The characterization of interpolating sequences $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ for $\mathcal{D}$ and its multiplier algebra $M_{\mathcal{D}}$ in terms of separation of the points z_j and embedding of the Dirichlet space in a Lebesgue space $L^2(\mu_Z)$, where $\mu_Z = \sum_{j=1}^{\infty} \frac{1}{1+\beta(0,z_j)} \delta_{z_j}$;
2. A characterization of the holomorphic functions b (called symbols) for which the bilinear form $B_b(f,g) \equiv \langle fg, b \rangle_{\mathcal{D}}$ is bounded on $\mathcal{D} \times \mathcal{D}$.

Overview

- In these lectures we discuss two theorems regarding the function theory of the classical Dirichlet space $\mathcal{D}$:

1. The characterization of interpolating sequences $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ for $\mathcal{D}$ and its multiplier algebra $M_{\mathcal{D}}$ in terms of separation of the points z_j and embedding of the Dirichlet space in a Lebesgue space $L^2(\mu_Z)$, where $\mu_Z = \sum_{j=1}^{\infty} \frac{1}{1+\beta(0,z_j)} \delta_{z_j}$;
2. A characterization of the holomorphic functions b (called symbols) for which the bilinear form $B_b(f,g) \equiv \langle fg, b \rangle_{\mathcal{D}}$ is bounded on $\mathcal{D} \times \mathcal{D}$.

- These theorems have some counterparts for $p \neq 2$ and $n > 1$, but the proofs are often more difficult and the results incomplete.

Theorem

$Z = \{z_j\}_{j=1}^{\infty}$ is interpolating for $\mathcal{D}$, equivalently $M_{\mathcal{D}}$, if and only if Z is separated and $\mu_Z \equiv \sum_{j=1}^{\infty} \frac{1}{1+\beta(0,z_j)}$ is a Carleson measure.

- A sequence of points $Z = \{z_j\}_{j=1}^{\infty}$ in the unit disk $\mathbb{D}$ is said to be *interpolating* for $\mathcal{D}$ if the weighted restriction map $\mathcal{R}_Z : \mathcal{D} \to \ell^{\infty}$ given by

$$\mathcal{R}_Z f \equiv \left\{ \frac{f(z_j)}{\sqrt{1+\beta(0,z_j)}} \right\}_{j=1}^{\infty}, \qquad \beta(0,z_j) \approx \ln \frac{1}{1-|z_j|},$$

maps *into* and *onto* ℓ^2; and interpolating for the multiplier algebra $M_{\mathcal{D}}$ if $\mathcal{R} : M_{\mathcal{D}} \to \ell^{\infty}$ is *onto* where $\mathcal{R}f = \{f(z_j)\}_{j=1}^{\infty}$.

Theorem

$Z = \{z_j\}_{j=1}^{\infty}$ is interpolating for $\mathcal{D}$, equivalently $M_{\mathcal{D}}$, if and only if Z is separated and $\mu_Z \equiv \sum_{j=1}^{\infty} \frac{1}{1+\beta(0,z_j)}$ is a Carleson measure.

- A sequence of points $Z = \{z_j\}_{j=1}^{\infty}$ in the unit disk $\mathbb{D}$ is said to be *interpolating* for $\mathcal{D}$ if the weighted restriction map $\mathcal{R}_Z : \mathcal{D} \to \ell^\infty$ given by

$$\mathcal{R}_Z f \equiv \left\{ \frac{f(z_j)}{\sqrt{1+\beta(0,z_j)}} \right\}_{j=1}^{\infty}, \qquad \beta(0,z_j) \approx \ln \frac{1}{1-|z_j|},$$

  maps *into* and *onto* ℓ^2; and interpolating for the multiplier algebra $M_{\mathcal{D}}$ if $\mathcal{R} : M_{\mathcal{D}} \to \ell^\infty$ is *onto* where $\mathcal{R}f = \{f(z_j)\}_{j=1}^{\infty}$.
- The sequence Z is *separated* if $\inf_{i \neq j} \beta(z_i, z_j) > 0$.

Theorem

$Z = \{z_j\}_{j=1}^{\infty}$ is interpolating for $\mathcal{D}$, equivalently $M_{\mathcal{D}}$, if and only if Z is separated and $\mu_Z \equiv \sum_{j=1}^{\infty} \frac{1}{1+\beta(0,z_j)}$ is a Carleson measure.

- A sequence of points $Z = \{z_j\}_{j=1}^{\infty}$ in the unit disk $\mathbb{D}$ is said to be *interpolating* for $\mathcal{D}$ if the weighted restriction map $\mathcal{R}_Z : \mathcal{D} \to \ell^{\infty}$ given by

$$\mathcal{R}_Z f \equiv \left\{ \frac{f(z_j)}{\sqrt{1+\beta(0,z_j)}} \right\}_{j=1}^{\infty}, \qquad \beta(0,z_j) \approx \ln \frac{1}{1-|z_j|},$$

  maps *into* and *onto* ℓ^2; and interpolating for the multiplier algebra $M_{\mathcal{D}}$ if $\mathcal{R} : M_{\mathcal{D}} \to \ell^{\infty}$ is *onto* where $\mathcal{R}f = \{f(z_j)\}_{j=1}^{\infty}$.
- The sequence Z is *separated* if $\inf_{i \neq j} \beta(z_i, z_j) > 0$.
- A positive measure μ is a Carleson measure if $\mathcal{D} \subset L^2(\mu)$.

- For a holomorphic *symbol function* b define the bilinear form

$$T_b(f,g) \equiv \langle fg, b\rangle_{\mathcal{D}} = fg\overline{b}(0) + \int_{\mathbb{D}} (f'g + fg')\,\overline{b'}.$$

- For a holomorphic *symbol function* b define the bilinear form

$$T_b(f,g) \equiv \langle fg, b\rangle_{\mathcal{D}} = fg\overline{b}(0) + \int_{\mathbb{D}} (f'g + fg')\,\overline{b'}.$$

- A result of Rochberg and Wu is that the *half* forms $\int_{\mathbb{D}} (f'g)\,\overline{b'}$ and $\int_{\mathbb{D}} (fg')\,\overline{b'}$ are each bounded on $\mathcal{D} \times \mathcal{D}$ *if and only if* $b \in \mathcal{X}$, where $\mathcal{X}$ is the space of holomorphic functions with norm

$$\|b\|_{\mathcal{X}} \equiv |b(0)| + \left\| |b'(z)|^2\, dA \right\|_{CM(\mathcal{D})}^{\frac{1}{2}} < \infty.$$

- For a holomorphic *symbol function b* define the bilinear form

$$T_b(f,g) \equiv \langle fg, b\rangle_{\mathcal{D}} = fg\overline{b}(0) + \int_{\mathbb{D}} (f'g + fg')\,\overline{b'}.$$

- A result of Rochberg and Wu is that the *half* forms $\int_{\mathbb{D}} (f'g)\,\overline{b'}$ and $\int_{\mathbb{D}} (fg')\,\overline{b'}$ are each bounded on $\mathcal{D}\times\mathcal{D}$ *if and only if* $b \in \mathcal{X}$, where $\mathcal{X}$ is the space of holomorphic functions with norm

$$\|b\|_{\mathcal{X}} \equiv |b(0)| + \left\| |b'(z)|^2\, dA \right\|_{CM(\mathcal{D})}^{\frac{1}{2}} < \infty.$$

- The question arises as to whether or not there is significant cancellation in the sum of the half forms, and the answer is NO:

$$\|b\|_{\mathcal{X}} \approx \|T_b\|_{\mathcal{D}\times\mathcal{D}} \equiv \sup_{\|f\|_{\mathcal{D}},\|g\|_{\mathcal{D}}\leq 1} |T_b(f,g)|\,.$$

- Splitting a bilinear form B into natural pieces B_1 and B_2, and then asking if the pieces B_i are each bounded when B is, is a question that arises often.

- Splitting a bilinear form B into natural pieces B_1 and B_2, and then asking if the pieces B_i are each bounded when B is, is a question that arises often.
- For example, the usual attack (initiated by Nazarov, Treil and Volberg) on the two weight norm inequality for the Hilbert transform

$$\left|\langle H(f\sigma), g\rangle_\omega\right| \lesssim \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},$$

begins by splitting the bilinear form on the left according to the length of the intervals in the Haar decompositions $f = \sum_{I \text{ dyadic}} \langle f, h_I^\sigma\rangle h_I^\sigma$ and $g = \sum_{J \text{ dyadic}} \langle g, h_J^\omega\rangle h_J^\omega$:

$$\langle H(f\sigma), g\rangle_\omega = \left(\sum_{|I|\leq|J|} + \sum_{|I|>|J|}\right) \langle f, h_I^\sigma\rangle \ \langle H(h_I^\sigma \sigma), h_J^\omega\rangle_\omega \ \overline{\langle g, h_J^\omega\rangle}.$$

- Splitting a bilinear form B into natural pieces B_1 and B_2, and then asking if the pieces B_i are each bounded when B is, is a question that arises often.
- For example, the usual attack (initiated by Nazarov, Treil and Volberg) on the two weight norm inequality for the Hilbert transform

$$\left|\langle H(f\sigma), g\rangle_\omega\right| \lesssim \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},$$

begins by splitting the bilinear form on the left according to the length of the intervals in the Haar decompositions $f = \sum_{I \text{ dyadic}} \langle f, h_I^\sigma\rangle h_I^\sigma$ and $g = \sum_{J \text{ dyadic}} \langle g, h_J^\omega\rangle h_J^\omega$:

$$\langle H(f\sigma), g\rangle_\omega = \left(\sum_{|I|\leq|J|} + \sum_{|I|>|J|}\right) \langle f, h_I^\sigma\rangle \ \langle H(h_I^\sigma \sigma), h_J^\omega\rangle_\omega \ \overline{\langle g, h_J^\omega\rangle}.$$

- It is not known if the boundedness of $B_1 = \sum_{|I|\leq|J|}$ and $B_2 = \sum_{|I|>|J|}$ follow from that of $B = \langle H(f\sigma), g\rangle_\omega$.

Part 2

Preliminaries

The unit disk

Automorphisms and invariance

- Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Let dz be Lebesgue measure on $\mathbb{C}$ and let $d\lambda(z) = \frac{dz}{\pi(1-|z|^2)^2}$ be the invariant measure on the disk, i.e.,

$$\int_{\mathbb{D}} (f \circ \varphi_a)(z)\, d\lambda(z) = \int_{\mathbb{D}} f(z)\, d\lambda(z), \qquad a \in \mathbb{D}, f \in H(\mathbb{D}),$$

where

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad a, z \in \mathbb{D},$$

are the automorphisms of the disk.

- Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Let dz be Lebesgue measure on $\mathbb{C}$ and let $d\lambda(z) = \frac{dz}{\pi(1-|z|^2)^2}$ be the invariant measure on the disk, i.e.,

$$\int_{\mathbb{D}} (f \circ \varphi_a)(z)\, d\lambda(z) = \int_{\mathbb{D}} f(z)\, d\lambda(z), \qquad a \in \mathbb{D}, f \in H(\mathbb{D}),$$

  where

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \qquad a, z \in \mathbb{D},$$

  are the automorphisms of the disk.
- The Poincaré/Bergman metric is

$$\beta(z,w) \equiv \frac{1}{2} \ln \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \qquad z, w \in \mathbb{D}.$$

Cauchy's formula

Cauchy's formula yields

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f\left(e^{i\theta}\right)}{e^{i\theta}-z} ie^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{f\left(e^{i\theta}\right)}{1-e^{-i\theta}z}d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} f\overline{k_z}d\theta, \end{aligned}$$

for $f \in H(\mathbb{D}) \cap C\left(\overline{\mathbb{D}}\right)$, where

$$k_z(w) \equiv \frac{1}{1-\overline{z}w}, \qquad z \in \mathbb{D}, w \in \overline{\mathbb{D}}.$$

Magic Bullet #1

- We have the following identities for $a, z, w \in \mathbb{D}$:

$$1 - \varphi_a(z)\overline{\varphi_a(w)} = \frac{\left(1 - |a|^2\right)(1 - z\overline{w})}{(1 - a\overline{w})(1 - z\overline{a})} = \frac{k_w(a)\, k_a(z)}{k_w(z)\, k_a(a)},$$

$$1 - |\varphi_a(z)|^2 = \frac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{|1 - z\overline{a}|^2} = \frac{|k_a(z)|^2}{k_z(z)\, k_a(a)}.$$

Magic Bullet #1

- We have the following identities for $a, z, w \in \mathbb{D}$:

$$1 - \varphi_a(z)\overline{\varphi_a(w)} = \frac{\left(1 - |a|^2\right)(1 - z\overline{w})}{(1 - a\overline{w})(1 - z\overline{a})} = \frac{k_w(a)\,k_a(z)}{k_w(z)\,k_a(a)},$$

$$1 - |\varphi_a(z)|^2 = \frac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{|1 - z\overline{a}|^2} = \frac{|k_a(z)|^2}{k_z(z)\,k_a(a)}.$$

- With the definitions $d(z_i, z_j) \equiv \left|\frac{z_i - z_j}{1 - \overline{z_j} z_i}\right|$ and $\widetilde{k_z}(w) \equiv \frac{k_z(w)}{\sqrt{k_z(z)}}$, the latter can be rewritten,

$$d(z_i, z_j)^2 + \left|\left\langle \widetilde{k_{z_i}}, \widetilde{k_{z_j}} \right\rangle\right|^2 = 1. \tag{1}$$

The pseudohyperbolic metric

- The function d is called the pseudohyperbolic metric on $\mathbb{D}$, and can be generalized to the Hilbert function spaces treated below.

The pseudohyperbolic metric

- The function d is called the pseudohyperbolic metric on $\mathbb{D}$, and can be generalized to the Hilbert function spaces treated below.
- Because of the identity (1), $d(z_i, z_j)$ can be thought of as the sine of the angle θ_{ij} between k_{z_i} and k_{z_j}. This interpretation leads to the following cute proof that d is a metric.

The pseudohyperbolic metric

- The function d is called the pseudohyperbolic metric on $\mathbb{D}$, and can be generalized to the Hilbert function spaces treated below.
- Because of the identity (1), $d(z_i, z_j)$ can be thought of as the sine of the angle θ_{ij} between k_{z_i} and k_{z_j}. This interpretation leads to the following cute proof that d is a metric.
- From geometry we have $\theta_{i\ell} \leq \theta_{ij} + \theta_{j\ell}$. If the right side is at most $\frac{\pi}{2}$, then

$$\sin \theta_{i\ell} \leq \sin\left(\theta_{ij} + \theta_{j\ell}\right) \leq \sin \theta_{ij} + \sin \theta_{j\ell};$$

otherwise, we have

$$\sin \theta_{i\ell} \leq 1 \leq \sin \theta_{ij} + \sin \theta_{j\ell}.$$

The pseudohyperbolic metric

- The function d is called the pseudohyperbolic metric on $\mathbb{D}$, and can be generalized to the Hilbert function spaces treated below.
- Because of the identity (1), $d(z_i, z_j)$ can be thought of as the sine of the angle θ_{ij} between k_{z_i} and k_{z_j}. This interpretation leads to the following cute proof that d is a metric.
- From geometry we have $\theta_{i\ell} \leq \theta_{ij} + \theta_{j\ell}$. If the right side is at most $\frac{\pi}{2}$, then

$$\sin\theta_{i\ell} \leq \sin\left(\theta_{ij} + \theta_{j\ell}\right) \leq \sin\theta_{ij} + \sin\theta_{j\ell};$$

otherwise, we have

$$\sin\theta_{i\ell} \leq 1 \leq \sin\theta_{ij} + \sin\theta_{j\ell}.$$

- Finally, there is a formula relating the Bergman and pseudohyperbolic metrics:

$$\beta(z, w) = \frac{1}{2}\log\frac{1 + d(z, w)}{1 - d(z, w)}.$$

The Dirichlet space

- The classical Dirichlet space $\mathcal{D}$ of holomorphic functions f on the unit disk $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{D}^*} = \left\{ \int_{\mathbb{D}} \left| f'(z) \right|^2 dxdy \right\}^{\frac{1}{2}} = \sqrt{Area\left(f\left(\Omega\right)\right)} < \infty,$$

occupies a pivotal endpoint niche in the theory of Hilbert spaces of holomorphic functions satisfying Sobolev type conditions.

The Dirichlet space

- The classical Dirichlet space $\mathcal{D}$ of holomorphic functions f on the unit disk $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{D}^*} = \left\{ \int_{\mathbb{D}} \left| f'(z) \right|^2 dxdy \right\}^{\frac{1}{2}} = \sqrt{Area\left(f\left(\Omega\right)\right)} < \infty,$$

  occupies a pivotal endpoint niche in the theory of Hilbert spaces of holomorphic functions satisfying Sobolev type conditions.

- As such, $\mathcal{D}$ inherits much of the character of the space BMO of functions of bounded mean oscillation on the real line $\mathbb{R}$, which in turn occupies a pivotal endpoint niche among the somewhat different scale of Lebesgue spaces on the line.

The Dirichlet space

- The classical Dirichlet space $\mathcal{D}$ of holomorphic functions f on the unit disk $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{D}^*} = \left\{ \int_{\mathbb{D}} \left| f'(z) \right|^2 dxdy \right\}^{\frac{1}{2}} = \sqrt{Area\left(f(\Omega)\right)} < \infty,$$

occupies a pivotal endpoint niche in the theory of Hilbert spaces of holomorphic functions satisfying Sobolev type conditions.

- As such, $\mathcal{D}$ inherits much of the character of the space *BMO* of functions of bounded mean oscillation on the real line $\mathbb{R}$, which in turn occupies a pivotal endpoint niche among the somewhat different scale of Lebesgue spaces on the line.
- For all automorphisms φ of the disk, there is the invariance

$$\|f \circ \varphi\|_{\mathcal{D}^*} = \int_{\mathbb{D}} \left| f'(\varphi(z)) \right|^2 \left| \varphi'(z) \right|^2 dz = \int_{\mathbb{D}} \left| f'(w) \right|^2 dw = \|f\|_{\mathcal{D}^*} .$$

Consequences of finite area

- If B is a finite Blaschke product in the disk,

$$B(z) = z^k \prod_{n=1}^{N-k} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}, \qquad 0 \leq k \leq N,$$

then $B\left(e^{i\theta}\right)$ wraps around the circle $\mathbb{T} = \partial\mathbb{D}$ exactly N times and so the area (counting multiplicities) of the image $B(\mathbb{D})$ is $N\pi$.

Consequences of finite area

- If B is a finite Blaschke product in the disk,

$$B(z) = z^k \prod_{n=1}^{N-k} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}, \qquad 0 \leq k \leq N,$$

  then $B\left(e^{i\theta}\right)$ wraps around the circle $\mathbb{T} = \partial\mathbb{D}$ exactly N times and so the area (counting multiplicities) of the image $B(\mathbb{D})$ is $N\pi$.

- A thorny consequence of this is that the Dirichlet space contains no *infinite* Blaschke products (since their images cover the disk infinitely often), and hence the zeroes of a Dirichlet space function cannot be factored out as is the case for a Hardy space function.

Carleson measures

- A geometric characterization of when the Dirichlet space $\mathcal{D}$ embeds in the Lebesgue space $L^2(\mu)$ is the *testing condition*:

$$\int_{S(z)} \mu(S(w))^2 \frac{dw}{\left(1-|w|^2\right)^2} \leq C_{testing}\,\mu(S(z)), \qquad z \in \mathbb{D}.$$

Carleson measures

- A geometric characterization of when the Dirichlet space $\mathcal{D}$ embeds in the Lebesgue space $L^2(\mu)$ is the *testing condition*:

$$\int_{S(z)} \mu(S(w))^2 \frac{dw}{\left(1-|w|^2\right)^2} \leq C_{testing}\,\mu(S(z)), \qquad z \in \mathbb{D}.$$

- An earlier *capacity condition* characterization of Stegenga is

$$\mu\left(\bigcup_{z\in F} S(z)\right) \lesssim C_{capacity}\, Cap\left(\bigcup_{z\in F} I(z)\right), \qquad I(z) = \partial S(z) \cap \mathbb{T}.$$

Carleson measures

- A geometric characterization of when the Dirichlet space $\mathcal{D}$ embeds in the Lebesgue space $L^2(\mu)$ is the *testing condition*:

$$\int_{S(z)} \mu(S(w))^2 \frac{dw}{\left(1-|w|^2\right)^2} \leq C_{testing}\, \mu(S(z)), \qquad z \in \mathbb{D}.$$

- An earlier *capacity condition* characterization of Stegenga is

$$\mu\left(\bigcup_{z\in F} S(z)\right) \lesssim C_{capacity}\, Cap\left(\bigcup_{z\in F} I(z)\right), \qquad I(z) = \partial S(z) \cap \mathbb{T}.$$

- We denote by $\|\mu\|_{CM(\mathcal{D})}$ the square of the norm of the embedding so that

$$\|\mu\|_{CM(\mathcal{D})} \approx C_{testing} \approx C_{capacity}.$$

A connection between conditions

- Upon passing to boundary values, the *capacity condition* is equivalent to the weak type potential inequality

$$\left\| I_{\frac{1}{2}} f \right\|_{L^{2,\infty}(\mu)} \lesssim \|f\|_{L^2(\mathbb{T})},$$

which by duality is equivalent to the restricted strong type inequality

$$\left\| I_{\frac{1}{2}} (g\mu) \right\|_{L^2(\mathbb{T})} \lesssim \|g\|_{L^{2,1}(\mu)},$$

which by definition holds if and only if

$$\left\| I_{\frac{1}{2}} (\mathbf{1}_E \mu) \right\|_{L^2(\mathbb{T})} \lesssim \|\mathbf{1}_E\|_{L^{2,1}(\mu)} = \sqrt{|E|_\mu}, \qquad \text{all sets } E \subset \mathbb{T}.$$

A connection between conditions

- Upon passing to boundary values, the *capacity condition* is equivalent to the weak type potential inequality

$$\left\| I_{\frac{1}{2}} f \right\|_{L^{2,\infty}(\mu)} \lesssim \|f\|_{L^2(\mathbb{T})},$$

which by duality is equivalent to the restricted strong type inequality

$$\left\| I_{\frac{1}{2}} (g\mu) \right\|_{L^2(\mathbb{T})} \lesssim \|g\|_{L^{2,1}(\mu)},$$

which by definition holds if and only if

$$\left\| I_{\frac{1}{2}} (\mathbf{1}_E \mu) \right\|_{L^2(\mathbb{T})} \lesssim \|\mathbf{1}_E\|_{L^{2,1}(\mu)} = \sqrt{|E|_\mu}, \qquad \text{all sets } E \subset \mathbb{T}.$$

- On the other hand, the boundary equivalent of the *testing condition* is

$$\left\| I_{\frac{1}{2}} (\mathbf{1}_I \mu) \right\|_{L^2(\mathbb{T})} \lesssim \|\mathbf{1}_I\|_{L^{2,1}(\mu)} = \sqrt{|I|_\mu}, \qquad \text{all arcs } I \subset \mathbb{T},$$

which gives the inequality $C_{testing} \lesssim C_{capacity}$.

The tree Dirichlet space

It turns out that the Dirichlet space $\mathcal{D}(\mathbb{D})$ can be effectively modeled on the tree $\mathcal{T}$ by the following Hilbert space of complex-valued functions $f : \mathcal{T} \to \mathbb{C}$ on $\mathcal{T}$:

$$\mathcal{D}(\mathcal{T}) = \left\{ f = (f(\alpha))_{\alpha \in \mathcal{T}} : \sum_{\alpha \in \mathcal{T}} |\triangle f(\alpha)|^2 < \infty \right\},$$

with inner product

$$\langle f, g \rangle = \sum_{\alpha \in \mathcal{T}} \triangle f(\alpha) \overline{\triangle g(\alpha)},$$

and where the backward difference operator $\triangle$ is defined on functions f by

$$\triangle f(\alpha) = \begin{cases} f(o) & \text{if} \quad \alpha = o \\ f(\alpha) - f(P\alpha) & \text{if} \quad \alpha \neq o \end{cases}.$$

Continuity of the restriction map

- The restriction map $\mathcal{R} : \mathcal{D}(\mathbb{D}) \to \mathcal{D}(\mathcal{T})$ defined by $\mathcal{R}f = (f(c(\alpha)))_{\alpha \in \mathcal{T}}$ for $f \in \mathcal{D}(\mathbb{D})$ turns out to be *continuous*.

Continuity of the restriction map

- The restriction map $\mathcal{R} : \mathcal{D}(\mathbb{D}) \to \mathcal{D}(\mathcal{T})$ defined by $\mathcal{R}f = (f(c(\alpha)))_{\alpha \in \mathcal{T}}$ for $f \in \mathcal{D}(\mathbb{D})$ turns out to be *continuous*.
- To see this let $\alpha \in \mathcal{T}$, and denote by B_α the largest ball contained in $K(\alpha)$ that is centered at $c(\alpha)$. In addition denote by H_α the convex hull of B_α and $B_{P\alpha}$. Then the mean value property for holomorphic functions, the fundamental theorem of calculus and the change of variable $\omega = tz + (1-t)\zeta$ give the following chain of (in)equalities:

Chain of (in)equalities

$$
\begin{aligned}
\left|f(\alpha) - f(P\alpha)\right| &= \left|f(c(\alpha)) - f(c(P\alpha))\right| \\
&= \left|\frac{1}{|B_\alpha|}\int_{B_\alpha} f(z)\,dz - \frac{1}{|B_{P\alpha}|}\int_{B_{P\alpha}} f(\zeta)\,d\zeta\right| \\
&= \left|\frac{1}{|B_\alpha|}\frac{1}{|B_{P\alpha}|}\int_{B_\alpha}\int_{B_{P\alpha}} [f(z) - f(\zeta)]\,dzd\zeta\right| \\
&= \left|\frac{1}{|B_\alpha|}\frac{1}{|B_{P\alpha}|}\int_{B_\alpha}\int_{B_{P\alpha}}\int_0^1 (z-\zeta)\cdot\nabla f(tz + (1-t)\zeta)\ d\right. \\
&\leq diam(H_\alpha)\frac{1}{|B_\alpha|}\frac{1}{|B_{P\alpha}|}\int_{B_\alpha}\int_{B_{P\alpha}}\int_0^1 \left|f'(tz + (1-t)\zeta)\right| \\
&\leq Cdiam(H_\alpha)\frac{1}{|H_\alpha|}\int_{H_\alpha}\left|f'(\omega)\right| d\omega.
\end{aligned}
$$

- Now we compute that

$$\begin{aligned}\|\mathcal{R}f\|^2_{\mathcal{D}(\mathcal{T})} &= |f(o)|^2 + \sum_{\alpha\in\mathcal{T}} |f(\alpha) - f(P\alpha)|^2 \\ &\leq |f(0)|^2 + C\sum_{\alpha\in\mathcal{T}} \frac{diam(H_\alpha)^2}{|H_\alpha|}\int_{H_\alpha} |f'(\omega)|^2\, d\omega \\ &\leq |f(0)|^2 + C\int_{\mathbb{D}} |f'(\omega)|^2\, d\omega \leq C\,\|f\|^2_{\mathcal{D}(\mathbb{D})},\end{aligned}$$

since $diam(H_\alpha)^2 \approx |H_\alpha|$ and the sets H_α have finite overlap at most two in the disk.

Restriction and Carleson measures

- Now we compute that

$$\begin{aligned}\|\mathcal{R}f\|^2_{\mathcal{D}(\mathcal{T})} &= |f(o)|^2 + \sum_{\alpha\in\mathcal{T}} |f(\alpha) - f(P\alpha)|^2 \\ &\leq |f(0)|^2 + C\sum_{\alpha\in\mathcal{T}} \frac{diam(H_\alpha)^2}{|H_\alpha|}\int_{H_\alpha} |f'(\omega)|^2 d\omega \\ &\leq |f(0)|^2 + C\int_{\mathbb{D}} |f'(\omega)|^2 d\omega \leq C\|f\|^2_{\mathcal{D}(\mathbb{D})},\end{aligned}$$

  since $diam(H_\alpha)^2 \approx |H_\alpha|$ and the sets H_α have finite overlap at most two in the disk.
- A major advantage of the model space $\mathcal{D}(\mathcal{T})$ is that the so-called Carleson measures for $\mathcal{D}(\mathcal{T})$ are easily calculated; these are the positive measures μ on $\mathcal{T}$, which here are the same as the nonnegative functions μ on $\mathcal{T}$, for which we have an embedding of $\mathcal{D}(\mathcal{T})$ into $L^2(\mu)$, i.e.

$$\|f\|^2_{L^2(\mu)} \leq C\|f\|^2_{\mathcal{D}(\mathcal{T})}, \qquad f\in\mathcal{D}(\mathcal{T}). \tag{2}$$

Reflections on trees

- Trees have been used in analysis for some time, but possibly the first instance of their use in the spirit above occurs in the atomic decomposition of spaces of holomorphic functions in Coifman and Rochberg. The above tree model has an equally simple and effective analogue in the case of the spaces $B_2^{\sigma}(\mathbb{D})$ when $0 \leq \sigma < \frac{1}{2}$. However, the model must be significantly changed in order to be of use for the Hardy space $B_2^{\frac{1}{2}}(\mathbb{D}) = H^2(\mathbb{D})$.

Reflections on trees

- Trees have been used in analysis for some time, but possibly the first instance of their use in the spirit above occurs in the atomic decomposition of spaces of holomorphic functions in Coifman and Rochberg. The above tree model has an equally simple and effective analogue in the case of the spaces $B_2^{\sigma}(\mathbb{D})$ when $0 \leq \sigma < \frac{1}{2}$. However, the model must be significantly changed in order to be of use for the Hardy space $B_2^{\frac{1}{2}}(\mathbb{D}) = H^2(\mathbb{D})$.
- In higher dimensions, one can construct an analogue $\mathcal{T}_n$ for the ball $\mathbb{B}_n$ of the tree $\mathcal{T}$ constructed above for the disk, but the construction is necessarily messy due to the fact that the sphere $\mathbb{S}^k$ is not neatly tiled when $k > 1$. While the corresponding tree space $\mathcal{D}(\mathcal{T}_n)$ remains effective for calculating the Carleson measures of the Dirichlet space $B_2^0(\mathbb{B}_n) = \mathcal{D}(\mathbb{B}_n)$ on the ball, it is no longer an adequate model for characterizing interpolation for the Dirichlet space since the corresponding restriction map $\mathcal{R}$ fails to be continuous from $\mathcal{D}(\mathbb{B}_n)$ to $\mathcal{D}(\mathcal{T}_n)$ when $n > 1$.

Reflections on trees 2

- Instead one can introduce a holomophic structure on the tree $\mathcal{T}_n$ (that mirrors the holomorphic geometry of the ball) and redefine the model space $\mathcal{D}(\mathcal{T}_n)$ to take this structure into account.

Reflections on trees 2

- Instead one can introduce a holomophic structure on the tree $\mathcal{T}_n$ (that mirrors the holomorphic geometry of the ball) and redefine the model space $\mathcal{D}(\mathcal{T}_n)$ to take this structure into account.
- The result is that the restriction operator is now continuous, and using this with some other special properties of the model, the Carleson measures and interpolating sequences for $\mathcal{D}(\mathbb{B}_n)$ can be characterized.

Reflections on trees 2

- Instead one can introduce a holomophic structure on the tree $\mathcal{T}_n$ (that mirrors the holomorphic geometry of the ball) and redefine the model space $\mathcal{D}(\mathcal{T}_n)$ to take this structure into account.
- The result is that the restriction operator is now continuous, and using this with some other special properties of the model, the Carleson measures and interpolating sequences for $\mathcal{D}(\mathbb{B}_n)$ can be characterized.
- Finally, the unstructured model $\mathcal{D}(\mathcal{T}_n)$ extends to an effective model for calculating Carleson measures for the spaces $B_2^{\sigma}(\mathbb{B}_n)$ with $0 \leq \sigma < \frac{1}{2}$. But again, this model breaks down at the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n) = H_n^2$.

Reflections on trees 2

- Instead one can introduce a holomophic structure on the tree $\mathcal{T}_n$ (that mirrors the holomorphic geometry of the ball) and redefine the model space $\mathcal{D}(\mathcal{T}_n)$ to take this structure into account.
- The result is that the restriction operator is now continuous, and using this with some other special properties of the model, the Carleson measures and interpolating sequences for $\mathcal{D}(\mathbb{B}_n)$ can be characterized.
- Finally, the unstructured model $\mathcal{D}(\mathcal{T}_n)$ extends to an effective model for calculating Carleson measures for the spaces $B_2^{\sigma}(\mathbb{B}_n)$ with $0 \leq \sigma < \frac{1}{2}$. But again, this model breaks down at the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n) = H_n^2$.
- Yet a different geometric structure is needed on the tree $\mathcal{T}_n$ to compute the Carleson measures for the Drury-Arveson Hardy space H_n^2.

Part 3

Interpolating sequences

The most satisfying proof solves the interpolating problem for a large collection of Hilbert spaces, those with the complete Nevanlinna-Pick property, so we begin with a discussion of Hilbert function spaces.

Reproducing kernels

- For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the inner product corresponding to the norm $\sqrt{\|f\|_{H^2}^2 + \|f\|_{\mathcal{D}}^2}$ satisfies

$$\begin{aligned} \langle f, g \rangle_{\mathcal{D}(\mathbb{D})} &= \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} dm(\zeta) + \frac{1}{\pi} \int_{\mathbb{D}} f'(z) \overline{g'(z)} dxdy \\ &= \sum_{n=0}^{\infty} (n+1) a_n \overline{b_n}, \qquad f, g \in \mathcal{D}(\mathbb{D}), \end{aligned}$$

Reproducing kernels

- For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the inner product corresponding to the norm $\sqrt{\|f\|_{H^2}^2 + \|f\|_{\mathcal{D}}^2}$ satisfies

$$\begin{aligned} \langle f, g \rangle_{\mathcal{D}(\mathbb{D})} &= \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} dm(\zeta) + \frac{1}{\pi} \int_{\mathbb{D}} f'(z) \overline{g'(z)} dxdy \\ &= \sum_{n=0}^{\infty} (n+1) a_n \overline{b_n}, \qquad f, g \in \mathcal{D}(\mathbb{D}), \end{aligned}$$

- The reproducing kernel $k_z(w)$ for the Dirichlet space is given by

$$k_z(w) = \frac{1}{\overline{z}w} \log \frac{1}{1 - \overline{z}w} = \sum_{n=0}^{\infty} \frac{1}{n+1} \overline{z}^n w^n,$$

where the branch of log is taken to satisfy $\log 1 = 0$. Indeed, with $g = k_z$ we have $b_n = \frac{1}{n+1} \overline{z}^n$ for $n \geq 0$ and so

$$\langle f, k_z \rangle_{\mathcal{D}(\mathbb{D})} = \sum_{n=0}^{\infty} (n+1) a_n \overline{\frac{1}{n+1} \overline{z}^n} = \sum_{n=0}^{\infty} a_n z^n = f(z).$$

Hilbert function spaces

- A Hilbert space $\mathcal{H}$ is said to be a *Hilbert function space* (aka a reproducing kernel Hilbert space - RKHS) on a set Ω if the elements of $\mathcal{H}$ are complex-valued functions f on Ω with the usual vector space structure, such that each point evaluation on $\mathcal{H}$ is a nonzero continuous linear functional, i.e. for every $x \in \Omega$ there is a positive constant C_x such that

$$|f(x)| \leq C_x \|f\|_{\mathcal{H}}, \qquad f \in \mathcal{H}, \tag{3}$$

and there is some f with $f(x) \neq 0$.

Hilbert function spaces

- A Hilbert space $\mathcal{H}$ is said to be a *Hilbert function space* (aka a reproducing kernel Hilbert space - RKHS) on a set Ω if the elements of $\mathcal{H}$ are complex-valued functions f on Ω with the usual vector space structure, such that each point evaluation on $\mathcal{H}$ is a nonzero continuous linear functional, i.e. for every $x \in \Omega$ there is a positive constant C_x such that

$$|f(x)| \leq C_x \|f\|_{\mathcal{H}}, \qquad f \in \mathcal{H}, \tag{3}$$

  and there is some f with $f(x) \neq 0$.
- The Riesz theorem shows there is a unique element $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle \text{ for all } x \in \Omega.$$

Hilbert function spaces

- A Hilbert space $\mathcal{H}$ is said to be a *Hilbert function space* (aka a reproducing kernel Hilbert space - RKHS) on a set Ω if the elements of $\mathcal{H}$ are complex-valued functions f on Ω with the usual vector space structure, such that each point evaluation on $\mathcal{H}$ is a nonzero continuous linear functional, i.e. for every $x \in \Omega$ there is a positive constant C_x such that

$$|f(x)| \leq C_x \|f\|_{\mathcal{H}}, \qquad f \in \mathcal{H}, \tag{3}$$

  and there is some f with $f(x) \neq 0$.
- The Riesz theorem shows there is a unique element $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle \text{ for all } x \in \Omega.$$

- The element k_x is called the reproducing kernel at x, and satisfies

$$k_x(y) = \langle k_y, k_x \rangle, \qquad x, y \in \Omega.$$

Positive semidefinite kernels

- Recall that a matrix $A = [a_{ij}]_{i,j=1}^N$ is semipositive definite, written $A \succeq 0$, if

$$\xi \cdot A\xi = \sum_{i.j=1}^{N} \xi_i \overline{\xi_j} a_{ij} \geq 0, \qquad \xi \in \mathbb{C}^N.$$

Positive semidefinite kernels

- Recall that a matrix $A = [a_{ij}]_{i,j=1}^{N}$ is semipositive definite, written $A \succeq 0$, if

$$\xi \cdot A\xi = \sum_{i.j=1}^{N} \xi_i \overline{\xi_j} a_{ij} \geq 0, \qquad \xi \in \mathbb{C}^N.$$

- The function $k(x,y) \equiv \langle k_y, k_x \rangle = k_x(y)$ is self-adjoint ($k(x,y) = \overline{k(y,x)}$), and for every finite subset $\{x_i\}_{i=1}^{N}$ of Ω, the matrix $[k(x_i,x_j)]_{1 \leq i,j \leq N}$ is positive semidefinite:

$$\sum_{i,j=1}^{N} \xi_i \overline{\xi_j} k(x_i,x_j) = \sum_{i,j=1}^{N} \xi_i \overline{\xi_j} \langle k_{x_j}, k_{x_i} \rangle$$

$$= \left\langle \sum_{j=1}^{N} \overline{\xi_j} k_{x_j}, \sum_{i=1}^{N} \overline{\xi_i} k_{x_i} \right\rangle = \left\| \sum_{i=1}^{N} \overline{\xi_i} k_{x_i} \right\|_{\mathcal{H}}^2 \geq 0.$$

The connection with inner products

Given a kernel function k on $\Omega \times \Omega$, define an inner product on finite linear combinations $\sum_{i=1}^{N} \xi_i k_{x_i}$ of the functions $k_{x_i}(\zeta) = k(\zeta, x_i)$, $\zeta \in \Omega$, by

$$\left\langle \sum_{i=1}^{N} \xi_i k_{x_i}, \sum_{j=1}^{N} \eta_j k_{x_j} \right\rangle = \sum_{i.j=1}^{N} \xi_i \overline{\eta_j} k(x_j, x_i),$$

and define the associated Hilbert function space $\mathcal{H}_k$ to be the completion of the functions $\sum_{i=1}^{N} \xi_i k_{x_i}$ under the norm corresponding to the above inner product.

Theorem

(E. H. Moore) The Hilbert space $\mathcal{H}_k$ has kernel k. If $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert function spaces on Ω that have the same kernel function k, then there is an isometry from $\mathcal{H}$ onto $\mathcal{H}'$ that preserves the kernel functions k_x, $x \in \Omega$.

Pointwise multipliers

- A function $\varphi : \Omega \to \mathbb{C}$ is said to be a *pointwise multiplier* on a Hilbert function space $\mathcal{H}$ if $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$. From the closed graph theorem we see that the operator $\mathcal{M}_\varphi : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{M}_\varphi f \equiv \varphi f$ is bounded. The linear space of all such functions is denoted $\mathcal{M}_{\mathcal{H}}$.

Pointwise multipliers

- A function $\varphi : \Omega \to \mathbb{C}$ is said to be a *pointwise multiplier* on a Hilbert function space $\mathcal{H}$ if $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$. From the closed graph theorem we see that the operator $\mathcal{M}_\varphi : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{M}_\varphi f \equiv \varphi f$ is bounded. The linear space of all such functions is denoted $\mathcal{M}_{\mathcal{H}}$.
- Now assume that $\mathcal{H}$ contains the constant functions. Then $\mathcal{M}_{\mathcal{H}} \subset \mathcal{H}$ since $\varphi = \varphi 1$. Moreover, the supremum norm of φ, namely $\|\varphi\|_\infty \equiv \sup_{x \in \Omega} |\varphi(x)|$, is bounded by the operator norm of M_φ.

Pointwise multipliers

- A function $\varphi : \Omega \to \mathbb{C}$ is said to be a *pointwise multiplier* on a Hilbert function space $\mathcal{H}$ if $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$. From the closed graph theorem we see that the operator $\mathcal{M}_\varphi : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{M}_\varphi f \equiv \varphi f$ is bounded. The linear space of all such functions is denoted $\mathcal{M}_\mathcal{H}$.
- Now assume that $\mathcal{H}$ contains the constant functions. Then $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$ since $\varphi = \varphi 1$. Moreover, the supremum norm of φ, namely $\|\varphi\|_\infty \equiv \sup_{x \in \Omega} |\varphi(x)|$, is bounded by the operator norm of M_φ.
- But much more is actually true, namely that for each $x \in \Omega$, the reproducing kernel k_x is an eigenvector of the adjoint operator $M_\varphi^* : \mathcal{H} \to \mathcal{H}$ with corresponding eigenvalue $\overline{\varphi(x)}$.

Magic Bullet #2

Suppose $\mathcal{H}$ is a Hilbert function space on Ω. For $\varphi \in \mathcal{M}_{\mathcal{H}}$, $f \in \mathcal{H}$ and $x \in \Omega$,

$$\begin{aligned}\left\langle f, M_\varphi^* k_x \right\rangle &= \left\langle M_\varphi f, k_x \right\rangle = \left(M_\varphi f\right)(x) \\ &= \varphi(x) f(x) \\ &= \varphi(x) \langle f, k_x \rangle = \left\langle f, \overline{\varphi(x)} k_x \right\rangle,\end{aligned}$$

which implies $M_\varphi^* k_x = \overline{\varphi(x)} k_x$, and in particular,

$$|\varphi(x)| \, \|k_x\| = \left\| \overline{\varphi(x)} k_x \right\| = \left\| M_\varphi^* k_x \right\| \leq \left\| M_\varphi^* \right\| \|k_x\| = \left\| M_\varphi \right\| \|k_x\| .$$

The Nevanlinna-Pick interpolation problem

- Suppose that H is a Hilbert function space of analytic functions on Ω with reproducing kernel $k_w(z)$. Let $Z = \{z_j\}_{j=1}^J$ be a finite set of points in Ω and consider the Nevanlinna-Pick interpolation problem: For which sequences of data $\{\xi_j\}_{j=1}^J \subset \mathbb{C}$ is there $\varphi \in M_H$ with muliplier norm one satisfying

$$\varphi(z_j) = \xi_j, \qquad 1 \leq j \leq J? \tag{4}$$

The Nevanlinna-Pick interpolation problem

- Suppose that H is a Hilbert function space of analytic functions on Ω with reproducing kernel $k_w(z)$. Let $Z = \{z_j\}_{j=1}^J$ be a finite set of points in Ω and consider the Nevanlinna-Pick interpolation problem: For which sequences of data $\{\xi_j\}_{j=1}^J \subset \mathbb{C}$ is there $\varphi \in M_H$ with mulipler norm one satisfying

$$\varphi(z_j) = \xi_j, \qquad 1 \leq j \leq J? \tag{4}$$

- There is an easy necessary condition for the data in terms of a certain matrix being positive semidefinite. If $\|\mathcal{M}_\varphi\| \equiv \|\varphi\|_{M_H} \leq 1$ then $\left\|\mathcal{M}_\varphi^*\right\| \leq 1$ and for every choice of scalars $\{\lambda_j\}_{j=1}^J \subset \mathbb{C}$ we have

$$0 \leq \left\|\sum_{j=1}^J \lambda_j k_{z_j}\right\|^2 - \left\|\mathcal{M}_\varphi^*\left(\sum_{j=1}^J \lambda_j k_{z_j}\right)\right\|^2 = \sum_{j,m=1}^J \left(1 - \xi_j \overline{\xi_m}\right) k_{z_j}(z_m) \lambda_j \overline{\lambda}$$

which is

$$\left[\left(1 - \xi_j \overline{\xi_m}\right) k_{z_j}(z_m)\right]_{j,m=1}^J \succeq 0. \tag{5}$$

The Nevanlinna-Pick property and extremal problems

We say that the Hilbert space H (more precisely the *inner product* of H) has the Nevanlinna-Pick property (NPP) if the implication above can be reversed.

Definition

The Hilbert space H has the *Nevanlinna-Pick property* if whenever (5) holds, there is $\varphi \in M_H$ with muliplier norm one satisfying (4).

There is a stronger notion called the *complete* Nevanlinna-Pick property (CNPP) that asserts the analogous property for *matrix-valued* multipliers mapping $H \otimes \mathbb{C}^s$ to $H \otimes \mathbb{C}^t$, and for all positive integers $s, t \in \mathbb{N}$.

An extremal problem

- There is a surprising consequence of the Nevanlinna-Pick property for certain extremal problems. Let $Z = \{z_j\}_{j=1}^{\infty}$ and $z_0 \notin Z$. Let f_0 be the unique solution to the extremal problem

$$\mathrm{Re}\, f_0\left(z_0\right) = \left\{\mathrm{Re}\, f\left(z_0\right) : f\left(z_j\right) = 0 \text{ for } 1 \leq j < \infty \text{ and } \|f\| \leq 1\right\}. \tag{6}$$

- There is a surprising consequence of the Nevanlinna-Pick property for certain extremal problems. Let $Z = \{z_j\}_{j=1}^{\infty}$ and $z_0 \notin Z$. Let f_0 be the unique solution to the extremal problem

$$\mathrm{Re}\, f_0(z_0) = \{\mathrm{Re}\, f(z_0) : f(z_j) = 0 \text{ for } 1 \leq j < \infty \text{ and } \|f\| \leq 1\}. \tag{6}$$

- Note that the solution exists and is unique because for each real t, there is a unique element of minimal norm in the closed convex set

$$E_t = \{f \in H : \mathrm{Re}\, f(z_0) = t, f(z_j) = 0 \text{ for } 1 \leq j < \infty \text{ and } \|f\| \leq 1\}.$$

Solving the extremal problem

- From the definition of f_0 we have

$$\left|\lambda_0 f_0(z_0)\right| = \left|\left\langle \sum_{j=0}^{\infty} \lambda_j k_{z_j}, f_0 \right\rangle\right| \leq \left\|\sum_{j=0}^{\infty} \lambda_j k_{z_j}\right\|,$$

which in terms of the data $\xi_0 = \frac{|f_0(z_0)|}{\|k_{z_0}\|}$ and $\xi_j = 0$ for $1 \leq j < \infty$ can be rewritten as

$$0 \leq \left\|\sum_{j=0}^{\infty} \lambda_j k_{z_j}\right\|^2 - \left|\lambda_0 f_0(z_0)\right|^2 = \sum_{j,m=0}^{\infty} \left(1 - \xi_j \overline{\xi_m}\right) k_{z_j}(z_m) \lambda_j \overline{\lambda_m}.$$

Solving the extremal problem

- From the definition of f_0 we have

$$\left|\lambda_0 f_0\left(z_0\right)\right| = \left|\left\langle \sum_{j=0}^{\infty} \lambda_j k_{z_j}, f_0 \right\rangle\right| \leq \left\|\sum_{j=0}^{\infty} \lambda_j k_{z_j}\right\|,$$

which in terms of the data $\xi_0 = \frac{|f_0(z_0)|}{\|k_{z_0}\|}$ and $\xi_j = 0$ for $1 \leq j < \infty$ can be rewritten as

$$0 \leq \left\|\sum_{j=0}^{\infty} \lambda_j k_{z_j}\right\|^2 - \left|\lambda_0 f_0\left(z_0\right)\right|^2 = \sum_{j,m=0}^{\infty} \left(1 - \xi_j \overline{\xi_m}\right) k_{z_j}\left(z_m\right) \lambda_j \overline{\lambda_m}.$$

- Since H has the Nevanlinna-Pick property, there is $\varphi_0 \in M_H$ with norm at most one satisfying

$$\varphi_0\left(z_0\right) = \xi_0 = \frac{\left|f_0\left(z_0\right)\right|}{\left\|k_{z_0}\right\|} \text{ and } \varphi_0\left(z_j\right) = 0 \text{ for } 1 \leq j < \infty.$$

A remarkable identity

- Thus the function $\rho(z) \equiv \varphi_0(z) \frac{k_{z_0}(z)}{\|k_{z_0}\|}$ satisfies

$$\|\rho\| = \left\| \varphi_0 \frac{k_{z_0}}{\|k_{z_0}\|} \right\| \leq \|\mathcal{M}_\varphi\| \left\| \frac{k_{z_0}}{\|k_{z_0}\|} \right\| \leq 1,$$

and

$$\operatorname{Re} \rho(z_0) = \operatorname{Re} \left(\varphi_0(z_0) \frac{k_{z_0}(z_0)}{\|k_{z_0}\|} \right) = \frac{|f_0(z_0)|}{\|k_{z_0}\|} \frac{\|k_{z_0}\|^2}{\|k_{z_0}\|} = |f_0(z_0)|$$

and $\rho(z_j) = 0$ for $1 \leq j < \infty$.

A remarkable identity

- Thus the function $\rho(z) \equiv \varphi_0(z) \frac{k_{z_0}(z)}{\|k_{z_0}\|}$ satisfies

$$\|\rho\| = \left\| \varphi_0 \frac{k_{z_0}}{\|k_{z_0}\|} \right\| \leq \|\mathcal{M}_\varphi\| \left\| \frac{k_{z_0}}{\|k_{z_0}\|} \right\| \leq 1,$$

  and

$$\mathrm{Re}\,\rho(z_0) = \mathrm{Re}\left(\varphi_0(z_0) \frac{k_{z_0}(z_0)}{\|k_{z_0}\|} \right) = \frac{|f_0(z_0)|}{\|k_{z_0}\|} \frac{\|k_{z_0}\|^2}{\|k_{z_0}\|} = |f_0(z_0)|$$

  and $\rho(z_j) = 0$ for $1 \leq j < \infty$.

- By the uniqueness of the solution to the extremal problem (6), we obtain the remarkable identity,

$$f_0(z) = \varphi_0(z) \frac{k_{z_0}(z)}{\|k_{z_0}\|}. \tag{7}$$

Consequences of the remarkable identity

- Every zero set of a function in H is included in a zero set of a function in M_H. Indeed, if $Z = \{z_j\}_{j=1}^{\infty}$ is the zero set of $f \in H$, then the extremal problem (6) has a solution provided $z_0 \notin Z$. But then $\varphi_0 \in M_H$ vanishes on Z as well.

Consequences of the remarkable identity

- Every zero set of a function in H is included in a zero set of a function in M_H. Indeed, if $Z = \{z_j\}_{j=1}^{\infty}$ is the zero set of $f \in H$, then the extremal problem (6) has a solution provided $z_0 \notin Z$. But then $\varphi_0 \in M_H$ vanishes on Z as well.
- Every interpolating set Z for H, Definition: $\mathcal{R}_Z : H \to \ell^2$ is bounded and onto where $\mathcal{R}_Z f = \left\{ \frac{f(z_j)}{\left\| k_{z_j} \right\|} \right\}_{j=1}^{\infty}$, is also an interpolating set for M_H, Definition: $\mathcal{R}(M_H) = \ell^{\infty}$. Note that these definitions agree with those given earlier in the case $H = \mathcal{D}$ since

$$\|k_z\|_{\mathcal{D}}^2 = \langle k_z, k_z \rangle_{\mathcal{D}} = k_z(z) = \frac{1}{|z|^2} \ln \frac{1}{1 - |z|^2} \approx 1 + \beta(0, z).$$

Equivalence of interpolating sequences for NPP spaces

Theorem

Suppose H is a Hilbert function space with the Nevanlinna-Pick property. Then a set Z is interpolating for H if and only if Z is interpolating for M_H.

Proof: If Z is interpolating for H, then $\left\{k_{z_j}\right\}_{j=1}^{\infty}$ is a Riesz basis, $\left\|\sum_{j=1}^{\infty} a_j k_{z_j}\right\| \approx \|\{a_j\}\|_{\ell^2}$, and consequently satisfies the unconditional basic sequence condition: if $|a_j| \leq |b_j|$, then

$$\left\|\sum_{j=1}^{\infty} a_j k_{z_j}\right\| \leq C \left\|\{a_j\}_{j=1}^{\infty}\right\|_{\ell^2(\mu_Z)} \leq C \left\|\{b_j\}_{j=1}^{\infty}\right\|_{\ell^2(\mu_Z)} \leq C \left\|\sum_{j=1}^{\infty} b_j k_{z_j}\right\|.$$

The proof continued

- We seek to solve the interpolation

$$\varphi(z_j) = \xi_j, \qquad 1 \leq j < \infty,$$

with $\varphi \in M_H$ of norm at most one whenever $\left\| \{\xi_j\}_{j=1}^{\infty} \right\|_{\infty} \leq \delta$, with $\delta > 0$ sufficiently small.

The proof continued

- We seek to solve the interpolation

$$\varphi\left(z_j\right)=\xi_j,\qquad 1\leq j<\infty,$$

with $\varphi\in M_H$ of norm at most one whenever $\left\|\left\{\xi_j\right\}_{j=1}^{\infty}\right\|_{\infty}\leq\delta$, with $\delta>0$ sufficiently small.

- But for $\delta\leq\frac{1}{C}$ we have $\left|\xi_j\lambda_j\right|\leq\frac{|\lambda_j|}{C}$, and the unconditional basic sequence condition implies

$$0\leq C^2\left\|\sum_{j=1}^{\infty}\frac{\lambda_j}{C}k_{z_j}\right\|^2-\left\|\sum_{j=1}^{\infty}\xi_j\lambda_jk_{z_j}\right\|^2=\sum_{j,m=1}^{\infty}\left(1-\xi_j\overline{\xi_m}\right)k_{z_j}\left(z_m\right)\lambda_j\overline{\lambda_m}.$$

The proof continued

- We seek to solve the interpolation

$$\varphi(z_j) = \xi_j, \qquad 1 \leq j < \infty,$$

with $\varphi \in M_H$ of norm at most one whenever $\left\| \{\xi_j\}_{j=1}^{\infty} \right\|_{\infty} \leq \delta$, with $\delta > 0$ sufficiently small.

- But for $\delta \leq \frac{1}{C}$ we have $\left|\xi_j \lambda_j\right| \leq \frac{|\lambda_j|}{C}$, and the unconditional basic sequence condition implies

$$0 \leq C^2 \left\| \sum_{j=1}^{\infty} \frac{\lambda_j}{C} k_{z_j} \right\|^2 - \left\| \sum_{j=1}^{\infty} \xi_j \lambda_j k_{z_j} \right\|^2 = \sum_{j,m=1}^{\infty} \left(1 - \xi_j \overline{\xi_m}\right) k_{z_j}(z_m) \lambda_j \overline{\lambda_m}.$$

- The Nevanlinna-Pick property now yields the desired solution $\varphi \in M_H$.

The proof continued

- Conversely, multiplier interpolation implies that the normalized reproducing kernels corresponding to Z are an unconditional basic sequence: Given $|b_j| \leq |a_j|$, choose $\varphi \in M_H$ such that $b_j = \overline{\varphi(z_j)} a_j$. Then Magic Bullet #2 gives

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} b_j \frac{k_{z_j}}{\|k_{z_j}\|} \right\| &= \left\| \sum_{j=1}^{\infty} \overline{\varphi(z_j)} a_j \frac{k_{z_j}}{\|k_{z_j}\|} \right\| \\ &= \left\| \mathcal{M}_{\varphi}^{*} \left(\sum_{j=1}^{\infty} a_j \frac{k_{z_j}}{\|k_{z_j}\|} \right) \right\| \leq \left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}}{\|k_{z_j}\|} \right\|. \end{aligned}$$

The proof continued

- Conversely, multiplier interpolation implies that the normalized reproducing kernels corresponding to Z are an unconditional basic sequence: Given $|b_j| \leq |a_j|$, choose $\varphi \in M_H$ such that $b_j = \overline{\varphi(z_j)} a_j$. Then Magic Bullet #2 gives

$$\begin{aligned}\left\| \sum_{j=1}^{\infty} b_j \frac{k_{z_j}}{\|k_{z_j}\|} \right\| &= \left\| \sum_{j=1}^{\infty} \overline{\varphi(z_j)} a_j \frac{k_{z_j}}{\|k_{z_j}\|} \right\| \\ &= \left\| \mathcal{M}_{\varphi}^{*} \left(\sum_{j=1}^{\infty} a_j \frac{k_{z_j}}{\|k_{z_j}\|} \right) \right\| \leq \left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}}{\|k_{z_j}\|} \right\|.\end{aligned}$$

- Now the following expectation calculation shows that $\left\{ \frac{k_{z_j}}{\|k_{z_j}\|} \right\}_{j=1}^{\infty}$ is a Riesz basis, which is equivalent to H interpolation.

UBS impies RB

- To show that **UBS** implies **RB**, we use the fact that for any finite collection of vectors $\{v_n\}_{n=1}^N$ in a Hilbert space H there is $\theta \in [0, 2\pi)$ such that

$$\left\| \sum_{n=1}^{N} e^{in\theta} v_n \right\|^2 = \sum_{n=1}^{N} \|v_n\|^2 . \tag{8}$$

UBS impies RB

- To show that **UBS** implies **RB**, we use the fact that for any finite collection of vectors $\{v_n\}_{n=1}^N$ in a Hilbert space H there is $\theta \in [0, 2\pi)$ such that

$$\left\| \sum_{n=1}^N e^{in\theta} v_n \right\|^2 = \sum_{n=1}^N \|v_n\|^2 . \tag{8}$$

- Indeed, we simply compute the expectation,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left\| \sum_{n=1}^N e^{in\theta} v_n \right\|^2 d\theta &= \sum_{m,n=1}^N \langle v_m, v_n \rangle \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} d\theta \\ &= \sum_{n=1}^N \|v_n\|^2 , \end{aligned}$$

and then use the intermediate value theorem with the continuity of $\sum_{n=1}^N e^{in\theta} v_n$ in θ when $N < \infty$.

UBS impies RB 2

- From (8) we thus obtain

$$\sum_{n=1}^{N} |a_n|^2 = \sum_{n=1}^{N} \left\| a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 = \left\| \sum_{n=1}^{N} e^{in\theta} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 ,$$

and hence from **UBS** that

$$\left\| \sum_{n=1}^{N} e^{in\theta} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 \leq C \left\| \sum_{n=1}^{N} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 ,$$

and

$$\left\| \sum_{n=1}^{N} e^{-in\theta} \left(e^{in\theta} a_n \right) \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 \leq C \left\| \sum_{n=1}^{N} e^{in\theta} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 .$$

UBS impies RB 2

- From (8) we thus obtain

$$\sum_{n=1}^{N} |a_n|^2 = \sum_{n=1}^{N} \left\| a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 = \left\| \sum_{n=1}^{N} e^{in\theta} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 ,$$

and hence from **UBS** that

$$\left\| \sum_{n=1}^{N} e^{in\theta} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 \leq C \left\| \sum_{n=1}^{N} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 ,$$

and

$$\left\| \sum_{n=1}^{N} e^{-in\theta} \left(e^{in\theta} a_n \right) \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 \leq C \left\| \sum_{n=1}^{N} e^{in\theta} a_n \frac{k_{z_n}}{\|k_{z_n}\|_{H^2}} \right\|_{H^2}^2 .$$

- Now let $N \to \infty$ to obtain **RB**.

The classical spaces

- For an integer $m \geq 0$, and for $0 \leq \sigma < \infty$, $m + \sigma > 1/2$ the analytic Besov-Sobolev spaces $B_2^\sigma(\mathbb{D})$ consist of those holomorphic functions f on the disk such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{D}} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda(z) \right\}^{\frac{1}{2}} < \infty. \tag{9}$$

The spaces $B_2^\sigma(\mathbb{D})$ are independent of m and are Hilbert spaces with inner product $\langle f, g \rangle$ given by

$$\sum_{k=0}^{m-1} f^{(k)}(0)\, \overline{g^{(k)}(0)} + \int_{\mathbb{D}} \left(1 - |z|^2\right)^{2(m+\sigma)} f^{(m)}(z)\, \overline{g^{(m)}(z)} d\lambda(z).$$

The classical spaces

- For an integer $m \geq 0$, and for $0 \leq \sigma < \infty$, $m + \sigma > 1/2$ the analytic Besov-Sobolev spaces $B_2^\sigma(\mathbb{D})$ consist of those holomorphic functions f on the disk such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{D}} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda(z) \right\}^{\frac{1}{2}} < \infty. \tag{9}$$

The spaces $B_2^\sigma(\mathbb{D})$ are independent of m and are Hilbert spaces with inner product $\langle f, g \rangle$ given by

$$\sum_{k=0}^{m-1} f^{(k)}(0)\overline{g^{(k)}(0)} + \int_{\mathbb{D}} \left(1 - |z|^2\right)^{2(m+\sigma)} f^{(m)}(z)\overline{g^{(m)}(z)} d\lambda(z).$$

- The space $B_2^\sigma(\mathbb{D})$ is a Hilbert function space on $\mathbb{D}$, and has reproducing kernel $k_z^\sigma(w)$ given by

$$k_z^\sigma(w) \equiv \begin{cases} \left(\frac{1}{1-w\overline{z}}\right)^{2\sigma} & \text{if} \quad 0 < \sigma < \frac{1}{2} \\ \frac{1}{w\overline{z}} \log \frac{1}{1-w\overline{z}} & \text{if} \quad \sigma = 0 \end{cases}, \qquad z \in \mathbb{D}, w \in \overline{\mathbb{D}}.$$

Magic Bullet #3

For $0 \leq \sigma \leq \frac{1}{2}$ the spaces $B_2^{\sigma}(\mathbb{D})$ have the complete Nevanlinna-Pick property (CNPP). This includes the Dirichlet space $\mathcal{D}(\mathbb{D}) = B_2^0(\mathbb{D})$.

Certain spaces with NP kernel

Theorem

Suppose $0 \leq \sigma \leq \frac{1}{2}$, $Z \subset \mathbb{D}$ *and* $\mu_Z = \sum_{z \in Z} k_z^\sigma(z)^{-\frac{1}{2}} \delta_z$. *Then* Z *is an interpolating sequence for* $B_2^\sigma(\mathbb{D})$ *if and only if* Z *is an interpolating sequence for the multiplier algebra* $M_{B_2^\sigma(\mathbb{D})}$ *if and only if* Z *satisfies the separation condition* $\inf_{i \neq j} \beta(z_i, z_j) > 0$ *and* μ_Z *is a* $B_2^\sigma(\mathbb{D})$*-Carleson measure.*

- We invoke a theorem of B. Böe which says that for certain Hilbert spaces with reproducing kernel, in the presence of the separation condition, a necessary and sufficient condition for a sequence to be interpolating is that the Grammian matrix $G \equiv \left[\left\langle \frac{k_{z_i}}{\left\|k_{z_i}\right\|}, \frac{k_{z_j}}{\left\|k_{z_j}\right\|}\right\rangle\right]_{i,j=1}^{\infty}$ associated with Z is bounded.

The Technical Property

- The spaces to which Böe's Theorem applies are those where the kernel has the Nevanlinna-Pick property, and which have the following additional Technical Property. Whenever we have a sequence for which the matrix G is bounded on ℓ^2 then the matrix with absolute values $\left[\left|\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|} \right\rangle\right|\right]_{i,j=1}^{\infty}$ is also bounded on ℓ^2.

The Technical Property

- The spaces to which Böe's Theorem applies are those where the kernel has the Nevanlinna-Pick property, and which have the following additional Technical Property. Whenever we have a sequence for which the matrix G is bounded on ℓ^2 then the matrix with absolute values $\left[\left|\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|}\right\rangle\right|\right]_{i,j=1}^{\infty}$ is also bounded on ℓ^2.
- For $0 \le \sigma < \frac{1}{2}$ the Technical Property holds because $\mathrm{Re}\left(\frac{1}{1-\overline{z_j}z_i}\right)^{2\sigma} \approx \left|\frac{1}{1-\overline{z_j}z_i}\right|^{2\sigma}$, which insures that the Gramm matrix has the desired property. For $\sigma = 0$ a slightly different ending will be given to the proof.

The Technical Property

- The spaces to which Böe's Theorem applies are those where the kernel has the Nevanlinna-Pick property, and which have the following additional Technical Property. Whenever we have a sequence for which the matrix G is bounded on ℓ^2 then the matrix with absolute values $\left[\left| \left\langle \frac{k_{z_i}}{\left\| k_{z_i} \right\|}, \frac{k_{z_j}}{\left\| k_{z_j} \right\|} \right\rangle \right| \right]_{i,j=1}^{\infty}$ is also bounded on ℓ^2.
- For $0 \leq \sigma < \frac{1}{2}$ the Technical Property holds because $\mathrm{Re}\left(\frac{1}{1-\overline{z_j} z_i} \right)^{2\sigma} \approx \left| \frac{1}{1-\overline{z_j} z_i} \right|^{2\sigma}$, which insures that the Gramm matrix has the desired property. For $\sigma = 0$ a slightly different ending will be given to the proof.
- Finally, the boundedness on ℓ^2 of the Grammian matrix is equivalent to $\mu_Z = \sum_{j=1}^{\infty} \left\| k_{z_j} \right\|^{-2} \delta_{z_j} = \sum_{j=1}^{\infty} (1 - |z_j|^2)^{2\sigma} \delta_{z_j}$ being a Carleson measure, so matters are reduced to Böe's Theorem once we know $B_2^{\sigma}(\mathbb{D})$ has the NPP.

Boundedness of the Grammian

- The Grammian matrix G is bounded on ℓ^2 if and only if μ_Z is a Carleson measure for H. To see this let $T : H \to \ell^2$ be the normalized restriction map $Tf = \left\{ \frac{f(z_j)}{\left\| k_{z_j} \right\|} \right\}_{j=1}^{\infty}$. Then μ_Z is a Carleson measure for H if and only if T is bounded.

Boundedness of the Grammian

- The Grammian matrix G is bounded on ℓ^2 if and only if μ_Z is a Carleson measure for H. To see this let $T : H \to \ell^2$ be the normalized restriction map $Tf = \left\{ \frac{f(z_j)}{\left\| k_{z_j} \right\|} \right\}_{j=1}^{\infty}$. Then μ_Z is a Carleson measure for H if and only if T is bounded.
- But $T^* \{\xi_j\}_{j=1}^{\infty} = \sum_{j=1}^{\infty} \xi_j \frac{k_{z_j}}{\left\| k_{z_j} \right\|}$ and so the matrix representation of TT^* relative to the standard basis $\{\mathbf{e}_j\}_{j=1}^{\infty}$ of ℓ^2 is the Grammian:

$$\left[\langle TT^* \mathbf{e}_i, \mathbf{e}_j \rangle \right]_{i,j=1}^{\infty} = \left[\left\langle T \left(\frac{k_{z_i}}{\| k_{z_i} \|} \right), \mathbf{e}_j \right\rangle \right]_{i,j=1}^{\infty}$$

$$= \left[\left\langle \frac{k_{z_i}(z_j)}{\| k_{z_i} \|} \right\rangle \right]_{i,j=1}^{\infty} = \left[\left\langle \frac{k_{z_i}}{\| k_{z_i} \|}, \frac{k_{z_j}}{\| k_{z_j} \|} \right\rangle \right]_{i,j=1}^{\infty}.$$

Boundedness of the Grammian

- The Grammian matrix G is bounded on ℓ^2 if and only if μ_Z is a Carleson measure for H. To see this let $T : H \to \ell^2$ be the normalized restriction map $Tf = \left\{ \frac{f(z_j)}{\|k_{z_j}\|} \right\}_{j=1}^{\infty}$. Then μ_Z is a Carleson measure for H if and only if T is bounded.
- But $T^* \{\xi_j\}_{j=1}^{\infty} = \sum_{j=1}^{\infty} \xi_j \frac{k_{z_j}}{\|k_{z_j}\|}$ and so the matrix representation of TT^* relative to the standard basis $\{\mathbf{e}_j\}_{j=1}^{\infty}$ of ℓ^2 is the Grammian:

$$
\begin{aligned}
\left[\langle TT^* \mathbf{e}_i, \mathbf{e}_j \rangle\right]_{i,j=1}^{\infty} &= \left[\left\langle T\left(\frac{k_{z_i}}{\|k_{z_i}\|} \right), \mathbf{e}_j \right\rangle \right]_{i,j=1}^{\infty} \\
&= \left[\left\langle \frac{k_{z_i}(z_j)}{\|k_{z_i}\|} \right\rangle \right]_{i,j=1}^{\infty} = \left[\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|} \right\rangle \right]_{i,j=1}^{\infty}.
\end{aligned}
$$

- Now use that T is bounded if and only if TT^* is bounded.

Certain Besov-Sobolev spaces have the NPP

- Agler and McCarthy showed that a reproducing kernel k has the complete Nevanlinna-Pick property if and only if for any finite set $\{z_1, z_2, ..., z_m\}$, the matrix H_m of reciprocals of inner products of reproducing kernels k_{z_i} for z_i, i.e.

$$H_m = \left[\frac{1}{\langle k_{z_i}, k_{z_j} \rangle} \right]_{i,j=1}^{m},$$

has exactly one positive eigenvalue counting multiplicities.

Certain Besov-Sobolev spaces have the NPP

- Agler and McCarthy showed that a reproducing kernel k has the complete Nevanlinna-Pick property if and only if for any finite set $\{z_1, z_2, ..., z_m\}$, the matrix H_m of reciprocals of inner products of reproducing kernels k_{z_i} for z_i, i.e.

$$H_m = \left[\frac{1}{\left\langle k_{z_i}, k_{z_j} \right\rangle} \right]_{i,j=1}^{m},$$

has exactly one positive eigenvalue counting multiplicities.

- Expand $\left\langle k_{z_i}, k_{z_j} \right\rangle^{-1}$ by the binomial theorem as

$$\left(1 - \overline{z_j} z_i\right)^{2\sigma} = 1 - \sum_{\ell=1}^{\infty} c_\ell \left(\overline{z_j} z_i\right)^{\ell},$$

where $0 \leq c_\ell = (-1)^{\ell+1} \binom{2\sigma}{\ell}$ for $\ell \geq 1$ and $0 < 2\sigma < 1$.

The proof continued

- The matrix $[\overline{z_j} z_i]_{i,j=1}^{m}$ is nonnegative semidefinite since

$$\sum_{i,j=1}^{m} \zeta_i \left(\overline{z_j} z_i\right) \overline{\zeta_i} = \left|(\zeta_1 z_1, ..., \zeta_m z_m)\right|^2 \geq 0.$$

The proof continued

- The matrix $[\overline{z_j} z_i]_{i,j=1}^{m}$ is nonnegative semidefinite since

$$\sum_{i,j=1}^{m} \zeta_i \left(\overline{z_j} z_i\right) \overline{\zeta_i} = \left|(\zeta_1 z_1, \ldots, \zeta_m z_m)\right|^2 \geq 0.$$

- Thus by Schur's Theorem so is $\left[\left(\overline{z_j} z_i\right)^{\ell}\right]_{i,j=1}^{m}$ for every $\ell \geq 1$, and hence, also, so is the sum with positive coefficients.

The proof continued

- The matrix $[\overline{z_j} z_i]_{i,j=1}^m$ is nonnegative semidefinite since

$$\sum_{i,j=1}^{m} \zeta_i \left(\overline{z_j} z_i\right) \overline{\zeta_i} = \left|(\zeta_1 z_1, \ldots, \zeta_m z_m)\right|^2 \geq 0.$$

- Thus by Schur's Theorem so is $\left[\left(\overline{z_j} z_i\right)^\ell\right]_{i,j=1}^m$ for every $\ell \geq 1$, and hence, also, so is the sum with positive coefficients.
- Thus the positive part of the matrix H_m is $[1]_{i,j=1}^m$ which has rank 1, and hence the sole positive eigenvalue of H_m is m.

Böe's Theorem

Theorem

Suppose H is a Hilbert space of analytic functions with a Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Suppose also that the Grammian has the Technical Property: whenever $\{z_j\}_{j=1}^{\infty}$ is a sequence for which the matrix G is bounded on ℓ^2 then the matrix with absolute values is also bounded on ℓ^2. Then a sequence $Z = \{z_j\}_{j=1}^{\infty}$ is interpolating for H if and only if Z is separated and $\mu_Z = \sum_{j=1}^{\infty} \left\| k_{z_j} \right\|^{-2} \delta_{z_j}$ is a Carleson measure for H.

Proof of Böe's Theorem

- If Z is interpolating for H, standard arguments show that Z is separated and that μ_Z is a Carleson measure for H.

Proof of Böe's Theorem

- If Z is interpolating for H, standard arguments show that Z is separated and that μ_Z is a Carleson measure for H.
- Conversely, the Grammian matrix G is bounded on ℓ^2. To show that Z is interpolating for H it suffices to show that $\left\{\widetilde{k_{z_i}}\right\}_{j=1}^{\infty}$ is a Riesz basis, where $\widetilde{k_{z_i}} = \frac{k_{z_i}}{\|k_{z_i}\|}$ is the normalized reproducing kernel for H. Let $\{f_j\}_{j=1}^{\infty}$ be the biorthogonal functions defined as the unique minimal norm solutions of

$$\frac{f_n(z_m)}{\|k_{z_m}\|} = \left\langle f_n, \widetilde{k_{z_m}} \right\rangle = \delta_m^n.$$

Proof of BT continued 2

- If P denotes projection onto the closed linear span $\vee_{j=1}^{\infty} k_{z_j}$ of the k_{z_j}, then $\left\langle Pf_n, \widetilde{k_{z_m}} \right\rangle = \left\langle f_n, \widetilde{k_{z_m}} \right\rangle = \delta_m^n$ and so $f_n = Pf_n \in \vee_{j=1}^{\infty} k_{z_j}$. By Bari's Theorem, $\left\{ \widetilde{k_{z_i}} \right\}_{j=1}^{\infty}$ is a Riesz basis if and only if both $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle \right]_{m,n=1}^{\infty}$ and $[\langle f_n, f_m \rangle]_{m,n=1}^{\infty}$ are bounded matrices on ℓ^2. We already know that $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle \right]_{m,n=1}^{\infty}$ is bounded, so it remains to show that $[\langle f_n, f_m \rangle]_{m,n=1}^{\infty}$ is also.

Proof of BT continued 2

- If P denotes projection onto the closed linear span $\vee_{j=1}^{\infty} k_{z_j}$ of the k_{z_j}, then $\left\langle Pf_n, \widetilde{k_{z_m}} \right\rangle = \left\langle f_n, \widetilde{k_{z_m}} \right\rangle = \delta_m^n$ and so $f_n = Pf_n \in \vee_{j=1}^{\infty} k_{z_j}$. By Bari's Theorem, $\left\{ \widetilde{k_{z_i}} \right\}_{j=1}^{\infty}$ is a Riesz basis if and only if both $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle \right]_{m,n=1}^{\infty}$ and $\left[\langle f_n, f_m \rangle \right]_{m,n=1}^{\infty}$ are bounded matrices on ℓ^2. We already know that $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle \right]_{m,n=1}^{\infty}$ is bounded, so it remains to show that $\left[\langle f_n, f_m \rangle \right]_{m,n=1}^{\infty}$ is also.
- For $A \subset Z = \{z_j\}_{j=1}^{\infty}$ let $H_A = \{f \in H : f(a) = 0 \text{ for } a \in A\}$. If $k_w^A(z)$ is the reproducing kernel for H_A, then $\left\| k_w^A \right\|^2 = k_w^A(w)$ and

$$k_w^A(w) = \sup \left\{ |f(w)| : f \in H_A \text{ with } \|f\| = \left\| k_w^A \right\| \right\}.$$

Proof of BT continued 3

- It follows that with $Z_n = Z \setminus \{z_n\}$, we have

$$f_n(z) = \frac{\|k_{z_n}\|}{\left\|k_{z_n}^{Z_n}\right\|^2} k_{z_n}^{Z_n}(z), \qquad n \geq 1.$$

Note in particular that

$$\|f_n\| = \frac{\|k_{z_n}\|}{\left\|k_{z_n}^{Z_n}\right\|} \text{ and } \frac{k_{z_n}^{Z_n}(z_m)}{\left\|k_{z_n}^{Z_n}\right\| \|k_{z_m}\|} = \frac{f_n(z_m)}{\|k_{z_m}\| \|f_n\|} = \frac{\delta_m^n}{\|f_n\|}.$$

Proof of BT continued 3

- It follows that with $Z_n = Z \setminus \{z_n\}$, we have

$$f_n(z) = \frac{\|k_{z_n}\|}{\left\|k_{z_n}^{Z_n}\right\|^2} k_{z_n}^{Z_n}(z), \qquad n \geq 1.$$

  Note in particular that

$$\|f_n\| = \frac{\|k_{z_n}\|}{\left\|k_{z_n}^{Z_n}\right\|} \text{ and } \frac{k_{z_n}^{Z_n}(z_m)}{\left\|k_{z_n}^{Z_n}\right\| \|k_{z_m}\|} = \frac{f_n(z_m)}{\|k_{z_m}\| \|f_n\|} = \frac{\delta_m^n}{\|f_n\|}.$$

- We now compute the entries $\langle f_n, f_m \rangle$ in the biorthogonal Grammian $\left[\langle f_n, f_m \rangle\right]_{m,n=1}^{\infty}$ in terms of the corresponding entries $\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle$ in the Grammian $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle\right]_{m,n=1}^{\infty}$. We have

$$\langle f_n, f_m \rangle = \frac{\|k_{z_n}\| \|k_{z_m}\|}{\left\|k_{z_n}^{Z_n}\right\|^2 \left\|k_{z_m}^{Z_m}\right\|^2} \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_m} \right\rangle. \tag{10}$$

Proof of BT continued 4

- Now we use that the reproducing kernels $k_w^{A\cup\{a\}}$ for $H_{A\cup\{a\}}$ are given in terms of those k_w^A for H_A by the formula

$$k_w^{A\cup\{a\}}(z) = k_w^A(z) - \frac{k_a^A(z)\,k_w^A(a)}{k_a^A(a)}.$$

Proof of BT continued 4

- Now we use that the reproducing kernels $k_w^{A\cup\{a\}}$ for $H_{A\cup\{a\}}$ are given in terms of those k_w^A for H_A by the formula

$$k_w^{A\cup\{a\}}(z) = k_w^A(z) - \frac{k_a^A(z)\, k_w^A(a)}{k_a^A(a)}.$$

- If we set

$$Z_{m,n} = Z \setminus \{z_m, z_n\} = Z_n \setminus \{z_m\} = Z_m \setminus \{z_n\},$$

we thus obtain

$$k_{z_n}^{Z_n}(z) = k_{z_n}^{Z_{m,n}}(z) - \frac{k_{z_m}^{Z_{m,n}}(z)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_m}^{Z_{m,n}}(z_m)}, \tag{11}$$

and the same formula with m and n interchanged.

Proof of BT continued 5

Then we have

$$
\begin{aligned}
\left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_m} \right\rangle &= \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_{m,n}} - \frac{k_{z_n}^{Z_{m,n}} k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} \right\rangle \\
&= \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_{m,n}} \right\rangle - \frac{k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} \left\langle k_{z_n}^{Z_n}, k_{z_n}^{Z_{m,n}} \right\rangle \\
&= k_{z_n}^{Z_n}(z_m) - \frac{k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} k_{z_n}^{Z_n}(z_n).
\end{aligned}
$$

Proof of BT continued 6

- Now from (11) we have

$$k_{z_n}^{Z_n}(z_n) = k_{z_n}^{Z_{m,n}}(z_n) - \frac{k_{z_m}^{Z_{m,n}}(z_n)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_m}^{Z_{m,n}}(z_m)} = \sigma_m^n k_{z_n}^{Z_{m,n}}(z_n),$$

where

$$\sigma_m^n = \frac{k_{z_n}^{Z_n}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} = \frac{\left\| k_{z_n}^{Z_n} \right\|^2}{\left\| k_{z_n}^{Z_{m,n}} \right\|^2} = 1 - \frac{k_{z_m}^{Z_{m,n}}(z_n)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_n}^{Z_{m,n}}(z_n)\, k_{z_m}^{Z_{m,n}}(z_m)}. \qquad (12)$$

Proof of BT continued 6

- Now from (11) we have

$$k_{z_n}^{Z_n}(z_n) = k_{z_n}^{Z_{m,n}}(z_n) - \frac{k_{z_m}^{Z_{m,n}}(z_n)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_m}^{Z_{m,n}}(z_m)} = \sigma_m^n k_{z_n}^{Z_{m,n}}(z_n),$$

where

$$\sigma_m^n = \frac{k_{z_n}^{Z_n}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} = \frac{\left\| k_{z_n}^{Z_n} \right\|^2}{\left\| k_{z_n}^{Z_{m,n}} \right\|^2} = 1 - \frac{k_{z_m}^{Z_{m,n}}(z_n)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_n}^{Z_{m,n}}(z_n)\, k_{z_m}^{Z_{m,n}}(z_m)}. \tag{12}$$

- This is at most 1 since

$$\left| k_{z_m}^{Z_{m,n}}(z_n) \right| = \left| \left\langle k_{z_m}^{Z_{m,n}}, k_{z_n}^{Z_{m,n}} \right\rangle \right| \leq \left\| k_{z_m}^{Z_{m,n}} \right\| \left\| k_{z_n}^{Z_{m,n}} \right\| = \sqrt{k_{z_m}^{Z_{m,n}}(z_m)\, k_{z_n}^{Z_{m,n}}(}$$

by Cauchy-Schwarz.

Proof of BT continued 7

- Note that $\left\| k_{z_n}^{Z_n} \right\|^2 = \sigma_m^n \left\| k_{z_n}^{Z_{m,n}} \right\|^2$. Combining equalities yields

$$\begin{aligned} \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_m} \right\rangle &= k_{z_n}^{Z_n}(z_m) - \frac{k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} k_{z_n}^{Z_n}(z_n) \\ &= k_{z_n}^{Z_n}(z_m) - \frac{k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} \sigma_m^n k_{z_n}^{Z_{m,n}}(z_n) \\ &= k_{z_n}^{Z_n}(z_m) - \sigma_m^n k_{z_m}^{Z_{m,n}}(z_n), \end{aligned} \tag{13}$$

and

$$\|f_n\| = \frac{\|k_{z_n}\|}{\left\| k_{z_n}^{Z_n} \right\|} \text{ and } \sigma_m^n = \frac{\left\| k_{z_n}^{Z_n} \right\|^2}{\left\| k_{z_n}^{Z_{m,n}} \right\|^2}.$$

Proof of BT continued 7

- Note that $\left\| k_{z_n}^{Z_n} \right\|^2 = \sigma_m^n \left\| k_{z_n}^{Z_{m,n}} \right\|^2$. Combining equalities yields

$$\left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_m} \right\rangle = k_{z_n}^{Z_n}(z_m) - \frac{k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} k_{z_n}^{Z_n}(z_n) \tag{13}$$

$$= k_{z_n}^{Z_n}(z_m) - \frac{k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} \sigma_m^n k_{z_n}^{Z_{m,n}}(z_n)$$

$$= k_{z_n}^{Z_n}(z_m) - \sigma_m^n k_{z_m}^{Z_{m,n}}(z_n),$$

and

$$\|f_n\| = \frac{\|k_{z_n}\|}{\left\| k_{z_n}^{Z_n} \right\|} \text{ and } \sigma_m^n = \frac{\left\| k_{z_n}^{Z_n} \right\|^2}{\left\| k_{z_n}^{Z_{m,n}} \right\|^2}.$$

- Note that $k_{z_n}^{Z_n}(z_m) = 0$ for $m \neq n$.

Proof of BT continued 8

- From the solution (7) to the extremal problem (6) with $Z_{m,n}$ in place of Z, and z_m in place of z_0, we obtain after renormalizing φ_0,

$$\frac{k_{z_m}^{Z_{m,n}}(z)}{\left\|k_{z_m}^{Z_{m,n}}\right\|^2} = \varphi_n^m(z)\,\frac{k_{z_m}(z)}{\left\|k_{z_m}\right\|^2}, \tag{14}$$

where $\varphi_n^m \in M_H$ is the unique extremal solution to

$$C_{M_H}(m,n) = \inf\left\{\left\|\varphi\right\|_{M_H} : \varphi(z_m) = 1 \text{ and } \varphi(z_j) = 0 \text{ for } j \in Z_{m,n}\right\}.$$

Proof of BT continued 8

- From the solution (7) to the extremal problem (6) with $Z_{m,n}$ in place of Z, and z_m in place of z_0, we obtain after renormalizing φ_0,

$$\frac{k_{z_m}^{Z_{m,n}}(z)}{\left\| k_{z_m}^{Z_{m,n}} \right\|^2} = \varphi_n^m(z) \frac{k_{z_m}(z)}{\| k_{z_m} \|^2}, \tag{14}$$

where $\varphi_n^m \in M_H$ is the unique extremal solution to

$$C_{M_H}(m,n) = \inf \left\{ \|\varphi\|_{M_H} : \varphi(z_m) = 1 \text{ and } \varphi(z_j) = 0 \text{ for } j \in Z_{m,n} \right\}.$$

- Before turning to a bound for $C_{M_H}(m,n)$, we complete the calculation of the biorthogonal Grammian $\left[\langle f_n, f_m \rangle\right]_{m,n=1}^{\infty}$.

For $m \neq n$ we have $k_{z_n}^{Z_n}(z_m) = 0$, and hence from (10), (13) and (14) we obtain

$$\begin{aligned}
\langle f_n, f_m \rangle &= \frac{\|k_{z_n}\| \, \|k_{z_m}\|}{\left\|k_{z_n}^{Z_n}\right\|^2 \left\|k_{z_m}^{Z_m}\right\|^2} \left\{ -\sigma_m^n k_{z_m}^{Z_{m,n}}(z_n) \right\} \\
&= -\frac{\|k_{z_n}\| \, \|k_{z_m}\|}{\left\|k_{z_n}^{Z_n}\right\|^2 \left\|k_{z_m}^{Z_m}\right\|^2} \sigma_m^n \left\|k_{z_m}^{Z_{m,n}}\right\|^2 \varphi_n^m(z_n) \frac{k_{z_m}(z_n)}{\|k_{z_m}\|^2} \\
&= -\|f_n\|^2 \frac{\sigma_m^n}{\sigma_n^m} \varphi_n^m(z_n) \frac{k_{z_m}(z_n)}{\|k_{z_m}\| \, \|k_{z_n}\|} \\
&= -\|f_n\|^2 \varphi_n^m(z_n) \left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle ,
\end{aligned}$$

since $\sigma_m^n = \sigma_n^m$ by (12).

Proof of BT continued 10

Generalized Blaschke products

- Using the Nevanlinna-Pick property and the identity (1) for H, there is a unique multiplier $\psi = \psi_{z_1}^{z_0} = \varphi_0 \in M_H$ of norm at most one satisfying the interpolation,

$$\psi(z_0) = d(z_0, z_1) = \sqrt{1 - \frac{|\langle k_{z_0}, k_{z_1} \rangle|^2}{\|k_{z_0}\|^2 \|k_{z_1}\|^2}} \text{ and } \psi(z_1) = 0,$$

and moreover, it is given by,

$$\psi_{z_1}^{z_0}(z) = d(z_0, z_1)^{-1} \left(1 - \frac{\langle k_{z_0}, k_{z_1} \rangle k_{z_1}(z)}{\langle k_{z_1}, k_{z_1} \rangle k_{z_0}(z)} \right). \tag{15}$$

Proof of BT continued 10

Generalized Blaschke products

- Using the Nevanlinna-Pick property and the identity (1) for H, there is a unique multiplier $\psi = \psi_{z_1}^{z_0} = \varphi_0 \in M_H$ of norm at most one satisfying the interpolation,

$$\psi(z_0) = d(z_0, z_1) = \sqrt{1 - \frac{\left|\langle k_{z_0}, k_{z_1}\rangle\right|^2}{\left\|k_{z_0}\right\|^2 \left\|k_{z_1}\right\|^2}} \text{ and } \psi(z_1) = 0,$$

and moreover, it is given by,

$$\psi_{z_1}^{z_0}(z) = d(z_0, z_1)^{-1}\left(1 - \frac{\langle k_{z_0}, k_{z_1}\rangle k_{z_1}(z)}{\langle k_{z_1}, k_{z_1}\rangle k_{z_0}(z)}\right). \tag{15}$$

- We will refer to $\psi_{z_1}^{z_0}$ as the *generalized Blaschke function* associated to the pair of points (z_0, z_1). It vanishes at z_1 and is positive at z_0. More generally, for $Z = \{z_n\}_{n=1}^{\infty}$, we will refer to the infinite product $B_Z^{z_0}(z) = \prod_{n=1}^{\infty} \psi_{z_n}^{z_0}(z)$ as the *generalized Blaschke product* in M_H associated to the set $Z = \{z_n\}_{n=1}^{\infty}$ with pole at $z_0 \notin Z$.

Proof of BT continued 11

The Blaschke condition

Theorem

Suppose H is a Hilbert space of analytic functions with a Nevanlinna-Pick reproducing kernel $k(x,y)$. Fix a sequence $Z = \{z_j\}_{j=1}^{\infty}$ and $z_0 \notin Z$. Then $B_Z^{z_0}(z)$ is not identically zero if and only if

$$B_Z^{z_0}(z_0)^2 \equiv \prod_{n=1}^{\infty} d(z_0, z_n)^2 > 0 \text{ if and only if } \mu_Z \text{ is a finite measure.}$$

Indeed, if the sequence $\{z_0\} \cup Z$ is separated and the measure μ_Z is finite,

$$\frac{|\langle k_n, k_{z_m}\rangle|}{\|k_{z_n}\| \, \|k_{z_m}\|} \leq (1-\varepsilon),$$

$$\sum_{n=1}^{\infty} \frac{|k_{z_0}(z_n)|^2}{\|k_{z_0}\|^2 \, \|k_{z_n}\|^2} = \sum_{n=1}^{\infty} \int \left|\widetilde{k_{z_0}}(z)\right|^2 d\mu_Z(z) = C_{z_0},$$

$$B_Z^{z_0}(z_0)^2 = \prod_{n=1}^{\infty} \psi_{z_n}^{z_0}(z_0)^2 = \prod_{n=1}^{\infty} d(z_0, z_n)^2 > 0.$$

Proof of BT continued 12

- We now claim the inequality

$$C_{M_H}(m, n) \leq C, \qquad m, n \geq 1. \tag{16}$$

Proof of BT continued 12

- We now claim the inequality

$$C_{M_H}(m,n) \le C, \qquad m,n \ge 1. \tag{16}$$

- Indeed, since $\psi_{z_j}^{z_m}(z_m) = d(z_m, z_j)$ and $\psi_{z_j}^{z_m}(z_j) = 0$, the generalized Blaschke product with pole z_m associated with $Z_{m,n}$, is

$$\begin{aligned} B_{Z_{m,n}}^{z_m}(z) &= \prod_{j \notin \{m,n\}} \psi_{z_j}^{z_m}(z) \\ &= \left\{ \prod_{j \notin \{m,n\}} d(z_m, z_j) \right\} \prod_{j \notin \{m,n\}} d(z_m, z_j)^{-1} \psi_{z_j}^{z_m}(z) \\ &= \left\{ \prod_{j \notin \{m,n\}} d(z_m, z_j) \right\} \varphi_n^m(z). \end{aligned}$$

Proof of BT continued 13

Since $B_{Z_{m,n}}^{z_m}$ is a multiplier of norm at most one, we then have

$$\begin{aligned} C_{M_H}(m,n) &\leq \prod_{j\notin\{m,n\}} d(z_m,z_j)^{-1} \\ &\leq \prod_{j\notin\{m,n\}} \left(1-\frac{\left|\left\langle k_{z_j},k_{z_m}\right\rangle\right|^2}{\left\|k_{z_j}\right\|^2\left\|k_{z_m}\right\|^2}\right)^{-1} \\ &\leq \sup_{m\geq 1}\prod_{j\neq m} \left(1-\frac{\left|\left\langle k_{z_j},k_{z_m}\right\rangle\right|^2}{\left\|k_{z_j}\right\|^2\left\|k_{z_m}\right\|^2}\right)^{-1}. \end{aligned}$$

Proof of BT continued 14

- By the Carleson condition applied to $\widetilde{k_{z_m}} = \frac{k_{z_m}}{\|k_{z_m}\|}$, we obtain

$$C = C\left\|\widetilde{k_{z_m}}\right\|^2 \geq \int \left|\widetilde{k_{z_m}}(z)\right|^2 d\mu_Z(z) = \sum_{j=1}^{\infty} \frac{\left|k_{z_m}(z_j)\right|^2}{\left\|k_{z_m}\right\|^2 \left\|k_{z_j}\right\|^2},$$

uniformly in m.

Proof of BT continued 14

- By the Carleson condition applied to $\widetilde{k_{z_m}} = \frac{k_{z_m}}{\|k_{z_m}\|}$, we obtain

$$C = C \left\| \widetilde{k_{z_m}} \right\|^2 \geq \int \left| \widetilde{k_{z_m}} (z) \right|^2 d\mu_Z (z) = \sum_{j=1}^{\infty} \frac{\left| k_{z_m} (z_j) \right|^2}{\left\| k_{z_m} \right\|^2 \left\| k_{z_j} \right\|^2},$$

uniformly in m.

- This together with separation, i.e. $\frac{|k_{z_m}(z_j)|^2}{\|k_{z_m}\|^2 \left\| k_{z_j} \right\|^2} \leq 1 - \varepsilon$ for some $\varepsilon > 0$, yield

$$\prod_{j \neq m} \left(1 - \frac{\left| \langle k_{z_j}, k_{z_m} \rangle \right|^2}{\left\| k_{z_j} \right\|^2 \left\| k_{z_m} \right\|^2} \right) \geq c > 0, \qquad m \geq 1,$$

and hence (16).

Completion of proof of BT

- At this point we use (16) to conclude that $|\langle f_n, f_m \rangle| \leq C \left| \left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle \right|$ for all m, n.

Completion of proof of BT

- At this point we use (16) to conclude that $|\langle f_n, f_m\rangle| \leq C \left|\left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle\right|$ for all m, n.
- Our hypothesis on the Grammian $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle\right]_{m,n=1}^{\infty}$ shows that $\left[\left|\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle\right|\right]_{m,n=1}^{\infty}$ is bounded on ℓ^2, and thus so is $[|\langle f_n, f_m\rangle|]_{m,n=1}^{\infty}$, hence $[\langle f_n, f_m\rangle]_{m,n=1}^{\infty}$. This completes the proof of Böe's Theorem.

Completion of proof of BT

- At this point we use (16) to conclude that $|\langle f_n, f_m \rangle| \leq C \left| \left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle \right|$ for all m, n.
- Our hypothesis on the Grammian $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle \right]_{m,n=1}^{\infty}$ shows that $\left[\left| \left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle \right| \right]_{m,n=1}^{\infty}$ is bounded on ℓ^2, and thus so is $[|\langle f_n, f_m \rangle|]_{m,n=1}^{\infty}$, hence $[\langle f_n, f_m \rangle]_{m,n=1}^{\infty}$. This completes the proof of Böe's Theorem.
- To obtain the case $\sigma = \frac{1}{2}$ of the interpolation theorem, one can calculate that when $\sigma = \frac{1}{2}$, the expression $-\|f_n\|^2 \varphi_n^m(z_n)$ factors as a product $\psi_m \overline{\psi_n}$ with $|\psi_m| \leq C$, and then the boundedness of $\langle f_n, f_m \rangle$ follows immediately from that of $\left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle$.

An open problem

- It is an open problem whether or not interpolating sequences for a (even complete) Nevanlinna-Pick kernel are characterized by the necessary conditions: separation and the Carleson condition. That the answer is YES has been conjectured both by Seip and by Agler and McCarthy.

An open problem

- It is an open problem whether or not interpolating sequences for a (even complete) Nevanlinna-Pick kernel are characterized by the necessary conditions: separation and the Carleson condition. That the answer is YES has been conjectured both by Seip and by Agler and McCarthy.
- The above proof of Böe uses a heavy hammer at the end by taking absolute values inside the sum and requiring the technical property of the Grammian $\left[\left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle\right]_{m,n=}^{\infty}$.

An open problem

- It is an open problem whether or not interpolating sequences for a (even complete) Nevanlinna-Pick kernel are characterized by the necessary conditions: separation and the Carleson condition. That the answer is YES has been conjectured both by Seip and by Agler and McCarthy.
- The above proof of Böe uses a heavy hammer at the end by taking absolute values inside the sum and requiring the technical property of the Grammian $\left[\left\langle \widetilde{k_{z_m}}, \widetilde{k_{z_n}} \right\rangle\right]_{m,n=}^{\infty}$.
- A recently posted result on the arxiv by Chalendar, Fricain and Timotin shows that a YES answer to this problem implies the Feichtinger Conjecture (every Bessel sequence is a finite union of Riesz sequences) for complete Nevanlinna-Pick kernels, which speaks to the difficulty of this problem.

Part 4

Bilinear Hankel forms

Hankel operators

- Hankel operators on the Hardy space of the disk, $H^2(\mathbb{D})$, can be studied as linear operators from $H^2(\mathbb{D})$ to its dual space, as conjugate linear operators from $H^2(\mathbb{D})$ to itself, or, in the viewpoint we will take here, as bilinear functionals on $H^2(\mathbb{D}) \times H^2(\mathbb{D})$.

Hankel operators

- Hankel operators on the Hardy space of the disk, $H^2(\mathbb{D})$, can be studied as linear operators from $H^2(\mathbb{D})$ to its dual space, as conjugate linear operators from $H^2(\mathbb{D})$ to itself, or, in the viewpoint we will take here, as bilinear functionals on $H^2(\mathbb{D}) \times H^2(\mathbb{D})$.
- In that formulation, given a holomorphic *symbol function* b we consider the bilinear Hankel form, defined initially for f, g in $\mathcal{P}(\mathbb{D})$, the space of polynomials, by

$$S_b(f, g) := \langle fg, b \rangle_{H^2}.$$

Hankel operators

- Hankel operators on the Hardy space of the disk, $H^2(\mathbb{D})$, can be studied as linear operators from $H^2(\mathbb{D})$ to its dual space, as conjugate linear operators from $H^2(\mathbb{D})$ to itself, or, in the viewpoint we will take here, as bilinear functionals on $H^2(\mathbb{D}) \times H^2(\mathbb{D})$.
- In that formulation, given a holomorphic *symbol function* b we consider the bilinear Hankel form, defined initially for f, g in $\mathcal{P}(\mathbb{D})$, the space of polynomials, by

$$S_b(f,g) := \langle fg, b\rangle_{H^2}.$$

- The norm of S_b is

$$\|S_b\|_{H^2 \times H^2} = \sup\left\{ |S_b(f,g)| : \|f\|_{H^2} = \|g\|_{H^2} = 1 \right\}.$$

Nehari's theorem on the Hardy space

- Nehari's classical criterion for the boundedness of S_b on the Hardy space H^2 can be cast in modern language using Fefferman's duality theorem.

Nehari's theorem on the Hardy space

- Nehari's classical criterion for the boundedness of S_b on the Hardy space H^2 can be cast in modern language using Fefferman's duality theorem.
- We say a positive measure μ on the disk is a Carleson measure for H^2 if

$$\|\mu\|_{CM(H^2)} := \sup\left\{\int_{\mathbb{D}} |f|^2 \, d\mu : \|f\|_{H^2} = 1\right\} < \infty$$

and that b is in the space BMO if

$$\|b\|_{BMO} := |b(0)| + \left\| |b'(z)|^2 (1 - |z|^2) dA \right\|_{CM(H^2)} < \infty.$$

Nehari's theorem on the Hardy space

- Nehari's classical criterion for the boundedness of S_b on the Hardy space H^2 can be cast in modern language using Fefferman's duality theorem.
- We say a positive measure μ on the disk is a Carleson measure for H^2 if

$$\|\mu\|_{CM(H^2)} := \sup\left\{\int_{\mathbb{D}} |f|^2 \, d\mu : \|f\|_{H^2} = 1\right\} < \infty$$

  and that b is in the space BMO if

$$\|b\|_{BMO} := |b(0)| + \left\| \left|b'(z)\right|^2 (1 - |z|^2) dA \right\|_{CM(H^2)} < \infty.$$

- Nehari's theorem is the equivalence $\|S_b\|_{H^2 \times H^2} \approx \|b\|_{BMO}$.

Dirichlet Hankel operators

- Our main result is an analogous statement for a similar class of bilinear forms on the Dirichlet space $\mathcal{D}(\mathbb{D}) = \mathcal{D}$. Recall that $\mathcal{D}$ is the Hilbert space of holomorphic functions on the disk with inner product

$$\langle f, g \rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA,$$

and normed by $\|f\|_{\mathcal{D}}^2 = \langle f, f \rangle_{\mathcal{D}}$.

Dirichlet Hankel operators

- Our main result is an analogous statement for a similar class of bilinear forms on the Dirichlet space $\mathcal{D}(\mathbb{D}) = \mathcal{D}$. Recall that $\mathcal{D}$ is the Hilbert space of holomorphic functions on the disk with inner product

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA,$$

  and normed by $\|f\|_{\mathcal{D}}^2 = \langle f, f\rangle_{\mathcal{D}}$.
- We consider a holomorphic *symbol function* b and define the associated bilinear form, initially for $f, g \in \mathcal{P}(\mathbb{D})$, by

$$T_b(f, g) := \langle fg, b\rangle_{\mathcal{D}}.$$

Dirichlet Hankel operators

- Our main result is an analogous statement for a similar class of bilinear forms on the Dirichlet space $\mathcal{D}(\mathbb{D}) = \mathcal{D}$. Recall that $\mathcal{D}$ is the Hilbert space of holomorphic functions on the disk with inner product

$$\langle f, g \rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA,$$

  and normed by $\|f\|_{\mathcal{D}}^2 = \langle f, f \rangle_{\mathcal{D}}$.
- We consider a holomorphic *symbol function* b and define the associated bilinear form, initially for $f, g \in \mathcal{P}(\mathbb{D})$, by

$$T_b(f, g) := \langle fg, b \rangle_{\mathcal{D}}.$$

- The norm of T_b is

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} = \sup\left\{|T_b(f, g)| : \|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1\right\}.$$

The main theorem

We say a positive measure μ on the disk is a Carleson measure for $\mathcal{D}$ if

$$\|\mu\|_{CM(\mathcal{D})} := \sup\left\{\int_{\mathbb{D}} |f|^2 \, d\mu : \|f\|_{\mathcal{D}} = 1\right\} < \infty,$$

and that the holomorphic function b is in the space $\mathcal{X}$ if

$$\|b\|_{\mathcal{X}} := |b(0)| + \left\| |b'(z)|^2 \, dA \right\|_{CM(\mathcal{D})} < \infty.$$

Our main result is

Theorem

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\mathcal{X}}$$

Outline of the proof

- It is easy to see that $\| T_b \|_{\mathcal{D} \times \mathcal{D}} \leq C \| b \|_{\mathcal{X}}$.

Outline of the proof

- It is easy to see that $\| T_b \|_{\mathcal{D}\times\mathcal{D}} \leq C \| b \|_{\mathcal{X}}$.
- To obtain the other inequality we must use the boundedness of T_b to show $|b'|^2 \, dA$ is a Carleson measure.

Outline of the proof

- It is easy to see that $\| T_b \|_{\mathcal{D}\times\mathcal{D}} \leq C \|b\|_{\mathcal{X}}$.
- To obtain the other inequality we must use the boundedness of T_b to show $|b'|^2 \, dA$ is a Carleson measure.
- Analysis of the capacity theoretic characterization of Carleson measures due to Stegenga allows us to focus attention on a certain set V in $\mathbb{D}$ and the relative sizes of $\int_V |b'|^2$ and the capacity of the set $\bar{V} \cap \partial\mathbb{D}$.

Outline of the proof 2

- To compare these quantities we construct $V_{\exp}$, an expanded version of the set V which satisfies two conflicting conditions.

Outline of the proof 2

- To compare these quantities we construct V_{exp}, an expanded version of the set V which satisfies two conflicting conditions.
- First, V_{exp} is not much larger than V, either when measured by $\int_{V_{\text{exp}}} |b'|^2$ or by the capacity of the $\overline{V_{\text{exp}}} \cap \partial\mathbb{D}$.

Outline of the proof 2

- To compare these quantities we construct V_{exp}, an expanded version of the set V which satisfies two conflicting conditions.
- First, V_{exp} is not much larger than V, either when measured by $\int_{V_{\mathrm{exp}}} |b'|^2$ or by the capacity of the $\overline{V_{\mathrm{exp}}} \cap \partial\mathbb{D}$.
- Second, $\mathbb{D} \setminus V_{\mathrm{exp}}$ is well separated from V in a way that allows the interaction of quantities supported on the two sets to be controlled.

Outline of the proof 3

- Once this is done we can construct a function $\Phi_V \in \mathcal{D}$ which is approximately one on V and which has Φ_V' approximately supported on $\mathbb{D} \setminus V_{\exp}$. Using Φ_V we build functions f and g with the property that

$$|T_b(f,g)| = \int_V |b'|^2 + \text{ error.}$$

Outline of the proof 3

- Once this is done we can construct a function $\Phi_V \in \mathcal{D}$ which is approximately one on V and which has Φ_V' approximately supported on $\mathbb{D} \setminus V_{\exp}$. Using Φ_V we build functions f and g with the property that
$$|T_b(f,g)| = \int_V |b'|^2 + \text{ error.}$$
- The technical estimates on Φ_V allow us to show that the error term is small and the boundedness of T_b then gives the required control of $\int_V |b'|^2$.

The Easy Direction of the proof

- Suppose that μ_b is a $\mathcal{D}$-Carleson measure. For $f, g \in \mathcal{P}(\mathbb{D})$, we have that $|T_b(f,g)|$ is at most

$$\left| f(0)\,g(0)\,\overline{b(0)} + \int_{\mathbb{D}} \left[f'(z)\,g(z) + f(z)\,g'(z)\right] \overline{b'(z)}\,dA \right|$$

$$\begin{aligned} &\leq \left|(fgb)(0)\right| + \|f\|_{\mathcal{D}} \left(\int_{\mathbb{D}} |g|^2\,d\mu_b\right)^{\frac{1}{2}} + \|g\|_{\mathcal{D}} \left(\int_{\mathbb{D}} |f|^2\,d\mu_b\right)^{\frac{1}{2}} \\ &\leq C\left(|b(0)| + \|\mu_b\|_{\mathcal{D}-Carleson}\right) \|f\|_{\mathcal{D}} \|g\|_{\mathcal{D}} = C\,\|b\|_{\mathcal{X}} \|f\|_{\mathcal{D}} \|g\|_{\mathcal{D}}. \end{aligned}$$

Preliminaries of the Hard Direction

- Setting $g = 1$ we obtain

$$\left|\langle f, b \rangle_{\mathcal{D}}\right| = \left|T_b(f, 1)\right| \leq \|T_b\| \, \|f\|_{\mathcal{D}} \, \|1\|_{\mathcal{D}}$$

for all polynomials $f \in \mathcal{P}(\mathbb{D})$, which shows that $b \in \mathcal{D}$ and

$$\|b\|_{\mathcal{D}} \leq C \, \|T_b\| \, . \tag{17}$$

- Setting $g = 1$ we obtain

$$|\langle f, b\rangle_{\mathcal{D}}| = |T_b(f, 1)| \leq \|T_b\| \|f\|_{\mathcal{D}} \|1\|_{\mathcal{D}}$$

for all polynomials $f \in \mathcal{P}(\mathbb{D})$, which shows that $b \in \mathcal{D}$ and

$$\|b\|_{\mathcal{D}} \leq C \|T_b\| . \tag{17}$$

- Let I_m be the midpoint of I and $z(I) = \left(1 - \frac{|I|}{2\pi}\right) z$ be the associated index point in the disk. Let $I(z)$ to be the interval such that $z(I(z)) = z$. We set $T(I)$, the tent over I to be the convex hull of I and $z(I)$ and let $T(z) = T(z(I)) \equiv T(I)$. More generally, for any open subset H of the circle $\mathbb{T}$, we set $T(H) = \cup_{I \subset H} T(I)$, called the *tent region* of H in the disk $\mathbb{D}$.

Preliminaries of the Hard Direction 2

- To complete the proof we will show that $\mu_b = |b'|^2 \, dA$ is a $\mathcal{D}$-Carleson measure by verifying a condition due to Stegenga: For any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b \left(\dot{\cup}_{j=1}^N T(I_j) \right) \leq C \; Cap_{\mathbb{D}} \left(\dot{\cup}_{j=1}^N I_j \right), \tag{18}$$

where for open $G \subset \mathbb{T}$ in any quadrant $\mathbb{Q}$,

$$\mathrm{Cap}_{\mathbb{Q}} \, G = \inf \left\{ \|\psi\|_{\mathcal{D}}^2 : \psi(0) = 0, \mathrm{Re}\, \psi(z) \geq 1 \text{ for } z \in G \right\}, \tag{19}$$

and in general, $Cap_{\mathbb{D}}(G) \equiv \sum \mathrm{Cap}_{\mathbb{Q}}(G \cap \mathbb{Q})$, where the sum is over the four quadrants.

Preliminaries of the Hard Direction 2

- To complete the proof we will show that $\mu_b = |b'|^2 \, dA$ is a $\mathcal{D}$-Carleson measure by verifying a condition due to Stegenga: For any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b\left(\dot{\cup}_{j=1}^N T(I_j)\right) \le C \; Cap_{\mathbb{D}}\left(\dot{\cup}_{j=1}^N I_j\right), \tag{18}$$

  where for open $G \subset \mathbb{T}$ in any quadrant $\mathbb{Q}$,

$$\mathrm{Cap}_{\mathbb{Q}}\, G = \inf\left\{\|\psi\|_{\mathcal{D}}^2 : \psi(0) = 0, \mathrm{Re}\,\psi(z) \ge 1 \text{ for } z \in G\right\}, \tag{19}$$

  and in general, $Cap_{\mathbb{D}}(G) \equiv \sum \mathrm{Cap}_{\mathbb{Q}}(G \cap \mathbb{Q})$, where the sum is over the four quadrants.
- We have equivalence with the logarithmic capacity $Cap_{\log}$:

$$Cap_{\mathbb{D}}(G) \approx Cap_{\log}(G), \qquad G \subset \mathbb{T}.$$

Preliminaries of the Hard Direction 2

- To complete the proof we will show that $\mu_b = |b'|^2 \, dA$ is a $\mathcal{D}$-Carleson measure by verifying a condition due to Stegenga: For any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b \left(\dot{\cup}_{j=1}^N T(I_j) \right) \leq C \; Cap_{\mathbb{D}} \left(\dot{\cup}_{j=1}^N I_j \right), \tag{18}$$

where for open $G \subset \mathbb{T}$ in any quadrant $\mathbb{Q}$,

$$\mathrm{Cap}_{\mathbb{Q}} \, G = \inf \left\{ \|\psi\|_{\mathcal{D}}^2 : \psi(0) = 0, \mathrm{Re}\, \psi(z) \geq 1 \text{ for } z \in G \right\}, \tag{19}$$

and in general, $Cap_{\mathbb{D}}(G) \equiv \sum \mathrm{Cap}_{\mathbb{Q}}(G \cap \mathbb{Q})$, where the sum is over the four quadrants.

- We have equivalence with the logarithmic capacity $Cap_{\log}$:

$$Cap_{\mathbb{D}}(G) \approx Cap_{\log}(G), \qquad G \subset \mathbb{T}.$$

- In our proof we use functions for which equality in a tree version of (19) is approximately attained.

Disk blowup and capacity

- For I an open arc and $0 < \rho \leq 1$, let I^ρ be the arc concentric with I having length $|I|^\rho$.

Definition

For G open in $\mathbb{T}$ let $G_{\mathbb{D}}^\rho \equiv \cup_{I \subset G} T(I^\rho)$ be the *disk blowup* (of order ρ) of the open set $G \subset \mathbb{T}$. The important feature of the disk blowup is that it achieves a good geometric separation between $G_{\mathbb{D}}^\rho$ and $T(G) = G_{\mathbb{D}}^0$.

Lemma

Let G be an open subset of the circle $\mathbb{T}$. Then

$$|z - w| \geq \left(1 - |w|^2\right)^\rho, \qquad w \in T(G) \text{ and } z \notin G_{\mathbb{D}}^\rho.$$

Disk blowup and capacity

- For I an open arc and $0 < \rho \leq 1$, let I^{ρ} be the arc concentric with I having length $|I|^{\rho}$.

Definition

For G open in $\mathbb{T}$ let $G_{\mathbb{D}}^{\rho} \equiv \cup_{I \subset G} T(I^{\rho})$ be the *disk blowup* (of order ρ) of the open set $G \subset \mathbb{T}$. The important feature of the disk blowup is that it achieves a good geometric separation between $G_{\mathbb{D}}^{\rho}$ and $T(G) = G_{\mathbb{D}}^{0}$.

Lemma

Let G be an open subset of the circle $\mathbb{T}$. Then

$$|z - w| \geq \left(1 - |w|^2\right)^{\rho}, \qquad w \in T(G) \text{ and } z \notin G_{\mathbb{D}}^{\rho}.$$

- The inequality follows from $G_{\mathbb{D}}^{\rho} = \cup_{I \subset G} T(I^{\rho})$ and

$$T(I^{\rho}) \subset \left\{z : |z - z(I)| < 2\left(1 - |z(I)|^2\right)^{\rho}\right\}.$$

The key asymptotic capacity estimate

- In addition to good geometric separation, the capacity of disk blowup is controlled by an inequality of Bishop:

$$Cap_{\mathbb{D}}\left(\cup_{I\subset G} I^{\rho}\right) \leq C_{\rho}\, Cap_{\mathbb{D}}\, G. \tag{20}$$

The key asymptotic capacity estimate

- In addition to good geometric separation, the capacity of disk blowup is controlled by an inequality of Bishop:

$$Cap_{\mathbb{D}}\left(\cup_{I\subset G} I^{\rho}\right) \leq C_{\rho}\, Cap_{\mathbb{D}}\, G. \tag{20}$$

- We do not know if the constant C_{ρ} in (20) satisfies the asymptotic estimate,

$$\lim_{\rho\to 1^-} C_{\rho} = 1. \tag{21}$$

The key asymptotic capacity estimate

- In addition to good geometric separation, the capacity of disk blowup is controlled by an inequality of Bishop:

$$Cap_{\mathbb{D}}\left(\cup_{I\subset G} I^{\rho}\right) \leq C_{\rho} Cap_{\mathbb{D}} G. \tag{20}$$

- We do not know if the constant C_{ρ} in (20) satisfies the asymptotic estimate,

$$\lim_{\rho\to 1^{-}} C_{\rho} = 1. \tag{21}$$

- It turns out that an asymptotic inequality such as (21) is the key to our proof below, in which we require that $\mu_b\left(G_{\mathbb{D}}^{\beta} \setminus T(G)\right)$ is small for an appropriate "extremal" set G.

The key asymptotic capacity estimate

- In addition to good geometric separation, the capacity of disk blowup is controlled by an inequality of Bishop:

$$Cap_{\mathbb{D}}\left(\cup_{I\subset G} I^{\rho}\right) \leq C_{\rho} Cap_{\mathbb{D}} G. \tag{20}$$

- We do not know if the constant C_{ρ} in (20) satisfies the asymptotic estimate,

$$\lim_{\rho\to 1^{-}} C_{\rho} = 1. \tag{21}$$

- It turns out that an asymptotic inequality such as (21) is the key to our proof below, in which we require that $\mu_b\left(G_{\mathbb{D}}^{\beta} \setminus T(G)\right)$ is small for an appropriate "extremal" set G.
- While (21) remains in doubt for disk blowups, it turns out to hold for certain "tree" blowups to which we now turn.

Tree capacities

- Consider a dyadic tree T together with the following notation.

Tree capacities

- Consider a dyadic tree T together with the following notation.
- If x is an element of the tree T, x^{-1} denotes its immediate predecessor in T.

Tree capacities

- Consider a dyadic tree T together with the following notation.
- If x is an element of the tree T, x^{-1} denotes its immediate predecessor in T.
- If z is an element of the sequence $Z \subset T$, Pz denotes its predecessor in Z: $Pz \in Z$ is the maximum element of $Z \cap [o, z)$ (we assume $o \in Z$ for convenience).

Tree capacities

- Consider a dyadic tree T together with the following notation.
- If x is an element of the tree T, x^{-1} denotes its immediate predecessor in T.
- If z is an element of the sequence $Z \subset T$, Pz denotes its predecessor in Z: $Pz \in Z$ is the maximum element of $Z \cap [o, z)$ (we assume $o \in Z$ for convenience).
- Let $Cap_T(E)$ be the tree capacity of E given by

$$\inf \left\{ \sum_{\kappa \in T} \triangle f(\kappa)^2 : f(o) = 0, \ f(\beta) \geq 1 \text{ for } \beta \in E \right\}. \qquad (22)$$

Tree condensers

- More generally, the capacity $Cap_T(E,F)$ of the pair (E,F), commonly known as a condenser (E,F), is given by

$$\inf\left\{\sum_{\kappa\in T}\triangle f(\kappa)^2 : f(\alpha)\leq 0 \text{ for } \alpha\in E,\ f(\beta)\geq 1 \text{ for } \beta\in F\right\}. \tag{23}$$

Tree condensers

- More generally, the capacity $Cap_T(E,F)$ of the pair (E,F), commonly known as a condenser (E,F), is given by

$$\inf\left\{\sum_{\kappa\in T}\triangle f(\kappa)^2 : f(\alpha)\leq 0 \text{ for } \alpha\in E,\ f(\beta)\geq 1 \text{ for } \beta\in F\right\}. \tag{23}$$

- We say that $S\subset T$ is a *stopping time* if every pair of distinct points in S are incomparable in T.

Tree condensers

- More generally, the capacity $Cap_T(E,F)$ of the pair (E,F), commonly known as a condenser (E,F), is given by

$$\inf\left\{\sum_{\kappa\in T}\triangle f(\kappa)^2 : f(\alpha)\leq 0 \text{ for } \alpha\in E,\ f(\beta)\geq 1 \text{ for } \beta\in F\right\}. \tag{23}$$

- We say that $S\subset T$ is a *stopping time* if every pair of distinct points in S are incomparable in T.
- Given stopping times $E,F\subset T$ we say that $E\succ F$ if for every $x\in E$ there is $y\in F$ with $y<x$.

Tree condensers

- More generally, the capacity $Cap_T(E, F)$ of the pair (E, F), commonly known as a condenser (E, F), is given by

$$\inf\left\{\sum_{\kappa\in T} \triangle f(\kappa)^2 : f(\alpha) \leq 0 \text{ for } \alpha \in E,\ f(\beta) \geq 1 \text{ for } \beta \in F\right\}. \tag{23}$$

- We say that $S \subset T$ is a *stopping time* if every pair of distinct points in S are incomparable in T.
- Given stopping times $E, F \subset T$ we say that $E \succ F$ if for every $x \in E$ there is $y \in F$ with $y < x$.
- For stopping times $E \succ F$ denote by $\mathcal{G}(E, F)$ the union of all those geodesics connecting a point of $x \in E$ to the point $y \in F$ lying above it, i.e. $y < x$.

Harmonic functions on trees

- Let $\Omega \subseteq T$. A point $x \in T$ is in the interior of Ω if $x, x^{-1}, x_+, x_- \in \Omega$. A function H is *harmonic* in Ω if

$$H(x) = \frac{1}{3}[H(x^{-1}) + H(x_+) + H(x_-)] \tag{24}$$

for every point x which is interior in Ω.

Harmonic functions on trees

- Let $\Omega \subseteq T$. A point $x \in T$ is in the interior of Ω if $x, x^{-1}, x_+, x_- \in \Omega$. A function H is *harmonic* in Ω if

$$H(x) = \frac{1}{3}[H(x^{-1}) + H(x_+) + H(x_-)] \tag{24}$$

  for every point x which is interior in Ω.

- Let $Ih\left(x\right) = \sum_{y \in [o,x]} h\left(y\right)$. If $H = Ih$ is harmonic in Ω, then we have the martingale property,

$$h(x) = h(x_+) + h(x_-), \tag{25}$$

  whenever x is in the interior of Ω.

Harmonic functions on trees

- Let $\Omega \subseteq T$. A point $x \in T$ is in the interior of Ω if $x, x^{-1}, x_+, x_- \in \Omega$. A function H is *harmonic* in Ω if

$$H(x) = \frac{1}{3}[H(x^{-1}) + H(x_+) + H(x_-)] \tag{24}$$

  for every point x which is interior in Ω.
- Let $Ih(x) = \sum_{y \in [o,x]} h(y)$. If $H = Ih$ is harmonic in Ω, then we have the martingale property,

$$h(x) = h(x_+) + h(x_-), \tag{25}$$

  whenever x is in the interior of Ω.
- Here is the main theorem on condensers in trees.

Harmonic theorem

Theorem

Let T be a dyadic tree and suppose that E and F are subsets as above.

1. There is an extremal function $H = Ih$ such that $Cap(E, F) = \|h\|_{\ell^2}^2$.
2. The function H is harmonic on $T \setminus (E \cup F)$.
3. If S is a stopping time in T, then $\sum_{\kappa \in S} |h(\kappa)| \leq 2Cap(E, F)$.
4. The function h is positive on $\mathcal{G}(E, F)$, and zero elsewhere.

Stopping time blowups

- An analogue of the disk blowup in trees is the *stopping time blowup*.

Definition

Given $0 \leq \rho \leq 1$ and a stopping time W in a tree T, define the *stopping time blowup* W_T^ρ of W in T as the set of minimal tree elements in $\{R^\rho \kappa : \kappa \in \mathcal{T}_\theta\}$, where $R^\rho \kappa$ denotes the unique element in the tree T satisfying

$$\begin{aligned} o &\leq R^\rho \kappa \leq \kappa, \\ \rho d(\kappa) &\leq d(R^\rho \kappa) < \rho d(\kappa) + 1. \end{aligned} \tag{26}$$

Stopping time blowups

- An analogue of the disk blowup in trees is the *stopping time blowup*.

Definition

Given $0 \leq \rho \leq 1$ and a stopping time W in a tree T, define the *stopping time blowup* W_T^ρ of W in T as the set of minimal tree elements in $\{R^\rho \kappa : \kappa \in \mathcal{T}_\theta\}$, where $R^\rho \kappa$ denotes the unique element in the tree T satisfying

$$\begin{aligned} o &\leq R^\rho \kappa \leq \kappa, \\ \rho d(\kappa) &\leq d(R^\rho \kappa) < \rho d(\kappa) + 1. \end{aligned} \qquad (26)$$

- Clearly W_T^ρ is a stopping time in T. Note that $R^1 \kappa = \kappa$. The element $R^\rho \kappa$ can be thought of as the "ρ^{th} root of κ" since in the Bergman tree model $\mathcal{T}$, $|R^\rho \kappa| = 2^{-d(R^\rho \kappa)} \approx 2^{-\rho d(\kappa)} = |\kappa|^\rho$.

Rotated tree capacities

- Now let $\mathcal{T}$ be the standard Bergman tree in $\mathbb{D}$. Let $\mathcal{T}_\theta$ be the rotation of the tree $\mathcal{T}$ by the angle θ, and let $Cap_{\mathcal{T}_\theta}$ be the tree capacity associated with $\mathcal{T}_\theta$ as in (22), and extend the definition to open subsets G of $\mathbb{T}$ by defining $Cap_{\mathcal{T}_\theta}(G)$ to be

$$\inf\left\{\sum_{\kappa\in\mathcal{T}_\theta}\triangle f(\kappa)^2 : f(o)=0,\ f(\beta)\geq 1 \text{ for } \beta\in\mathcal{T}_\theta,\ I(\beta)\subset G\right\}.$$

Rotated tree capacities

- Now let $\mathcal{T}$ be the standard Bergman tree in $\mathbb{D}$. Let $\mathcal{T}_\theta$ be the rotation of the tree $\mathcal{T}$ by the angle θ, and let $Cap_{\mathcal{T}_\theta}$ be the tree capacity associated with $\mathcal{T}_\theta$ as in (22), and extend the definition to open subsets G of $\mathbb{T}$ by defining $Cap_{\mathcal{T}_\theta}(G)$ to be

$$\inf \left\{ \sum_{\kappa \in \mathcal{T}_\theta} \triangle f(\kappa)^2 : f(o) = 0, \ f(\beta) \geq 1 \text{ for } \beta \in \mathcal{T}_\theta, \, I(\beta) \subset G \right\}.$$

- This is consistent with the definition of tree capacity of a stopping time W in $\mathcal{T}_\theta$ in the sense that if $G = \cup \{ I(\kappa) : \kappa \in W \}$, we have

$$Cap_{\mathcal{T}_\theta}(W) = Cap_{\mathcal{T}_\theta}(\{o\}, W) = Cap_{\mathcal{T}_\theta}(G).$$

Rotated tree capacities

- Now let $\mathcal{T}$ be the standard Bergman tree in $\mathbb{D}$. Let $\mathcal{T}_\theta$ be the rotation of the tree $\mathcal{T}$ by the angle θ, and let $Cap_{\mathcal{T}_\theta}$ be the tree capacity associated with $\mathcal{T}_\theta$ as in (22), and extend the definition to open subsets G of $\mathbb{T}$ by defining $Cap_{\mathcal{T}_\theta}(G)$ to be

$$\inf \left\{ \sum_{\kappa \in \mathcal{T}_\theta} \triangle f(\kappa)^2 : f(o) = 0,\ f(\beta) \geq 1 \text{ for } \beta \in \mathcal{T}_\theta,\ I(\beta) \subset G \right\}.$$

- This is consistent with the definition of tree capacity of a stopping time W in $\mathcal{T}_\theta$ in the sense that if $G = \cup \{I(\kappa) : \kappa \in W\}$, we have

$$Cap_{\mathcal{T}_\theta}(W) = Cap_{\mathcal{T}_\theta}(\{o\}, W) = Cap_{\mathcal{T}_\theta}(G).$$

- When the angle θ is not important, we will simply write $\mathcal{T}$ with the understanding that all results have analogues with $\mathcal{T}_\theta$ in place of $\mathcal{T}$.

Stopping times, arcs and tents

- There are natural bijections between the following three sets of objects:

- *stopping times* W in the tree $\mathcal{T}$;
- $\mathcal{T}$*-open subsets* G of the circle $\mathbb{T}$;
- $\mathcal{T}$*-tent regions* Γ of the disk $\mathbb{D}$.

Stopping times, arcs and tents

- There are natural bijections between the following three sets of objects:

- *stopping times* W in the tree $\mathcal{T}$;
- $\mathcal{T}$*-open subsets* G of the circle $\mathbb{T}$;
- $\mathcal{T}$*-tent regions* Γ of the disk $\mathbb{D}$.

- The bijections are given as follows. For W a *stopping time* in $\mathcal{T}$, its associated $\mathcal{T}$*-open set* in $\mathbb{T}$ is the $\mathcal{T}$-shadow $S_{\mathcal{T}}(W) = \cup \{I(\kappa) : \kappa \in W\}$ of W on the circle (this also *defines* the collection of $\mathcal{T}$-open sets). The associated $\mathcal{T}$*-tent region* in $\mathbb{D}$ is $T_{\mathcal{T}}(W) = \cup \{T(I(\kappa)) : \kappa \in W\}$ (this also *defines* the collection of $\mathcal{T}$-tent regions).

Stopping times, arcs and tents

- There are natural bijections between the following three sets of objects:

- *stopping times* W in the tree $\mathcal{T}$;
- $\mathcal{T}$*-open subsets* G of the circle $\mathbb{T}$;
- $\mathcal{T}$*-tent regions* Γ of the disk $\mathbb{D}$.

- The bijections are given as follows. For W a *stopping time* in $\mathcal{T}$, its associated $\mathcal{T}$*-open set* in $\mathbb{T}$ is the $\mathcal{T}$-shadow $S_{\mathcal{T}}(W) = \cup \{I(\kappa) : \kappa \in W\}$ of W on the circle (this also *defines* the collection of $\mathcal{T}$-open sets). The associated $\mathcal{T}$*-tent region* in $\mathbb{D}$ is $T_{\mathcal{T}}(W) = \cup \{T(I(\kappa)) : \kappa \in W\}$ (this also *defines* the collection of $\mathcal{T}$-tent regions).
- Note that for any open subset E of the circle $\mathbb{T}$, there is a unique $\mathcal{T}$-open set $G \subset E$ such that $E \setminus G$ is at most countable. We often informally identify the open sets E and G.

Condenser difficulty

- In order to simplify notation, we identify a stopping time $W = W_{\mathcal{T}}$ with its associated $\mathcal{T}$-shadow on the circle and its $\mathcal{T}$-tent region in the disk.

Condenser difficulty

- In order to simplify notation, we identify a stopping time $W = W_{\mathcal{T}}$ with its associated $\mathcal{T}$-shadow on the circle and its $\mathcal{T}$-tent region in the disk.
- We now investigate the tree analogue $G_{\mathcal{T}}^{\rho}$ of the disk blowup $G_{\mathbb{D}}^{\rho}$ of an open subset G of the circle $\mathbb{T}$. According to the natural bijections above, we can view $G_{\mathcal{T}}^{\rho}$ as a stopping time, an open subset of the circle, or as a $\mathcal{T}$-tent region in the disk.

Condenser difficulty

- In order to simplify notation, we identify a stopping time $W = W_{\mathcal{T}}$ with its associated $\mathcal{T}$-shadow on the circle and its $\mathcal{T}$-tent region in the disk.
- We now investigate the tree analogue $G^{\rho}_{\mathcal{T}}$ of the disk blowup $G^{\rho}_{\mathbb{D}}$ of an open subset G of the circle $\mathbb{T}$. According to the natural bijections above, we can view $G^{\rho}_{\mathcal{T}}$ as a stopping time, an open subset of the circle, or as a $\mathcal{T}$-tent region in the disk.
- It turns out that if W is a stopping time for $\mathcal{T}$ and $Z = W^{\rho}_{\mathcal{T}}$ is the stopping time blowup of W, then there is a good estimate for the tree capacity of Z, namely $Cap_{\mathcal{T}}(\{o\}, Z) < \frac{1}{\rho} Cap_{\mathcal{T}}(\{o\}, W)$, but no good condenser estimate of the form,

$$Cap_{\mathcal{T}}(Z, W) < C_{\rho} Cap_{\mathcal{T}}(\{o\}, W).$$

Capacitary blowup

Thus the stopping time blowup does not lead to a useful capacity estimate for the condenser $Cap_{\mathcal{T}}\left(W_{\mathcal{T}}^{\rho}, W\right)$. Instead we use a method based on a *capacitary* extremal and a comparison principle. Let W be a stopping time in $\mathcal{T}$. By Theorem 11, there is a unique extremal function $H = Ih$ such that

$$\begin{aligned} H(o) &= 0, \\ H(x) &= 1 \text{ for } x \in W, \\ Cap_{\mathcal{T}} W &= \|h\|_{\ell^2}^2, \end{aligned} \tag{27}$$

Definition

Given a stopping time W in $\mathcal{T}$, the corresponding extremal H satisfying (27), and $0 < \rho < 1$, define the *capacitary blowup* $\widehat{W_{\mathcal{T}}^{\rho}}$ (stopping time) of W by

$$\widehat{W_{\mathcal{T}}^{\rho}} = \left\{t \in \mathcal{G}\left(\{o\}, W\right) : H(t) \geq \rho \text{ and } H(x) \leq \rho \text{ for } x < t\right\}.$$

Capacitary blowup estimates

- The capacitary blowup satisfies an estimate with constant asymptotically equal to 1.

Lemma

$Cap_{\mathcal{T}} \widehat{W_{\mathcal{T}}^{\rho}} \leq \frac{1}{\rho^2} Cap_{\mathcal{T}} W.$

Capacitary blowup estimates

- The capacitary blowup satisfies an estimate with constant asymptotically equal to 1.

Lemma

$Cap_{\mathcal{T}} \widehat{W_{\mathcal{T}}^{\rho}} \leq \frac{1}{\rho^2} Cap_{\mathcal{T}} W.$

- **Proof**: Let $H^{\rho} = \frac{1}{\rho} H$ and $h^{\rho} = \frac{1}{\rho} h$ where $h = \triangle H$ and H is the extremal for W in (27). Then H^{ρ} is a candidate for the infimum in the definition of capacity of $\widehat{W_{\mathcal{T}}^{\rho}}$, and hence by the "comparison principle",

$$Cap_{\mathcal{T}} \widehat{W_{\mathcal{T}}^{\rho}} \leq \|h^{\rho}\|_{\ell^2}^2 = \left(\frac{1}{\rho}\right)^2 \|h\|_{\ell^2}^2 = \frac{1}{\rho^2} Cap_{\mathcal{T}} W.$$

Tree separation

- We also have good *tree* separation inherited from the stopping time blowup $W_{\mathcal{T}}^{\rho}$.

Lemma

$W_{\mathcal{T}}^{\rho} \subset \widehat{W_{\mathcal{T}}^{\rho}}$ as open subsets of the circle or as $\mathcal{T}$-tent regions in the disk. Consequently, $Cap_{\mathcal{T}} W_{\mathcal{T}}^{\rho} \leq \frac{1}{\rho^2} Cap_{\mathcal{T}} W$.

Tree separation

- We also have good *tree* separation inherited from the stopping time blowup $W_{\mathcal{T}}^{\rho}$.

Lemma

$W_{\mathcal{T}}^{\rho} \subset \widehat{W_{\mathcal{T}}^{\rho}}$ as open subsets of the circle or as $\mathcal{T}$-tent regions in the disk. Consequently, $Cap_{\mathcal{T}} W_{\mathcal{T}}^{\rho} \leq \frac{1}{\rho^2} Cap_{\mathcal{T}} W$.

- **Proof**: The restriction of h to a geodesic is a concave function of distance from the root, and so if $o < z < w \in W$, then

$$h(z) \geq \left(1 - \frac{d(z)}{d(w)}\right) h(o) + \frac{d(z)}{d(w)} h(w) = \frac{d(z)}{d(w)} \geq \rho, \qquad z \in \widehat{W_{\mathcal{T}}^{\rho}},$$

and this proves $W_{\mathcal{T}}^{\rho} \subset \widehat{W_{\mathcal{T}}^{\rho}}$. The inequality now follows from Lemma 14.

A good condenser estimate

- The capacitary blowup $\widehat{W^{\rho}_{\mathcal{T}}}$, unlike the stopping time blowup $W^{\rho}_{\mathcal{T}}$, does indeed satisfy a good condenser inequality. It suffices to obtain a condenser inequality only for those W with small capacity.

Lemma

$$Cap_{\mathcal{T}}\left(W, \widehat{W^{\rho}_{\mathcal{T}}}\right) \leq \frac{4}{(1-\rho)^2} Cap_{\mathcal{T}} W \ \ provided\ Cap_{\mathcal{T}} W \leq \frac{1}{4}(1-\rho)^2.$$

A good condenser estimate

- The capacitary blowup $\widehat{W_{\mathcal{T}}^{\rho}}$, unlike the stopping time blowup $W_{\mathcal{T}}^{\rho}$, does indeed satisfy a good condenser inequality. It suffices to obtain a condenser inequality only for those W with small capacity.

Lemma

$Cap_{\mathcal{T}}\left(W, \widehat{W_{\mathcal{T}}^{\rho}}\right) \leq \frac{4}{(1-\rho)^2} Cap_{\mathcal{T}} W$ *provided* $Cap_{\mathcal{T}} W \leq \frac{1}{4}(1-\rho)^2$.

- **Proof**: Let H be the extremal for W in (27). For $t \in \widehat{W_{\mathcal{T}}^{\rho}}$ we have by our assumption,

$$h(t) \leq \|h\|_{\ell^2} \leq \sqrt{Cap_{\mathcal{T}} W} \leq \frac{1}{2}(1-\rho),$$

and so

$$H(t) = H(At) + h(t) \leq \rho + \frac{1}{2}(1-\rho) = \frac{1+\rho}{2}.$$

Condensers continued

- If we define $\widetilde{H}(t) = \frac{2}{1-\rho}\left\{H(t) - \frac{1+\rho}{2}\right\}$, then $\widetilde{H} \leq 0$ on $\widehat{W_{\mathcal{T}}^{\rho}}$ and $\widetilde{H} = 1$ on W. Thus $\widetilde{H}$ is a candidate for the capacity of the condenser and so by the "comparison principle",

$$\begin{aligned} Cap_{\mathcal{T}}\left(W, \widehat{W_{\mathcal{T}}^{\rho}}\right) &\leq \left\|\triangle \widetilde{H}\right\|_{\ell^2\left(\mathcal{G}\left(W_{\mathcal{T}}^{\rho}, W\right)\right)}^2 \leq \left\|\triangle \widetilde{H}\right\|_{\ell^2(\mathcal{T}_1)}^2 \\ &= \left(\frac{2}{1-\rho}\right)^2 \|h\|_{\ell^2(\mathcal{T}_1)}^2 = \frac{4}{(1-\rho)^2} Cap_{\mathcal{T}} W. \end{aligned}$$

Condensers continued

- If we define $\widetilde{H}(t) = \frac{2}{1-\rho}\left\{H(t) - \frac{1+\rho}{2}\right\}$, then $\widetilde{H} \leq 0$ on $\widehat{W_{\mathcal{T}}^{\rho}}$ and $\widetilde{H} = 1$ on W. Thus $\widetilde{H}$ is a candidate for the capacity of the condenser and so by the "comparison principle",

$$\begin{aligned} Cap_{\mathcal{T}}\left(W, \widehat{W_{\mathcal{T}}^{\rho}}\right) &\leq \left\|\triangle \widetilde{H}\right\|^2_{\ell^2\left(\mathcal{G}\left(W_{\mathcal{T}}^{\rho}, W\right)\right)} \leq \left\|\triangle \widetilde{H}\right\|^2_{\ell^2(\mathcal{T}_1)} \\ &= \left(\frac{2}{1-\rho}\right)^2 \|h\|^2_{\ell^2(\mathcal{T}_1)} = \frac{4}{(1-\rho)^2} Cap_{\mathcal{T}} W. \end{aligned}$$

- The disk blowups have good geometric separation properties (useful when estimating Bergman type kernels) and the capacitary blowup has a good condenser estimate (useful in constructing holomorphic extremals).

- Now we define a holomorphic approximation Φ to the function $H = Ih$ on $\mathcal{T}_1$ constructed in Proposition 11 using a parameter $s > -1$.

- Now we define a holomorphic approximation Φ to the function $H = Ih$ on $\mathcal{T}_1$ constructed in Proposition 11 using a parameter $s > -1$.
- Define an ameliorating factor by $\varphi_\kappa(z) = \left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}$.

- Now we define a holomorphic approximation Φ to the function $H = Ih$ on $\mathcal{T}_1$ constructed in Proposition 11 using a parameter $s > -1$.
- Define an ameliorating factor by $\varphi_\kappa(z) = \left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}$.
- Define a holomorphic approximation by

$$\Phi(z) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\,\varphi_\kappa(z) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}. \qquad (28)$$

The difference

Note that

$$\sum_{\kappa\in\mathcal{T}_1} h(\kappa)\, I\delta_\kappa(z) = I\left(\sum_{\kappa\in\mathcal{T}_1} h(\kappa)\,\delta_\kappa\right)(z) = Ih(z) = H(z),$$

and so the difference of the holomorphic approximation Φ and the extremal H is

$$\Phi(z) - H(z) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\left\{\varphi_\kappa - I\delta_\kappa\right\}(z). \tag{29}$$

Holomorphic Approximate Extremals and Capacity Estimates

The projection operator

- We will also need to write Φ in terms of the projection operator

$$\Gamma_s h(z) = \int_{\mathbb{D}} h(\zeta) \frac{\left(1 - |\zeta|^2\right)^s}{\left(1 - \overline{\zeta} z\right)^{1+s}} dA. \qquad (30)$$

Holomorphic Approximate Extremals and Capacity Estimates

The projection operator

- We will also need to write Φ in terms of the projection operator

$$\Gamma_s h(z) = \int_{\mathbb{D}} h(\zeta) \frac{\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} dA. \tag{30}$$

- Namely, $\Phi = \Gamma_s g$ where

$$g(\zeta) = \sum_{\kappa \in \mathcal{T}_1} h(\kappa) \frac{1}{|B_\kappa|} \frac{\left(1-\overline{\zeta}\kappa\right)^{1+s}}{\left(1-|\zeta|^2\right)^s} \chi_{B_\kappa}(\zeta), \tag{31}$$

and B_κ is the Euclidean ball centered at κ with radius $c(1-|\kappa|)$ for a sufficiently small positive constant c to be chosen later.

- We will also need to write Φ in terms of the projection operator

$$\Gamma_s h(z) = \int_{\mathbb{D}} h(\zeta) \frac{\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} dA. \tag{30}$$

- Namely, $\Phi = \Gamma_s g$ where

$$g(\zeta) = \sum_{\kappa \in \mathcal{T}_1} h(\kappa) \frac{1}{|B_\kappa|} \frac{\left(1-\overline{\zeta}\kappa\right)^{1+s}}{\left(1-|\zeta|^2\right)^s} \chi_{B_\kappa}(\zeta), \tag{31}$$

and B_κ is the Euclidean ball centered at κ with radius $c(1-|\kappa|)$ for a sufficiently small positive constant c to be chosen later.

- The function Φ satisfies the following estimates.

Holomorphic Approximate Extremals and Capacity Estimates

Theorem

Let $E = \{w_k\}_k$ *be contained in a quadrant* $\mathbb{Q}$, *and* $F = \{w_k^*\}_k$ *where* $F = \widehat{E_{\mathcal{T}}^{\rho}}$. *Suppose* $Cap_{\mathcal{T}}(E, F)$ *is sufficiently small,* $z \in \mathbb{D}$ *and* $s > -1$. *Then we have*

$$\begin{cases} |\Phi(z) - \Phi(w_k)| & \leq & CCap_{\mathcal{T}}(E,F), & z \in T(w_k) \\ \operatorname{Re}\Phi(w_k) & \geq & c > 0, & k \geq 1 \\ |\Phi(w_k)| & \leq & C, & k \geq 1 \\ |\Phi(z)| & \leq & CCap_{\mathcal{T}}(E,F), & z \notin F \end{cases} . \tag{32}$$

Furthermore, if $s > -\frac{1}{2}$ *then* $\Phi = \Gamma_s g$ *where*

$$|g(\zeta)|^2 \, dA \leq C \; Cap_{\mathcal{T}}(E,F). \tag{33}$$

Proof of the Holomorphic Approximation Theorem

Corollary

For $s > \frac{1}{2}$,

$$\|\Phi\|_{\mathcal{D}}^2 \leq \int_{\mathbb{D}} |g(\zeta)|^2 \, dA \leq C \ Cap_{\mathcal{T}}(E, F). \tag{34}$$

Proof of the theorem: From (29) we have

$$\begin{aligned} |\Phi(z) - H(z)| &\leq \sum_{\kappa \in [o,z]} |h(\kappa)\{\varphi_\kappa(z) - 1\}| + \sum_{\kappa \notin [o,z]} |h(\kappa)\varphi_\kappa(z)| \\ &= I(z) + II(z). \end{aligned}$$

We also have that h is nonnegative and supported in $V_G^\gamma \setminus V_G^\alpha$. We first show that

$$II(z) \leq \sum_{\kappa \notin [o,z]} h(\kappa) \left| \frac{1 - |\kappa|^2}{1 - \bar{\kappa} z} \right|^{1+s} \leq C Cap(E, F).$$

Proof of the HA Theorem 2

- For $A > 1$ let

$$\Omega_k = \left\{ \kappa \in \mathcal{T} : A^{-k-1} < \left| \frac{1 - |\kappa|^2}{1 - \overline{\kappa} z} \right| \leq A^{-k} \right\}.$$

Proof of the HA Theorem 2

- For $A > 1$ let

$$\Omega_k = \left\{ \kappa \in \mathcal{T} : A^{-k-1} < \left| \frac{1 - |\kappa|^2}{1 - \overline{\kappa} z} \right| \leq A^{-k} \right\}.$$

- If we choose A sufficiently close to 1, then for every k the set Ω_k is a union of two disjoint stopping times for $\mathcal{T}$.

Proof of the HA Theorem 2

- For $A > 1$ let

$$\Omega_k = \left\{ \kappa \in \mathcal{T} : A^{-k-1} < \left| \frac{1 - |\kappa|^2}{1 - \overline{\kappa} z} \right| \leq A^{-k} \right\}.$$

- If we choose A sufficiently close to 1, then for every k the set Ω_k is a union of two disjoint stopping times for $\mathcal{T}$.
- Now we use the stopping time property 3 in Theorem 11 to obtain

$$\sum_{\kappa \in \Omega_k} h(\kappa) \leq C Cap_{\mathcal{T}}(E, F), \ k \geq 0.$$

Proof of the HA Theorem 2

- For $A > 1$ let

$$\Omega_k = \left\{ \kappa \in \mathcal{T} : A^{-k-1} < \left| \frac{1 - |\kappa|^2}{1 - \overline{\kappa} z} \right| \leq A^{-k} \right\}.$$

- If we choose A sufficiently close to 1, then for every k the set Ω_k is a union of two disjoint stopping times for $\mathcal{T}$.
- Now we use the stopping time property 3 in Theorem 11 to obtain

$$\sum_{\kappa \in \Omega_k} h(\kappa) \leq C Cap_{\mathcal{T}}(E, F), \ \ k \geq 0.$$

- Altogether we then have

$$II(z) \leq \sum_{k=0}^{\infty} \sum_{\kappa \in \Omega_k} h(\kappa) A^{-k(1+s)} \leq C_s Cap_{\mathcal{T}}(E, F).$$

Proof of the HA Theorem 3

- If $z \in \mathbb{D} \setminus F$, then $I(z) = 0$ and $H(z) = 0$ and we have

$$\left|\Phi(z)\right| = \left|\Phi(z) - H(z)\right| \leq II(z) \leq C_s Cap_{\mathcal{T}}(E, F),$$

which is the fourth line in (32).

Proof of the HA Theorem 3

- If $z \in \mathbb{D} \setminus F$, then $I(z) = 0$ and $H(z) = 0$ and we have

$$|\Phi(z)| = |\Phi(z) - H(z)| \leq II(z) \leq C_s Cap_{\mathcal{T}}(E, F),$$

which is the fourth line in (32).

- If $z \in T(w_k)$, then for $\kappa \notin [o, w_k]$ we have

$$|\varphi_\kappa(w_k)| \leq C |\varphi_\kappa(z)|,$$

and for $\kappa \in [o, z]$ we have

$$\begin{aligned} |\varphi_\kappa(z) - \varphi_\kappa(w_k)| &= \left| \left(\frac{1 - |\kappa|^2}{1 - \overline{\kappa} z} \right)^{1+s} - \left(\frac{1 - |\kappa|^2}{1 - \overline{\kappa} w_k} \right)^{1+s} \right| \\ &\leq C_s \frac{|z - w_k|}{1 - |\kappa|^2}. \end{aligned}$$

Proof of the HA Theorem 4

Thus for $z \in T\left(w_k^\alpha\right)$,

$$
\begin{aligned}
\left|\Phi(z) - \Phi(w_k)\right| &\leq \sum_{\kappa \in \left[o, w_k^\alpha\right]} h(\kappa) \left|\varphi_\kappa(z) - \varphi_\kappa(w_k)\right| + C \sum_{\kappa \notin [o,z]} h(\kappa) \left|\varphi_\kappa(\right. \\
&\leq C_s \sum_{\kappa \in \left[o, w_k^\alpha\right]} h(\kappa) \frac{|z - w_k|}{1 - |\kappa|^2} + CII(z) \\
&\leq C_s \, Cap_{\mathcal{T}}(E, F),
\end{aligned}
$$

since $h(\kappa) \leq C \; Cap_{\mathcal{T}}(E, F)$ and $\sum_{\kappa \in [o, w_k]} \frac{1}{1 - |\kappa|^2} \approx \frac{1}{1 - |w_k|^2}$. This proves the first line in (32).

Proof of the HA Theorem 5

- Moreover, we note that for $s = 0$ and $\kappa \in [o, w_k]$,

$$\operatorname{Re} \varphi_\kappa (w_k) = \operatorname{Re} \frac{1 - |\kappa|^2}{1 - \overline{\kappa} w_k} = \operatorname{Re} \frac{1 - |\kappa|^2}{|1 - \overline{\kappa} w_k|^2} (1 - \kappa \overline{w_k}) \geq c > 0.$$

Proof of the HA Theorem 5

- Moreover, we note that for $s = 0$ and $\kappa \in [o, w_k]$,

$$\operatorname{Re} \varphi_\kappa (w_k) = \operatorname{Re} \frac{1 - |\kappa|^2}{1 - \overline{\kappa} w_k} = \operatorname{Re} \frac{1 - |\kappa|^2}{|1 - \overline{\kappa} w_k|^2} (1 - \kappa \overline{w_k}) \geq c > 0.$$

- A similar result holds for $s > -1$ provided the Bergman tree $\mathcal{T}$ is constructed sufficiently thin depending on s.

Proof of the HA Theorem 5

- Moreover, we note that for $s = 0$ and $\kappa \in [o, w_k]$,

$$\operatorname{Re} \varphi_\kappa (w_k) = \operatorname{Re} \frac{1 - |\kappa|^2}{1 - \overline{\kappa} w_k} = \operatorname{Re} \frac{1 - |\kappa|^2}{|1 - \overline{\kappa} w_k|^2} (1 - \kappa \overline{w_k}) \geq c > 0.$$

- A similar result holds for $s > -1$ provided the Bergman tree $\mathcal{T}$ is constructed sufficiently thin depending on s.
- It then follows from $\sum_{\kappa \in [o, w_k]} h(\kappa) = 1$ that

$$\begin{aligned} \operatorname{Re} \Phi (w_k) &= \sum_{\kappa \in [o, w_k]} h(\kappa) \operatorname{Re} \varphi_\kappa (w_k) + \sum_{\kappa \notin [o, w_k]} h(\kappa) \operatorname{Re} \varphi_\kappa (w_k) \\ &\geq c \sum_{\kappa \in [o, w_k]} h(\kappa) - C \; Cap_{\mathcal{T}} (E, F) \geq c' > 0. \end{aligned}$$

Proof of the HA Theorem 6

- We trivially have

$$\left|\Phi\left(w_k\right)\right| \leq I\left(z\right) + II\left(z\right) \leq C \sum_{\kappa \in [o, w_k]} h\left(\kappa\right) + C\ Cap_{\mathcal{T}}\left(E, F\right) \leq C,$$

and this completes the proof of (32).

Proof of the HA Theorem 6

- We trivially have

$$\left|\Phi\left(w_k\right)\right| \leq I\left(z\right) + II\left(z\right) \leq C \sum_{\kappa\in[o,w_k]} h\left(\kappa\right) + C\ Cap_{\mathcal{T}}\left(E,F\right) \leq C,$$

  and this completes the proof of (32).

- Finally we prove (33). From property 1 of Theorem 11 we obtain

$$\begin{aligned}\int_{\mathbb{D}} \left|g\left(\zeta\right)\right|^2 dA &= \int_{\mathbb{D}} \left|\sum_{\kappa\in\mathcal{T}} h\left(\kappa\right) \frac{1}{|B_\kappa|} \frac{\left(1-\overline{\zeta}\kappa\right)^{1+s}}{\left(1-|\zeta|^2\right)^s} \chi_{B_\kappa}\left(\zeta\right)\right|^2 dA \\ &= \sum_{\kappa\in\mathcal{T}} \left|h\left(\kappa\right)\right|^2 \frac{1}{|B_\kappa|^2} \int_{B_\kappa} \frac{\left|1-\overline{\zeta}\kappa\right|^{2+2s}}{\left(1-|\zeta|^2\right)^{2s}} dA \\ &\approx \sum_{\kappa\in\mathcal{T}} \left|h\left(\kappa\right)\right|^2 \approx Cap_{\mathcal{T}}\left(E,F\right).\end{aligned}$$

Comparison of tree and disk capacities

- We can now compare the tree and disk capacities.

Corollary

Let G be a finite union of arcs in the circle $\mathbb{T}$. Then

$$Cap_{\mathcal{T}}(G) \approx Cap_{\mathbb{D}}(G), \tag{35}$$

where $Cap_{\mathbb{D}}$ denotes the disk capacity.

Comparison of tree and disk capacities

- We can now compare the tree and disk capacities.

Corollary

Let G be a finite union of arcs in the circle $\mathbb{T}$. Then

$$Cap_{\mathcal{T}}\left(G\right) \approx Cap_{\mathbb{D}}\left(G\right), \tag{35}$$

where $Cap_{\mathbb{D}}$ denotes the disk capacity.

- **Proof**: We may suppose that $G \subset \mathbb{Q} \cap \mathbb{T}$ for some quadrant $\mathbb{Q}$. The inequality $\lesssim$ in (35) follows easily from Theorem 17 which provides a candidate for testing the Stegenga capacity of G.

Comparison of tree and disk capacities

- We can now compare the tree and disk capacities.

Corollary

Let G be a finite union of arcs in the circle $\mathbb{T}$. Then

$$Cap_{\mathcal{T}}\left(G\right) \approx Cap_{\mathbb{D}}\left(G\right), \tag{35}$$

where $Cap_{\mathbb{D}}$ denotes the disk capacity.

- **Proof**: We may suppose that $G \subset \mathbb{Q} \cap \mathbb{T}$ for some quadrant $\mathbb{Q}$. The inequality $\lesssim$ in (35) follows easily from Theorem 17 which provides a candidate for testing the Stegenga capacity of G.
- We take $F = \{o\}$ and $E = G$ in Theorem 17.

Comparison of tree and disk capacities

- We can now compare the tree and disk capacities.

Corollary

Let G be a finite union of arcs in the circle $\mathbb{T}$. Then

$$Cap_{\mathcal{T}}(G) \approx Cap_{\mathbb{D}}(G), \tag{35}$$

where $Cap_{\mathbb{D}}$ denotes the disk capacity.

- **Proof**: We may suppose that $G \subset \mathbb{Q} \cap \mathbb{T}$ for some quadrant $\mathbb{Q}$. The inequality $\lesssim$ in (35) follows easily from Theorem 17 which provides a candidate for testing the Stegenga capacity of G.
- We take $F = \{o\}$ and $E = G$ in Theorem 17.
- Let c, C be the constants in Theorem 17, and suppose that $Cap(E, F) \leq \frac{c}{3C}$. Set $\Psi(z) = \frac{3}{c}(\Phi(z) - \Phi(0))$.

Proof of comparison 2

- Then $\Psi(0) = 0$,

$$\begin{aligned} \operatorname{Re}\Psi(z) &= \frac{3}{c}\left\{\operatorname{Re}\Phi(z) - \operatorname{Re}\Phi(0)\right\} \\ &\geq \frac{3}{c}\left\{c - 2C\operatorname{Cap}(E,F)\right\} \geq 1,\ z \in G, \end{aligned}$$

and by (34) we have

$$\|\Psi\|_{\mathcal{D}}^2 = \left(\frac{3}{c}\right)^2 \|\Phi\|_{\mathcal{D}}^2 \leq \left(\frac{3}{c}\right)^2 C\ Cap(E,F).$$

Proof of comparison 2

- Then $\Psi(0) = 0$,

$$\begin{aligned} \operatorname{Re}\Psi(z) &= \frac{3}{c}\{\operatorname{Re}\Phi(z) - \operatorname{Re}\Phi(0)\} \\ &\geq \frac{3}{c}\{c - 2C\operatorname{Cap}(E,F)\} \geq 1, \ z \in G, \end{aligned}$$

and by (34) we have

$$\|\Psi\|_{\mathcal{D}}^2 = \left(\frac{3}{c}\right)^2 \|\Phi\|_{\mathcal{D}}^2 \leq \left(\frac{3}{c}\right)^2 C \ Cap(E,F).$$

- Continuing with Lemma 16 we obtain that for $G \subset \mathbb{T}$,

$$\|\Psi\|_{\mathcal{D}}^2 \leq \left(\frac{3}{c}\right)^2 C \ Cap_{\mathbb{T}}(E,F) \leq C \ Cap_{\mathbb{T}} E = C \ Cap_{\mathbb{T}} G.$$

Proof of comparison 3

- Conversely, to obtain the inequality $\gtrapprox$ in (35), let $\psi \in \mathcal{D}$ be an extremal function for $Cap_{\mathbb{D}} G$.

Proof of comparison 3

- Conversely, to obtain the inequality $\gtrsim$ in (35), let $\psi \in \mathcal{D}$ be an extremal function for $Cap_{\mathbb{D}} G$.
- Define $h(o) = 0$ and

$$h(\kappa) = (1 - |\kappa|) \int_{Q(\kappa)} |\psi'(z)| \, d\lambda(z), \ \kappa \in \mathcal{T} \setminus \{o\},$$

where $Q_h(\kappa)$ is the hyperbolic cube corresponding to κ in $\mathcal{T}$, and $d\lambda(z)$ is invariant measure on the disk $\mathbb{D}$.

Proof of comparison 3

- Conversely, to obtain the inequality $\gtrsim$ in (35), let $\psi \in \mathcal{D}$ be an extremal function for $Cap_{\mathbb{D}} G$.
- Define $h(o) = 0$ and

$$h(\kappa) = (1 - |\kappa|) \int_{Q(\kappa)} |\psi'(z)| \, d\lambda(z), \ \kappa \in \mathcal{T} \setminus \{o\},$$

where $Q_h(\kappa)$ is the hyperbolic cube corresponding to κ in $\mathcal{T}$, and $d\lambda(z)$ is invariant measure on the disk $\mathbb{D}$.

- One easily verifies that $Ih(o) = 0$, and

$$\begin{aligned} \|Ih\|^2_{B_2(\mathcal{T}_1)} &= \|h\|^2_{\ell^2(\mathcal{T})} = \sum_{\kappa \in \mathcal{T}_1} (1 - |\kappa|)^2 \left(\int_{Q(\kappa)} |\psi'(z)| \, d\lambda(z) \right)^2 \\ &\leq C \sum_{\kappa \in \mathcal{T}_1} \int_{Q(\kappa)} |\psi'(z)| \, dA = C \|\psi\|^2_{\mathcal{D}}. \end{aligned}$$

Proof of comparison 4

- Moreover,

$$Ih(\beta) = \sum_{\kappa \in [o,\beta]} h(\kappa) \geq \operatorname{Re} \psi(\beta) \geq c > 0, \text{ for } S(\beta) \subset G,$$

Proof of comparison 4

- Moreover,

$$Ih(\beta) = \sum_{\kappa \in [o,\beta]} h(\kappa) \geq \operatorname{Re}\psi(\beta) \geq c > 0, \text{ for } S(\beta) \subset G,$$

- Indeed, if $B_h(\kappa, R)$ is the hyperbolic ball of radius R about κ, then for R large enough,

$$\begin{aligned} |\psi(\beta)| &\leq \sum_{\kappa \in [o,\beta]} \left|\psi(\kappa) - \psi\left(\kappa^{-1}\right)\right| \\ &\leq \sum_{\kappa \in [o,\beta]} \left|\frac{1}{|B_h(\kappa,1)|} \int_{B_h(\kappa,1)} \psi(z)\, dA - \frac{1}{|B_h(\kappa^{-1},1)|} \int_{B_h(\kappa^{-1},1} \right. \\ &\leq C \sum_{\kappa \in [o,\beta]} \frac{1-|\kappa|^2}{|B_h(\kappa,1)|} \int_{B_h(\kappa,R)} |\psi'(z)|\, dA \\ &\leq C \sum_{\kappa \in [o,\beta]} \left(1-|\kappa|^2\right) \int_{Q(\kappa)} |\psi'(z)|\, d\lambda(z) = C \sum_{\kappa \in [o,\beta]} h(\kappa), \end{aligned}$$

where the final inequality is the submean value property for $|\psi'(z)|$.

End of proof of comparison

It follows that

$$
\begin{aligned}
Cap_{\mathcal{T}}\, G &= \inf\left\{ \|H\|^2_{B_2(\mathcal{T})} : H(0) = 0, \operatorname{Re} H(\kappa) \geq 1 \text{ if } S(\kappa) \subset G \right\} \\
&\leq \left\| \frac{1}{c} Ih \right\|^2_{B_2(\mathcal{T})} \leq \frac{C}{c^2} \|\psi\|^2_{\mathcal{D}} = \frac{C}{c^2} Cap_{\mathbb{D}}\, G .
\end{aligned}
$$

Asymptotic capacity estimate on the disk

- A result of Bishop says that

$$Cap_{\mathbb{D}}\left(\cup_{j=1}^{N} I_j^{\rho}\right) \leq C_\rho \, Cap_{\mathbb{D}}\left(\cup_{j=1}^{N} I_j\right), \qquad (36)$$

for a constant C_ρ depending only on $0 < \rho < 1$.

Asymptotic capacity estimate on the disk

- A result of Bishop says that

$$Cap_{\mathbb{D}}\left(\cup_{j=1}^{N} I_j^{\rho}\right) \leq C_\rho \, Cap_{\mathbb{D}}\left(\cup_{j=1}^{N} I_j\right), \tag{36}$$

  for a constant C_ρ depending only on $0 < \rho < 1$.

- In the next Corollary we use the asymptotic versions of this that hold for tree capacities, i.e $C_\rho \searrow 1$ as $\rho \nearrow 1$, given by Lemma 14.

Asymptotic capacity estimate on the disk

- A result of Bishop says that

$$Cap_{\mathbb{D}}\left(\cup_{j=1}^{N} I_j^{\rho}\right) \leq C_\rho \, Cap_{\mathbb{D}}\left(\cup_{j=1}^{N} I_j\right), \tag{36}$$

  for a constant C_ρ depending only on $0 < \rho < 1$.

- In the next Corollary we use the asymptotic versions of this that hold for tree capacities, i.e $C_\rho \searrow 1$ as $\rho \nearrow 1$, given by Lemma 14.

- Let $d\theta$ be Lebesgue measure on $\mathbb{T}$ normalized to have mass one. Abbreviate $Cap_{\mathcal{T}_\theta}$ by Cap_θ, and let $T_\theta(E)$ be the $\mathcal{T}_\theta$-tent region corresponding to an open subset E of the circle $\mathbb{T}$. Recall that $T(E) = \cup_{I \subset E} T(I)$. Now define M by

$$M := \sup_{E \text{ open } \subset \mathbb{T}} \frac{\int_{\mathbb{T}} \mu_b\left(T_\theta(E)\right) d\theta}{\int_{\mathbb{T}} Cap_\theta(E)\, d\theta}. \tag{37}$$

Proof of the Carleson measure estimate

- The quantity M is comparable to the Carleson measure norm squared.

Corollary

With M as in (37) we have $\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx M$.

Proof of the Carleson measure estimate

- The quantity M is comparable to the Carleson measure norm squared.

Corollary

With M as in (37) we have $\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx M$.

- **Proof**: Using Corollary 19 and $T_\theta(E) \subset T(E)$, we have

$$\begin{aligned} M &\leq C \sup_{E \text{ open } \subset \mathbb{T}} \frac{\int_{\mathbb{T}} \mu_b(T(E))\, d\theta}{\int_{\mathbb{T}} Cap_{\mathbb{D}}(E)\, d\theta} \\ &= C \sup_{E \text{ open } \subset \mathbb{T}} \frac{\mu_b(T(E))}{Cap_{\mathbb{D}}(E)} \approx \|\mu_b\|^2_{\mathcal{D}-Carleson}, \end{aligned}$$

where the final comparison is Stegenga's theorem.

Proof of the CE 2

- Conversely, one can verify using the argument in (40) below that for $0 < \rho < 1$,

$$\begin{aligned} \mu_b\left(T\left(E\right)\right) &\leq C\int_{\mathbb{T}} \mu_b\left(T_\theta\left(E_{\mathbb{D}}^{\rho}\right)\right) d\theta \\ &\leq CM\int_{\mathbb{T}} Cap_\theta\left(E_{\mathbb{D}}^{\rho}\right) d\theta \\ &\approx CMCap_{\mathbb{D}}\left(E_{\mathbb{D}}^{\rho}\right) \\ &\leq CMCap_{\mathbb{D}}\left(E\right), \end{aligned}$$

where the third line uses (35) with $E_{\mathbb{D}}^{\rho}$ and $\mathcal{T}_1\left(\theta\right)$ in place of G and $\mathcal{T}_1$, and the final inequality follows from (36).

Proof of the CE 2

- Conversely, one can verify using the argument in (40) below that for $0 < \rho < 1$,

$$\begin{aligned}
\mu_b(T(E)) &\leq C \int_{\mathbb{T}} \mu_b\left(T_\theta\left(E_{\mathbb{D}}^\rho\right)\right) d\theta \\
&\leq CM \int_{\mathbb{T}} Cap_\theta\left(E_{\mathbb{D}}^\rho\right) d\theta \\
&\approx CMCap_{\mathbb{D}}\left(E_{\mathbb{D}}^\rho\right) \\
&\leq CMCap_{\mathbb{D}}(E),
\end{aligned}$$

  where the third line uses (35) with $E_{\mathbb{D}}^\rho$ and $\mathcal{T}_1(\theta)$ in place of G and $\mathcal{T}_1$, and the final inequality follows from (36).

- Thus from Stegenga's theorem we obtain

$$\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx \sup_{E \text{ open } \subset \mathbb{T}} \frac{\mu_b(T(E))}{Cap_{\mathbb{D}}(E)} \leq CM.$$

The crucial step in the proof

- Given $0 < \delta < 1$, let G be an open set in $\mathbb{T}$ such that

$$\frac{\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\theta}{\int_{\mathbb{T}} Cap_\theta \left(G \right) d\theta} \geq \delta M \tag{38}$$

The crucial step in the proof

- Given $0 < \delta < 1$, let G be an open set in $\mathbb{T}$ such that

$$\frac{\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\theta}{\int_{\mathbb{T}} Cap_\theta \left(G \right) d\theta} \geq \delta M \tag{38}$$

- We need to know that $\mu_b(V_G^\beta \setminus V_G)$ is small compared to $\mu_b \left(V_G \right)$.

The crucial step in the proof

- Given $0 < \delta < 1$, let G be an open set in $\mathbb{T}$ such that

$$\frac{\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\theta}{\int_{\mathbb{T}} Cap_\theta \left(G \right) d\theta} \geq \delta M \tag{38}$$

- We need to know that $\mu_b(V_G^\beta \setminus V_G)$ is small compared to $\mu_b\left(V_G\right)$.
- This is the crucial step of the proof and is the main reason we introduced tree capacities - namely so that the asymptotic capacity estimate holds in Lemma 15.

The crucial step in the proof

- Given $0 < \delta < 1$, let G be an open set in $\mathbb{T}$ such that

$$\frac{\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\theta}{\int_{\mathbb{T}} Cap_\theta \left(G \right) d\theta} \geq \delta M \tag{38}$$

- We need to know that $\mu_b(V_G^\beta \setminus V_G)$ is small compared to $\mu_b \left(V_G \right)$.
- This is the crucial step of the proof and is the main reason we introduced tree capacities - namely so that the asymptotic capacity estimate holds in Lemma 15.

Theorem

Given $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) < 1$ in (38) and $\beta = \beta(\varepsilon) < 1$ so that for any G satisfying (38), we have with $V_G^\beta = G_{\mathbb{D}}^\beta$ and $V_G = G_{\mathbb{D}}^1 = T\left(G \right)$,

$$\mu_b(V_G^\beta \setminus V_G) \leq \varepsilon \mu_b \left(V_G \right), \tag{39}$$

Proof of the crucial step

- Let $G^{\rho}(\theta) = G^{\rho}_{\mathcal{I}_\theta}$ and $Cap_\theta = Cap_{\mathcal{I}_\theta}$. Lemma 15 shows that $Cap_\theta(G^{\rho}(\theta)) \leq \rho^{-2} Cap_\theta(G)$, for $0 \leq \theta < 2\pi$, $0 < \rho < 1$, and if we integrate on $\mathbb{T}$ we obtain

$$\int_{\mathbb{T}} Cap_\theta(G^{\rho}(\theta))\, d\sigma \leq \frac{1}{\rho^2} \int_{\mathbb{T}} Cap_\theta(G)\, d\theta.$$

Proof of the crucial step

- Let $G^{\rho}(\theta) = G^{\rho}_{\mathcal{T}_\theta}$ and $Cap_\theta = Cap_{\mathcal{T}_\theta}$. Lemma 15 shows that $Cap_\theta(G^{\rho}(\theta)) \leq \rho^{-2} Cap_\theta(G)$, for $0 \leq \theta < 2\pi$, $0 < \rho < 1$, and if we integrate on $\mathbb{T}$ we obtain

$$\int_{\mathbb{T}} Cap_\theta(G^{\rho}(\theta))\, d\sigma \leq \frac{1}{\rho^2} \int_{\mathbb{T}} Cap_\theta(G)\, d\theta.$$

- From (37) and (38) we thus have

$$\begin{aligned}
\int_{\mathbb{T}} \mu_b(T_\theta(G^{\rho}(\theta)))\, d\sigma &\leq M \int_{\mathbb{T}} Cap_\theta(G^{\rho}(\theta))\, d\theta \\
&\leq M \frac{1}{\rho^2} \int_{\mathbb{T}} Cap_\theta(G)\, d\theta \\
&\leq \frac{1}{\delta\rho^2} \int_{\mathbb{T}} \mu_b(T_\theta(G))\, d\theta.
\end{aligned}$$

Proof of the crucial step 2

It follows that

$$\begin{aligned} & \int_{\mathbb{T}} \mu_b \left(T_\theta \left(G^\rho \left(\theta \right) \right) \setminus T_\theta \left(G \right) \right) d\theta \\ = \; & \int_{\mathbb{T}} \mu_b \left(T_\theta \left(G^\rho \left(\theta \right) \right) \right) d\sigma - \int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\theta \\ \leq \; & \left(\frac{1}{\delta \rho^2} - 1 \right) \int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\theta . \end{aligned}$$

Proof of the crucial step 3

- Now with $\eta = \frac{\rho+1}{2}$ halfway between ρ and 1,

$$\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G^\rho (\theta) \right) \setminus T_\theta (G) \right) d\theta = \int_{\mathbb{T}} \int_{T_\theta(G^\rho(\theta)) \setminus T_\theta(G)} d\mu_b (z) \, d\theta$$

$$\geq \int_{\mathbb{T}} \int_{T_\theta(G^\rho(\theta)) \setminus T(G)} d\mu_b (z) \, d\theta = \int_{\mathbb{D}} \left\{ \frac{1}{2\pi} \int_{\{\theta : z \in T_\theta(G^\rho(\theta)) \setminus T(G)\}} d\theta \right\}$$

$$\geq \frac{1}{2} \int_{T\left(G_{\mathbb{D}}^\eta\right) \setminus T(G)} d\mu_b (z) \, ,$$

since every $z \in T\left(G_{\mathbb{D}}^\eta\right)$ lies in $T_\theta \left(G^\rho (\theta) \right)$ for at least half of the θ's in $[0, 2\pi)$.

Proof of the crucial step 3

- Now with $\eta = \frac{\rho+1}{2}$ halfway between ρ and 1,

$$\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G^\rho (\theta) \right) \setminus T_\theta (G) \right) d\theta = \int_{\mathbb{T}} \int_{T_\theta(G^\rho(\theta)) \setminus T_\theta(G)} d\mu_b (z)\, d\theta$$

$$\geq \int_{\mathbb{T}} \int_{T_\theta(G^\rho(\theta)) \setminus T(G)} d\mu_b (z)\, d\theta = \int_{\mathbb{D}} \left\{ \frac{1}{2\pi} \int_{\{\theta : z \in T_\theta(G^\rho(\theta)) \setminus T(G)\}} d\theta \right\}$$

$$\geq \frac{1}{2} \int_{T\left(G_{\mathbb{D}}^\eta\right) \setminus T(G)} d\mu_b (z),$$

since every $z \in T\left(G_{\mathbb{D}}^\eta\right)$ lies in $T_\theta \left(G^\rho (\theta) \right)$ for at least half of the θ's in $[0, 2\pi)$.

- We may assume above that the components of $G_{\mathbb{D}}^\rho$ have small length since otherwise we trivially have $\int_{\mathbb{T}} Cap_{\mathcal{T}(\theta)} (G)\, d\sigma \geq c > 0$ and so then

$$M \leq \frac{1}{c} \int d\mu_b \leq \frac{1}{c} \|b\|_{\mathcal{D}}^2 \leq \frac{C}{c} \|T_b\|^2. \tag{41}$$

Proof of the crucial step 4

- Combining the above inequalities and using $\rho = 2\eta - 1$, $\frac{1}{2} \leq \rho < 1$, and choosing $\delta = \eta$, we obtain

$$\begin{aligned}
\mu_b\left(T\left(G_{\mathbb{D}}^{\eta}\right) \setminus T\left(G\right)\right) &\leq 2\left(\frac{1}{\delta\rho^2} - 1\right)\int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta \\
&= 2\left(\frac{1}{\eta\left(2\eta - 1\right)^2} - 1\right)\int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta \\
&\leq C\left(1 - \eta\right)\int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta,
\end{aligned}$$

for $\frac{3}{4} \leq \eta < 1$.

Proof of the crucial step 4

- Combining the above inequalities and using $\rho = 2\eta - 1$, $\frac{1}{2} \leq \rho < 1$, and choosing $\delta = \eta$, we obtain

$$\begin{aligned}\mu_b\left(T\left(G_{\mathbb{D}}^{\eta}\right) \setminus T(G)\right) &\leq 2\left(\frac{1}{\delta\rho^2} - 1\right)\int_{\mathbb{T}} \mu_b\left(T_\theta(G)\right) d\theta \\ &= 2\left(\frac{1}{\eta\left(2\eta-1\right)^2} - 1\right)\int_{\mathbb{T}} \mu_b\left(T_\theta(G)\right) d\theta \\ &\leq C\left(1-\eta\right)\int_{\mathbb{T}} \mu_b\left(T_\theta(G)\right) d\theta,\end{aligned}$$

  for $\frac{3}{4} \leq \eta < 1$.

- Recalling $V_G^{\eta} = T\left(G_{\mathbb{D}}^{\eta}\right)$ and $V_G = T(G)$ this becomes

$$\mu_b\left(V_G^{\eta} \setminus V_G\right) \leq C\left(1-\eta\right)\int_{\mathbb{T}} \mu_b\left(T_\theta(G)\right) d\theta \leq C\left(1-\eta\right)\mu_b\left(V_G\right),$$

  $3/4 \leq \eta < 1$, since $T_\theta(G) \subset T(G) = V_G$ for all θ. Thus given $\varepsilon > 0$ it is possible to select δ and β so that (39) holds.

Theorem

Let (X, μ), (Y, ν) and (Z, ω) be measure spaces and $H(x, y, z)$ be a nonnegative measurable function on $X \times Y \times Z$. Define

$$T(f, g)(x) = \int_{Y \times Z} H(x, y, z) f(y) \, d\nu(y) \, g(z) \, d\omega(z), \quad x \in X,$$

at least initially for nonnegative functions f, g. Then if $1 < p < \infty$, T is bounded from $L^p(\nu) \times L^p(\omega)$ to $L^p(\mu)$ if there are positive functions h, k and m on X, Y and Z respectively such that

The Schur theorem continued

Theorem

$$\int_{Y\times Z} H(x,y,z)\,k(y)^{p'}\,m(z)^{p'}\,d\nu(y)\,d\omega(z) \leq (Ah(x))^{p'},$$

for μ-a.e. $x \in X$, and

$$\int_X H(x,y,z)\,h(x)^p\,d\mu(x) \leq (Bk(y)\,m(z))^p,$$

for $\nu \times \omega$-a.e. $(y,z) \in Y \times Z$. Moreover, $\|T\|_{operator} \leq AB$.

Proof of Schur's Theorem

$$
\begin{aligned}
& \int_X |Tf(x)|^p \, d\mu(x) \\
\leq \; & \int_X \left(\int_{Y \times Z} H(x,y,z) \, k(y)^{p'} m(z)^{p'} \, d\nu(y) \, d\omega(z) \right)^{p/p'} \\
& \times \left(\int_{Y \times Z} H(x,y,z) \left(\frac{f(y)}{k(y)} \right)^p d\nu(y) \left(\frac{g(z)}{m(z)} \right)^p d\omega(z) \right) d\mu(x) \\
\leq \; & A^p \int_{Y \times Z} \left(\int_X H(x,y,z) \, h(x)^p \, d\mu(x) \right) \left(\frac{f(y)}{k(y)} \right)^p d\nu(y) \left(\frac{g(z)}{m(z)} \right)^p d \\
\leq \; & A^p B^p \int_{Y \times Z} k(y)^p \, m(z)^p \left(\frac{f(y)}{k(y)} \right)^p d\nu(y) \left(\frac{g(z)}{m(z)} \right)^p d\omega(z) \\
= \; & (AB)^p \int_Y f(y)^p \, d\nu(y) \int_Z g(z)^p \, d\omega(z) .
\end{aligned}
$$

Integral estimates

Schur's Theorem can be used along with the estimates

$$\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^t}{|1-\overline{w}z|^{2+t+c}} dw \approx \begin{cases} C_t & \text{if} \quad c<0,\ t>-1 \\ -C_t \log(1-|z|^2) & \text{if} \quad c=0,\ t>-1 \\ C_t(1-|z|^2)^{-c} & \text{if} \quad c>0,\ t>-1 \end{cases}, . \tag{42}$$

to prove the following Corollary which we will use later.

Lebesgue boundedness

Define

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-\overline{w}z)^{2+a+b}} f(w)\,dw,$$

$$Sf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-\overline{w}z|^{2+a+b}} f(w)\,dw.$$

Corollary

Suppose that $t \in \mathbb{R}$ and $1 \leq p < \infty$ and set

$$d\nu_t(z) = (1-|z|^2)^t dA.$$

Then T is bounded on $L^p(\mathbb{D}, d\nu_t)$ if and only if S is bounded on $L^p(\mathbb{D}, d\nu_t)$ if and only if

$$-pa < t+1 < p(b+1). \tag{43}$$

A Bilinear Lemma

We now apply Theorem 22 to prove a lemma about a bilinear operator mapping $\ell^2(\mathcal{A}) \times \ell^2(\mathcal{B})$ to $L^2(\mathbb{D})$ where $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{T}$ which are well separated.

Lemma

Suppose $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{T}$, $h \in \ell^2(\mathcal{A})$ and $k \in \ell^2(\mathcal{B})$, and $\frac{1}{2} < \alpha < 1$. Suppose further that $\mathcal{A}$ and $\mathcal{B}$ satisfy the separation condition: $\forall \kappa \in \mathcal{A}$, $\gamma \in \mathcal{B}$ we have

$$|\kappa - \gamma| \geq (1 - |\gamma|^2)^\alpha. \tag{44}$$

Then the bilinear map of (h, k) to functions on the disk given by

$$T(h, b^*)(z) = \left(\sum_{\kappa \in \mathcal{A}} h(\kappa) \frac{(1 - |\kappa|^2)^{1+s}}{|1 - \overline{\kappa} z|^{2+s}} \right) \left(\sum_{\gamma \in \mathcal{B}} b^*(\gamma) \frac{(1 - |\gamma|^2)^{1+s}}{|1 - \overline{\gamma} z|^{1+s}} \right)$$

is bounded from $\ell^2(\mathcal{A}) \times \ell^2(\mathcal{B})$ to $L^2(\mathbb{D})$.

Proof of the Bilinear Lemma

- We will verify the hypotheses of the previous theorem. The kernel function here is

$$H(z,\kappa,\gamma) = \frac{\left(1-|\kappa|^2\right)^{1+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}, \; z \in \mathbb{D}, \kappa \in \mathcal{A}, \gamma \in \mathcal{B},$$

with Lebesgue measure on $\mathbb{D}$, and counting measure on $\mathcal{A}$ and $\mathcal{B}$.

Proof of the Bilinear Lemma

- We will verify the hypotheses of the previous theorem. The kernel function here is

$$H(z,\kappa,\gamma) = \frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}, \; z \in \mathbb{D}, \kappa \in \mathcal{A}, \gamma \in \mathcal{B},$$

  with Lebesgue measure on $\mathbb{D}$, and counting measure on $\mathcal{A}$ and $\mathcal{B}$.
- We will take as Schur functions

$$h(z) = \left(1-|z|^2\right)^{-\frac{1}{4}}, \; k(\kappa) = \left(1-|\kappa|^2\right)^{\frac{1}{4}} \text{ and } m(\gamma) = (1-|\gamma|^2)^{\frac{\varepsilon}{2}},$$

  on $\mathbb{D}$, $\mathcal{A}$ and $\mathcal{B}$ respectively, where $\varepsilon > 0$ will be chosen sufficiently small later.

We must then verify

$$\sum_{\kappa\in\mathcal{A}}\sum_{\gamma\in\mathcal{B}}\frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}\leq A^2\left(1-|z|^2\right)^{-\frac{1}{2}}, \tag{45}$$

for $z\in\mathbb{D}$, and

$$\int_{\mathbb{D}}\frac{\left(1-|\kappa|^2\right)^{1+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{\left(1-|\gamma|^2\right)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}\left(1-|z|^2\right)^{-\frac{1}{2}}dA \tag{46}$$

$$\leq B^2\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\varepsilon},$$

for $\kappa\in\mathcal{A}$ and $\gamma\in\mathcal{B}$.

Proof of the BL 3

- To prove (45) we write

$$\sum_{\kappa\in\mathcal{A}}\sum_{\gamma\in\mathcal{B}}\frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}$$

$$=\left(\sum_{\kappa\in\mathcal{A}}\frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}}\right)\left(\sum_{\gamma\in\mathcal{B}}\frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}\right).$$

- To prove (45) we write

$$\sum_{\kappa\in\mathcal{A}}\sum_{\gamma\in\mathcal{B}}\frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}$$

$$=\left(\sum_{\kappa\in\mathcal{A}}\frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}}\right)\left(\sum_{\gamma\in\mathcal{B}}\frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}\right).$$

- Then from (42) we obtain (45):

$$\sum_{\kappa\in\mathcal{A}}\frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}} \leq C\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^{-\frac{1}{2}+s}}{|1-\overline{w}z|^{2+s}}dw \leq C\left(1-|z|^2\right)^{-\frac{1}{2}}$$

$$\sum_{\gamma\in\mathcal{B}}\frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}} \leq C\int_{\zeta\in V_G}\frac{\left(1-|\zeta|^2\right)^{-1+\varepsilon+s}}{|1-\overline{\zeta}z|^{1+s}}dA \leq C.$$

Proof of the BL 4

- The proof of (46) will use separation (44).

Proof of the BL 4

- The proof of (46) will use separation (44).
- We have

$$\int_{\mathbb{D}} \frac{\left(1-|\kappa|^2\right)^{1+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+s}}{|1-\overline{\gamma}z|^{1+s}} \left(1-|z|^2\right)^{-\frac{1}{2}} dA$$

$$= \int_{|z-\gamma^*|\leq 1-|\gamma|^2} + \int_{1-|\gamma|^2\leq|z-\gamma^*|\leq\frac{1}{2}|\kappa-\gamma|}$$

$$+ \int_{|z-\kappa^*|\leq 1-|\kappa|^2} + \int_{1-|\kappa|^2\leq|z-\kappa^*|\leq\frac{1}{2}|\kappa-\gamma|} + \int_{|z-\gamma^*|,|z-\kappa^*|\geq|\kappa-\gamma|} \ldots dA$$

$$= I + II + III + IV + V.$$

Proof of the BL 5

By (44) $|\kappa - \gamma| \geq \left(1 - |\gamma|^2\right)^{\alpha}$ and so

$$
\begin{aligned}
I &\approx \frac{\left(1-|\kappa|^2\right)^{1+s}}{|\kappa-\gamma|^{2+s}} \int_{|z-\gamma^*|\leq 1-|\gamma|^2} \left(1-|z|^2\right)^{-\frac{1}{2}} dA \\
&\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{\frac{3}{2}}}{|\kappa-\gamma|^{2+s}} \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\frac{3}{2}(1-\alpha)},
\end{aligned}
$$

Proof of the BL 6

and similarly

$$
\begin{aligned}
II &\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{2+s}} \int_{1-|\gamma|^2 \leq |z-\gamma^*| \leq \frac{1}{2}|\kappa-\gamma|} \frac{\left(1-|z|^2\right)^{-\frac{1}{2}}}{|z-\gamma^*|^{1+s}} dA \\
&\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{2+s}} \left(1-|\gamma|^2\right)^{\frac{1}{2}-s} \\
&= \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{\frac{3}{2}}}{|\kappa-\gamma|^{2+s}} \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\frac{3}{2}(1-\alpha)}.
\end{aligned}
$$

Proof of the BL 7

Continuing to use $|\kappa - \gamma| \geq \left(1 - |\gamma|^2\right)^{\alpha}$ we obtain

$$III \approx \frac{\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{1+s}} \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{(1+s)(1-\alpha)},$$

and similarly,

$$IV \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\varepsilon},$$

for some $\varepsilon > 0$.

Proof of the BL 8

Finally

$$
\begin{aligned}
V &\approx \int_{|z-\gamma^*|,|z-\kappa^*|\geq|\kappa-\gamma|} \frac{\left(1-|\kappa|^2\right)^{1+s}}{|z-\kappa^*|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+s}}{|z-\gamma^*|^{1+s}} \left(1-|z|^2\right)^{-\frac{1}{2}} dA \\
&\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{\frac{3}{2}+2s}} \\
&\leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{(1+s)(1-\alpha)}.
\end{aligned}
$$

Proof of the Main Result: discussion

- To complete the proof of our main result, we will show that μ_b is a $\mathcal{D}$-Carleson measure by verifying Stegenga's condition (18); that is, we will show that for any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b\left(\dot{\cup}_{j=1}^N T(I_j)\right) \leq C \; Cap_{\mathbb{D}}\left(\dot{\cup}_{j=1}^N I_j\right).$$

Proof of the Main Result: discussion

- To complete the proof of our main result, we will show that μ_b is a $\mathcal{D}$-Carleson measure by verifying Stegenga's condition (18); that is, we will show that for any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b\left(\dot{\cup}_{j=1}^N T\left(I_j\right)\right) \leq C \ Cap_{\mathbb{D}}\left(\dot{\cup}_{j=1}^N I_j\right).$$

- In fact we will see that it suffices to verify this for the sets $G = \dot{\cup}_{j=1}^N I_j$ described in (38) that are near extremals for (37). We will prove the inequality

$$\mu_b\left(V_G\right) \leq C \left\| T_b \right\|^2 Cap_{\mathbb{D}}\left(G\right). \tag{47}$$

Proof discussion continued

- Once we have this, Corollary 19 yields

$$M = \frac{\int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\sigma}{\int_{\mathbb{T}} Cap_\theta\left(G\right) d\sigma} \leq \frac{\mu_b\left(V_G\right)}{\int_{\mathbb{T}} Cap_\theta\left(G\right) d\sigma} \leq C \left\| T_b \right\|^2 .$$

Proof discussion continued

- Once we have this, Corollary 19 yields

$$M = \frac{\int_{\mathbb{T}} \mu_b \left(T_\theta \left(G \right) \right) d\sigma}{\int_{\mathbb{T}} Cap_\theta \left(G \right) d\sigma} \leq \frac{\mu_b \left(V_G \right)}{\int_{\mathbb{T}} Cap_\theta \left(G \right) d\sigma} \leq C \left\| T_b \right\|^2 .$$

- By Corollary 20 $\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx M$ which then completes the proof of Theorem 8.

Proof of the Main Result

- We now turn to the proof of the estimate (47). Let $\frac{1}{2} < \beta < \beta_1 < \gamma < \alpha < 1$ to be fixed later. Let G be an open subset of the circle $\mathbb{T}$ satisfying (38) with $\varepsilon > 0$ to be chosen below. Let $\mathcal{T}$ be a Bergman tree.

Proof of the Main Result

- We now turn to the proof of the estimate (47). Let $\frac{1}{2} < \beta < \beta_1 < \gamma < \alpha < 1$ to be fixed later. Let G be an open subset of the circle $\mathbb{T}$ satisfying (38) with $\varepsilon > 0$ to be chosen below. Let $\mathcal{T}$ be a Bergman tree.
- We define in succession the following regions in the disk,

$$\begin{aligned} V_G &= T_{\mathcal{T}}(G), \quad V_G^\alpha = G_{\mathbb{D}}^\alpha, \\ V_G^\gamma &= \widehat{(V_G^\alpha)_{\mathcal{T}}^{\frac{\gamma}{\alpha}}}, \quad V_G^\beta = \left(V_G^\gamma\right)_{\mathbb{D}}^{\frac{\beta}{\gamma}}, \end{aligned}$$

so that V_G is the $\mathcal{T}$-tent associated with G, V_G^α is a disk blowup of G, V_G^γ is a $\mathcal{T}$-capacitary blowup of V_G^α, and V_G^β is a disk blowup of V_G^γ.

Proof of the Main Result

- We now turn to the proof of the estimate (47). Let $\frac{1}{2} < \beta < \beta_1 < \gamma < \alpha < 1$ to be fixed later. Let G be an open subset of the circle $\mathbb{T}$ satisfying (38) with $\varepsilon > 0$ to be chosen below. Let $\mathcal{T}$ be a Bergman tree.
- We define in succession the following regions in the disk,

$$\begin{aligned} V_G &= T_{\mathcal{T}}(G), \quad V_G^\alpha = G_{\mathbb{D}}^\alpha, \\ V_G^\gamma &= \widehat{(V_G^\alpha)_{\mathcal{T}}^{\frac{\gamma}{\alpha}}}, \quad V_G^\beta = \left(V_G^\gamma\right)_{\mathbb{D}}^{\frac{\beta}{\gamma}}, \end{aligned}$$

  so that V_G is the $\mathcal{T}$-tent associated with G, V_G^α is a disk blowup of G, V_G^γ is a $\mathcal{T}$-capacitary blowup of V_G^α, and V_G^β is a disk blowup of V_G^γ.
- Using the natural bijections introduced above, we write

$$V_G = \{w_k\}_k \text{ and } V_G^\alpha = \{w_k^\alpha\}_k \text{ and } V_G^\gamma = \left\{w_k^\gamma\right\}_k \text{ and } V_G^\beta = \left\{w_k^\beta\right\}_k, \tag{48}$$

  with $w_k, w_k^\alpha, w_k^\gamma, w_k^\beta \in \mathcal{T}$. Following previous notation we write $E = V_G^\alpha$ and $F = V_G^\gamma$.

Proof of the Main Result 3

- We will obtain our estimate (47) by using the boundedness of T_b on certain functions f and g in $\mathcal{D}$. The function f will be approximately $b'\chi_{V_G}$, and the function g will be constructed using an approximate extremal function and will be approximately equal to χ_{V_G}.

Proof of the Main Result 3

- We will obtain our estimate (47) by using the boundedness of T_b on certain functions f and g in $\mathcal{D}$. The function f will be approximately $b'\chi_{V_G}$, and the function g will be constructed using an approximate extremal function and will be approximately equal to χ_{V_G}.
- Now define Φ as in (28) above, so that we have the estimates in Proposition 17 and Corollary 18. From Corollary 19 and (36) we obtain

$$Cap_{\mathcal{T}}(E,F) \leq C Cap_{\mathbb{D}} G. \tag{49}$$

Proof of the Main Result 3

- We will obtain our estimate (47) by using the boundedness of T_b on certain functions f and g in $\mathcal{D}$. The function f will be approximately $b'\chi_{V_G}$, and the function g will be constructed using an approximate extremal function and will be approximately equal to χ_{V_G}.
- Now define Φ as in (28) above, so that we have the estimates in Proposition 17 and Corollary 18. From Corollary 19 and (36) we obtain

$$Cap_{\mathcal{T}}(E,F) \leq CCap_{\mathbb{D}} G. \tag{49}$$

- We will use $g = \Phi^2$ and

$$f(z) = \Gamma_s \left(\frac{1}{(1+s)\overline{\zeta}} \chi_{V_G} b'(\zeta) \right)(z) \tag{50}$$

as our test functions in the bilinear inequality

$$|T_b(f,g)| \leq \|T_b\| \, \|f\|_{\mathcal{D}} \, \|g\|_{\mathcal{D}}. \tag{51}$$

Proof of the Main Result 4

- From (50) we have

$$f(z) = \int_{V_G} \frac{b'(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} \frac{dA}{(1+s)\,\overline{\zeta}}.$$

Proof of the Main Result 4

- From (50) we have

$$f(z) = \int_{V_G} \frac{b'(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} \frac{dA}{(1+s)\overline{\zeta}}.$$

- Thus

$$\begin{aligned} f'(z) &= \int_{V_G} \frac{b'(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA \\ &= b'(z) - \int_{\mathbb{D}\setminus V_G} \frac{b'(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA \\ &= b'(z) + \Lambda b'(z), \end{aligned}$$

by the reproducing property of the generalized Bergman kernels $\frac{\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}}$, and

- where

$$\Lambda b'(z) = -\int_{\mathbb{D}\setminus V_G} \frac{b'(\zeta)(1-|\zeta|)^s}{(1-\overline{\zeta}z)^{2+s}}\,dA. \tag{52}$$

Proof of the Main Result 5

- where
$$\Lambda b'(z) = -\int_{\mathbb{D}\setminus V_G} \frac{b'(\zeta)(1-|\zeta|)^s}{(1-\overline{\zeta}z)^{2+s}}\, dA. \qquad (52)$$

- Now if we plug f and $g = \Phi^2$ as above in $T_b(f,g)$ we obtain $T_b(f,g) = T_b(f,\Phi^2) = T_b(f\Phi,\Phi)$ which we analyze as

$$\begin{aligned}
&\int_{\mathbb{D}} \{f'(z)\Phi(z) + 2f(z)\Phi'(z)\}\Phi(z)\overline{b'(z)}dA + f(0)\Phi(0)^2\overline{b} \quad (5\\
=\ & f(0)\Phi(0)^2\overline{b(0)} + \int_{\mathbb{D}} |b'(z)|^2 \Phi(z)^2\, dA \\
&+2\int_{\mathbb{D}} \Phi(z)\Phi'(z) f(z)\overline{b'(z)}dA + \int_{\mathbb{D}} \Lambda b'(z)\overline{b'(z)}\Phi(z)^2\, dA \\
=\ & (1)+(2)+(3)+(4).
\end{aligned}$$

Proof of the Main Result 6

- Trivially, we have

$$\left|(1)\right| \leq C \left\|b\right\|_{\mathcal{D}}^{2} Cap_{\mathcal{T}}\left(E,F\right) \leq C \left\|T_b\right\|^{2} Cap_{\mathcal{T}}\left(E,F\right). \qquad (54)$$

Proof of the Main Result 6

- Trivially, we have

$$|(1)| \leq C \|b\|_{\mathcal{D}}^2 \, Cap_{\mathcal{T}}(E,F) \leq C \| T_b\|^2 \, Cap_{\mathcal{T}}(E,F). \tag{54}$$

- Now we write

$$\begin{aligned} (2) &= \int_{\mathbb{D}} |b'(z)|^2 \Phi(z)^2 \, dA \\ &= \left\{ \int_{V_G} + \int_{V_G^\beta \setminus V_G} + \int_{\mathbb{D} \setminus V_G^\beta} \right\} |b'(z)|^2 \Phi(z)^2 \, dA \\ &= (2_A) + (2_B) + (2_C). \end{aligned} \tag{55}$$

Proof of the Main Result 7

- The main term (2A) satisfies

$$\begin{aligned}(2_A) &= \mu_b(V_G) + \int_{V_G} \left|b'(z)\right|^2 \left(\Phi(z)^2 - 1\right) dA \\ &= \mu_b(V_G) + O\left(\|T_b\|^2 \, Cap(E,F)\right),\end{aligned} \qquad (56)$$

by (32) and (17).

Proof of the Main Result 7

- The main term (2A) satisfies

$$\begin{aligned} (2_A) &= \mu_b(V_G) + \int_{V_G} \left|b'(z)\right|^2 \left(\Phi(z)^2 - 1\right) dA \\ &= \mu_b(V_G) + O\left(\|T_b\|^2 \, Cap(E, F)\right), \end{aligned} \tag{56}$$

  by (32) and (17).
- For term (2B) we use (39) to obtain

$$|(2_B)| \le C\mu_b\left(V_G^\beta \setminus V_G\right) \le C\varepsilon\mu_b(V_G). \tag{57}$$

Proof of the Main Result 7

- The main term (2A) satisfies

$$\begin{aligned}(2_A) &= \mu_b(V_G) + \int_{V_G} |b'(z)|^2 \left(\Phi(z)^2 - 1\right) dA \qquad (56)\\ &= \mu_b(V_G) + O\left(\|T_b\|^2 \, Cap(E,F)\right),\end{aligned}$$

by (32) and (17).

- For term (2B) we use (39) to obtain

$$|(2_B)| \leq C\mu_b\left(V_G^\beta \setminus V_G\right) \leq C\varepsilon\mu_b(V_G). \qquad (57)$$

- Using (32) once more, we see that term (2C) satisfies

$$\begin{aligned}|(2_C)| &\leq \int_{\mathbb{D}\setminus V_G^\beta} |b'(z)|^2 \left(C_{\alpha,\beta,\rho} Cap_{\mathcal{T}}(E,F)\right)^3 dA \qquad (58)\\ &\leq C\|T_b\|^2 \, Cap_{\mathcal{T}}(E,F).\end{aligned}$$

Proof of the Main Result 8

- Altogether, using (54), (55), (56), (57) and (58) in (53) we have

$$\mu_b(V_G) \leq \left|T_b\left(f,\Phi^2\right)\right| + C\mu_b\left(V_G^\beta \setminus V_G\right) + C\left\|T_b\right\|^2 Cap_{\mathcal{T}}(E,F) + |(3)| + |(4)|. \tag{59}$$

Proof of the Main Result 8

- Altogether, using (54), (55), (56), (57) and (58) in (53) we have

$$\mu_b(V_G) \leq \left|T_b\left(f,\Phi^2\right)\right| + C\mu_b\left(V_G^\beta \setminus V_G\right) + C\left\|T_b\right\|^2 Cap_{\mathcal{T}}(E,F) + |(3)| + |(4)|. \tag{59}$$

- We estimate (3) using Cauchy-Schwarz with $\varepsilon > 0$ small as follows:

$$\begin{aligned} |(3)| &\leq 2\int_{\mathbb{D}} \left|\Phi(z)\, b'(z)\right| \left|\Phi'(z)\, f(z)\right| dA \\ &\leq \varepsilon \int_{\mathbb{D}} \left|\Phi(z)\, b'(z)\right|^2 dA + \frac{C}{\varepsilon}\int_{\mathbb{D}} \left|\Phi'(z)\, f(z)\right|^2 dA \\ &= (3_A) + (3_B). \end{aligned}$$

Proof of the Main Result 8

- Altogether, using (54), (55), (56), (57) and (58) in (53) we have

$$\mu_b(V_G) \leq \left|T_b(f,\Phi^2)\right| + C\mu_b\left(V_G^\beta \setminus V_G\right) + C\|T_b\|^2 Cap_{\mathcal{T}}(E,F) + |(3)| + |(4)|. \quad (59)$$

- We estimate (3) using Cauchy-Schwarz with $\varepsilon > 0$ small as follows:

$$\begin{aligned} |(3)| &\leq 2\int_{\mathbb{D}} \left|\Phi(z) b'(z)\right| \left|\Phi'(z) f(z)\right| dA \\ &\leq \varepsilon \int_{\mathbb{D}} \left|\Phi(z) b'(z)\right|^2 dA + \frac{C}{\varepsilon}\int_{\mathbb{D}} \left|\Phi'(z) f(z)\right|^2 dA \\ &= (3_A) + (3_B). \end{aligned}$$

- Using the decomposition and argument surrounding term (2) we obtain

$$\begin{aligned} |(3_A)| &\leq \varepsilon\left\{\int_{V_G} + \int_{V_G^\beta\setminus V_G} + \int_{\mathbb{D}\setminus V_G^\beta}\right\}\left|\Phi(z) b'(z)\right|^2 dA \quad (60) \\ &\leq C\varepsilon\left(\mu_b(V_G) + C\|T_b\|^2 Cap_{\mathcal{T}}(E,F).\right) \end{aligned}$$

To estimate term (3_B) we use

$$\begin{aligned}
|f(z)| &\leq \left| \Gamma_s \left(\frac{1}{(1+s)\overline{\zeta}} \chi_{V_G} b'(\zeta) \right)(z) \right| \\
&\leq \int_{V_G} \frac{\left(1-|\zeta|^2\right)^s}{\left|1-\overline{\zeta}z\right|^{1+s}} \left|b'(\zeta)\right| dA \\
&\approx \sum_{\gamma \in \mathcal{T}_1 \cap V_G} \frac{\left(1-|\gamma|^2\right)^{1+s}}{\left|1-\overline{\gamma}z\right|^{1+s}} \int_{B_\gamma} \left|b'(\zeta)\right| \left(1-|\zeta|^2\right) d\lambda(\zeta) \\
&= \sum_{\gamma \in \mathcal{T}_1 \cap V_G} \frac{\left(1-|\gamma|^2\right)^{1+s}}{\left|1-\overline{\gamma}z\right|^{1+s}} b^*(\gamma),
\end{aligned}$$

where

$$\sum_{\gamma \in \mathcal{T}_1 \cap V_G} b^*(\gamma)^2 \approx \sum_{\gamma \in \mathcal{T}_1 \cap V_G} \int_{B_\gamma} \left|b'(\zeta)\right|^2 \left(1-|\zeta|^2\right)^2 d\lambda(\zeta) = \int_{V_G} \left|b'(\zeta)\right|^2 d$$

Proof of the Main Result 10

- We now use the separation of $\mathbb{D} \setminus V_G^{\alpha}$ and V_G. The facts that $\mathcal{A} = supp\,(h) \subset \mathbb{D} \setminus V_G^{\alpha}$ and $\mathcal{B} = \mathcal{T}_1 \cap V_G \subset V_G$, together with Lemma 10, insure that (44) is satisfied.

Proof of the Main Result 10

- We now use the separation of $\mathbb{D} \setminus V_G^\alpha$ and V_G. The facts that $\mathcal{A} = supp\,(h) \subset \mathbb{D} \setminus V_G^\alpha$ and $\mathcal{B} = \mathcal{T}_1 \cap V_G \subset V_G$, together with Lemma 10, insure that (44) is satisfied.
- Hence we can use Lemma 25 and the representation of Φ in 28 to continue with

$$(3_B) = \int_{\mathbb{D}} \left| \Phi'(z) f(z) \right|^2 dA \leq C \left(\sum_{\kappa \in \mathcal{A}} h(\kappa)^2 \right) \left(\sum_{\gamma \in \mathcal{B}} b^*(\gamma)^2 \right),$$

- We now use the separation of $\mathbb{D} \setminus V_G^\alpha$ and V_G. The facts that $\mathcal{A} = supp\,(h) \subset \mathbb{D} \setminus V_G^\alpha$ and $\mathcal{B} = \mathcal{T}_1 \cap V_G \subset V_G$, together with Lemma 10, insure that (44) is satisfied.
- Hence we can use Lemma 25 and the representation of Φ in 28 to continue with

$$(3_B) = \int_{\mathbb{D}} \left|\Phi'(z)\, f(z)\right|^2 dA \leq C \left(\sum_{\kappa\in\mathcal{A}} h(\kappa)^2\right) \left(\sum_{\gamma\in\mathcal{B}} b^*(\gamma)^2\right),$$

- We also have from (17) and Corollary 18 that

$$\left(\sum_{\kappa\in\mathcal{A}} h(\kappa)^2\right) \left(\sum_{\gamma\in\mathcal{B}} b^*(\gamma)^2\right) \leq C Cap\,(E, F)\, \|T_b\|^2 .$$

Proof of the Main Result 11

- Altogether we then have

$$(3_B) \leq C \ Cap_{\mathcal{T}}(E,F) \|T_b\|^2, \tag{61}$$

and thus also

$$|(3)| \leq \varepsilon \int_{V_G} |b'(z)|^2 + C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F). \tag{62}$$

- Altogether we then have

$$\left(3_B\right) \le C \ Cap_{\mathcal{T}}\left(E, F\right) \left\| T_b \right\|^2 , \tag{61}$$

and thus also

$$\left|(3)\right| \le \varepsilon \int_{V_G} \left|b'(z)\right|^2 + C \left\| T_b \right\|^2 Cap_{\mathcal{T}}\left(E, F\right) . \tag{62}$$

- We begin our estimate of term (4) by

$$\begin{aligned} \left|(4)\right| &= \left| \int_{\mathbb{D}} \Lambda b'(z) \overline{b'(z)} \Phi(z)^2 \, dA \right| \\ &\le \sqrt{\int_{\mathbb{D}} \left|b'(z) \Phi(z)\right|^2 dA} \sqrt{\int_{\mathbb{D}} \left|\Lambda b'(z) \Phi(z)\right|^2 dA} \end{aligned} \tag{63}$$

where the first factor is $\sqrt{\frac{1}{\varepsilon}\left(3_A\right)}$.

Proof of the Main Result 12

- Now we claim the following estimate for $(4_A) = \|\Phi\Lambda b'\|_{L^2(\mathbb{D})}$:

$$
\begin{aligned}
(4_A) &= \int_{\mathbb{D}} \left|\Phi(z)\Lambda b'(z)\right|^2 dA && (64)\\
&\leq C\mu_b\left(V_G^{\beta} \setminus V_G\right) + C\|T_b\|^2 Cap_{\mathcal{T}}(E,F) && (65)\\
&\leq \varepsilon\mu_b(V_G) + C\|T_b\|^2 Cap_{\mathcal{T}}(E,F).
\end{aligned}
$$

- Now we claim the following estimate for $(4_A) = \|\Phi\Lambda b'\|_{L^2(\mathbb{D})}$:

$$\begin{aligned}(4_A) &= \int_{\mathbb{D}} \left|\Phi(z)\Lambda b'(z)\right|^2 dA \qquad (64)\\ &\leq C\mu_b\left(V_G^\beta \setminus V_G\right) + C\|T_b\|^2 Cap_{\mathcal{T}}(E,F) \qquad (65)\\ &\leq \varepsilon\mu_b(V_G) + C\|T_b\|^2 Cap_{\mathcal{T}}(E,F).\end{aligned}$$

- Indeed, the second inequality follows from (39), so we now turn to the first inequality.

From (52) we obtain

$$
\begin{aligned}
(4_A) &= \int_{\mathbb{D}} |\Phi(z)|^2 \left| \left\{ \int_{V_G^\beta \setminus V_G} + \int_{\mathbb{D} \setminus V_G^\beta} \right\} \frac{b'(\zeta)\,(1-|\zeta|)^s}{(1-\overline{\zeta} z)^{2+s}} dA \right|^2 dA \\
&\leq C \int_{\mathbb{D}} |\Phi(z)|^2 \left(\int_{V_G^\beta \setminus V_G} \frac{|b'(\zeta)|\,(1-|\zeta|)^s}{|1-\overline{\zeta} z|^{2+s}} dA \right)^2 dA \\
&\quad + C \int_{\mathbb{D}} |\Phi(z)|^2 \left| \int_{\mathbb{D} \setminus V_G^\beta} \frac{b'(\zeta)\,(1-|\zeta|)^s}{(1-\overline{\zeta} z)^{2+s}} dA \right|^2 dA \\
&= (4_{AA}) + (4_{AB}).
\end{aligned}
$$

Proof of the Main Result 14

- Corollary 24 shows that

$$
\begin{aligned}
\left|(4_{AA})\right| &\leq \int_{\mathbb{D}} \left(\int_{V_G^\beta \setminus V_G} \frac{\left|b'(\zeta)\right| (1-|\zeta|)^s}{\left|1-\overline{\zeta} z\right|^{2+s}} dA \right)^2 dA \\
&\leq C \int_{V_G^\beta \setminus V_G} \left|b'(\zeta)\right|^2 dA = C\mu_b \left(V_G^\beta \setminus V_G \right).
\end{aligned}
$$

Proof of the Main Result 14

- Corollary 24 shows that

$$\begin{aligned} |(4_{AA})| &\leq \int_{\mathbb{D}} \left(\int_{V_G^\beta \setminus V_G} \frac{|b'(\zeta)| (1-|\zeta|)^s}{|1-\overline{\zeta} z|^{2+s}} dA \right)^2 dA \\ &\leq C \int_{V_G^\beta \setminus V_G} |b'(\zeta)|^2 \, dA = C\mu_b \left(V_G^\beta \setminus V_G \right). \end{aligned}$$

- We write the second integral as

$$\begin{aligned} (4_{AB}) &= \left\{ \int_{V_G^\gamma} + \int_{\mathbb{D}\setminus V_G^\gamma} \right\} |\Phi(z)|^2 \left| \int_{\mathbb{D}\setminus V_G^\beta} \frac{b'(\zeta)(1-|\zeta|)^s}{(1-\overline{\zeta} z)^{2+s}} dA \right|^2 dA \\ &= (4_{ABA}) + (4_{ABB}). \end{aligned}$$

- By Corollary 24 again,

$$\begin{aligned} |(4_{ABB})| &\leq C \; Cap_{\mathcal{T}}(E,F)^2 \int_{\mathbb{D}} |b'(\zeta)|^2 \, dA \\ &\leq C \, \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F)^2 \\ &\leq C \, \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F) \, . \end{aligned}$$

Proof of the Main Result 15

- By Corollary 24 again,

$$\begin{aligned} |(4_{ABB})| &\leq C \; Cap_{\mathcal{T}}(E,F)^2 \int_{\mathbb{D}} |b'(\zeta)|^2 \, dA \\ &\leq C \, \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F)^2 \\ &\leq C \, \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F) \, . \end{aligned}$$

- Finally, with $\beta < \beta_1 < \gamma < \alpha < 1$, Corollary 24 shows that the term (4_{ABA}) satisfies the following estimate. Recall that $V_G^{\gamma} = \cup J_k^{\gamma}$ and $w_j^{\gamma} = z\left(J_k^{\gamma}\right)$. We set $A_{\ell} = \left\{k : J_k^{\gamma} \subset J_{\ell}^{\beta_1}\right\}$ and define $\ell(k)$ by the condition $k \in A_{\ell(k)}$. Then using the geometric separation of $\mathbb{D} \setminus V_G^{\beta}$ and V_G^{γ} in Lemma 10, we complete the proof of (64) as follows:

Proof of the Main Result 16

$$
\begin{aligned}
|(4_{ABA})| &\leq C \int_{V_G^{\gamma}} \left(\int_{\mathbb{D} \setminus V_G^{\beta}} \frac{|b'(\zeta)|\,(1-|\zeta|)^s}{\left|1-\overline{\zeta} z\right|^{2+s}} dA \right)^2 dA \\
&\approx C \sum_k \int_{J_k^{\gamma}} \left|J_k^{\gamma}\right| \left(\frac{|b'(\zeta)|\,(1-|\zeta|)^s}{\left|1-\overline{\zeta} w_k^{\gamma}\right|^{2+s}} dA \right)^2 dA \\
&= C \sum_k \frac{\left|J_k^{\gamma}\right|}{\left|J_{\ell(k)}^{\beta_1}\right|} \left|J_{\ell(k)}^{\beta_1}\right| \int_{J_k^{\gamma}} \left(\frac{|b'(\zeta)|\,(1-|\zeta|)^s}{\left|1-\overline{\zeta} w_k^{\gamma}\right|^{2+s}} dA \right)^2 dA
\end{aligned}
$$

Proof of the Main Result 17

$$
\begin{aligned}
&\approx \quad C\sum_{\ell}\frac{\sum_{k\in A_\ell}\left|J_k^{\gamma}\right|}{\left|J_\ell^{\beta_1}\right|}\int_{J_\ell^{\beta_1}}\left(\int_{\mathbb{D}\setminus V_G^{\beta}}\frac{\left|b'(\zeta)\right|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}}dA\right)^2 dA \\
&\leq \quad C\left|V_G^{\beta_1}\right|^{\varepsilon(\gamma-\beta_1)}\int_{V_G^{\beta_1}}\left(\int_{\mathbb{D}\setminus V_G^{\beta}}\frac{\left|b'(\zeta)\right|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}}dA\right)^2 dA \\
&\leq \quad C\left|V_G^{\beta_1}\right|^{\varepsilon(\gamma-\beta_1)}\|b\|_{\mathcal{D}}^2\leq C\left\|T_b\right\|^2 Cap_{\mathcal{T}}(E,F).
\end{aligned}
$$

Proof of the Main Result 18

Now we can estimate term (4) by

$$
\begin{aligned}
|(4)| &= \left| \int_{\mathbb{D}} \Lambda b'(z) \overline{b'(z)} \Phi(z)^2 \, dA \right| && (66) \\
&\leq \sqrt{\int_{\mathbb{D}} |b'(z) \Phi(z)|^2 \, dA} \sqrt{\int_{\mathbb{D}} |\Lambda b'(z) \Phi(z)|^2 \, dA} \\
&\leq \sqrt{(3_A)/\varepsilon} \sqrt{(4_A)} && (67) \\
&\leq \sqrt{C\mu_b(V_G) + C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F)} \\
&\quad \times \sqrt{\varepsilon\mu_b(V_G) + C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F)} \\
&\leq \sqrt{\varepsilon}\mu_b(V_G) + C\sqrt{\mu_b(V_G)} \sqrt{\|T_b\|^2 \, Cap_{\mathcal{T}}(E,F)} \\
&\quad + C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F),
\end{aligned}
$$

using (64) and the estimate (60) for (3_A) already proved above.

Proof of the Main Result 19

- Finally, we estimate $T_b\left(f, \Phi^2\right) = T_b\left(f\Phi, \Phi\right)$by

$$|T_b\left(f\Phi, \Phi\right)| \leq \|T_b\| \, \|\Phi\|_{\mathcal{D}} \, \|\Phi f\|_{\mathcal{D}} \leq C \, \|T_b\| \sqrt{Cap_{\mathcal{T}}\left(E, F\right)} \, \|\Phi f\|_{\mathcal{D}} .$$

Proof of the Main Result 19

- Finally, we estimate $T_b\left(f, \Phi^2\right) = T_b\left(f\Phi, \Phi\right)$by

  $$\left|T_b\left(f\Phi, \Phi\right)\right| \leq \|T_b\| \, \|\Phi\|_{\mathcal{D}} \, \|\Phi f\|_{\mathcal{D}} \leq C \, \|T_b\| \sqrt{Cap_{\mathcal{T}}\left(E, F\right)} \, \|\Phi f\|_{\mathcal{D}}\,.$$

- Now

  $$\begin{aligned} \|\Phi f\|_{\mathcal{D}}^2 &\leq C \int \left|\Phi'\left(z\right) f\left(z\right)\right|^2 dA + C \int \left|\Phi\left(z\right) f'\left(z\right)\right|^2 dA \\ &\leq C\left|3_A\right| + C\left|3_B\right| + C \int \left|\Phi\left(z\right) \Lambda b'\left(z\right)\right|^2 dA \\ &\leq C\mu_b\left(V_G\right) + C\,\|T_b\|^2 \, Cap_{\mathcal{T}}\left(E, F\right), \end{aligned}$$

  by (64) and the estimates (60) and (61) for $\left(3_A\right)$ and $\left(3_B\right)$.

Proof of the Main Result 19

- Finally, we estimate $T_b\left(f, \Phi^2\right) = T_b\left(f\Phi, \Phi\right)$by

$$\left| T_b\left(f\Phi, \Phi\right)\right| \leq \| T_b\| \, \|\Phi\|_{\mathcal{D}} \, \|\Phi f\|_{\mathcal{D}} \leq C \, \| T_b\| \sqrt{Cap_{\mathcal{T}}\left(E, F\right)} \, \|\Phi f\|_{\mathcal{D}} .$$

- Now

$$\begin{aligned} \|\Phi f\|_{\mathcal{D}}^2 &\leq C \int \left|\Phi'\left(z\right) f\left(z\right)\right|^2 dA + C \int \left|\Phi\left(z\right) f'\left(z\right)\right|^2 dA \\ &\leq C \left|3_A\right| + C\left|3_B\right| + C \int \left|\Phi\left(z\right) \Lambda b'\left(z\right)\right|^2 dA \\ &\leq C\mu_b\left(V_G\right) + C \, \| T_b\|^2 \, Cap_{\mathcal{T}}\left(E, F\right), \end{aligned}$$

by (64) and the estimates (60) and (61) for $\left(3_A\right)$ and $\left(3_B\right)$.

- When we plug this into the previous estimate we get that $\left| T_b\left(f, \Phi^2\right)\right|$ is at most

$$C \, \| T_b\| \sqrt{Cap_{\mathcal{T}}\left(E, F\right)} \sqrt{\mu_b\left(V_G\right) + \| T_b\|^2 \, Cap_{\mathcal{T}}\left(E, F\right)} \quad (68)$$

$$\leq C \sqrt{\| T_b\|^2 \, Cap_{\mathcal{T}}\left(E, F\right)} \left(\sqrt{\mu_b\left(V_G\right)} + \| T_b\| \, Cap_{\mathcal{T}}\left(E, F\right)^{\frac{1}{2}}\right).$$

Proof of the Main Result 20

- Using Proposition 21 and the estimates (62), (66) and (68) in (59) we obtain

$$\begin{aligned}\mu_b(V_G) &\leq \sqrt{\varepsilon}\mu_b(V_G) + C\|T_b\|^2 Cap(E,F) \\ &\quad + C\sqrt{\|T_b\|^2 Cap(E,F)}\sqrt{\mu_b(V_G)} \\ &\leq \sqrt{\varepsilon}\mu_b(V_G) + C\|T_b\|^2 Cap(E,F).\end{aligned}$$

- Using Proposition 21 and the estimates (62), (66) and (68) in (59) we obtain

$$\begin{aligned}\mu_b(V_G) &\leq \sqrt{\varepsilon}\mu_b(V_G) + C\|T_b\|^2 Cap(E,F) \\ &\quad + C\sqrt{\|T_b\|^2 Cap(E,F)}\sqrt{\mu_b(V_G)} \\ &\leq \sqrt{\varepsilon}\mu_b(V_G) + C\|T_b\|^2 Cap(E,F).\end{aligned}$$

- Absorbing the first term on the right side, and using (49), we finally obtain

$$\mu_b(V_G) \leq C\|T_b\|^2 Cap_{\mathcal{T}}(E,F) \leq C\|T_b\|^2 Cap_{\mathbb{D}} G,$$

which is (47).

An open problem

- The theorem for the Hilbert space $\mathcal{H} = \mathcal{D}$ proved above is similar in many respects to the result of Maz'ya and Verbitsky on Schrödinger forms on the Sobolev space $\mathcal{H} = W^{1,2}$, not involving function theory at all: Let Q be a complex-valued distribution on $\mathbb{R}^n$, $n \geq 3$.Then

$$\left| \int_{\mathbb{R}^n} u(x)\, v(x)\, \overline{Q(x)} dx \right| \lesssim \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},$$

holds *if and only if* $Q = \operatorname{div} \Gamma$ where

$$\int_{\mathbb{R}^n} |u(x)|^2 |\Gamma(x)|^2 dx \lesssim \|\nabla u\|_{L^2}^2.$$

An open problem

- The theorem for the Hilbert space $\mathcal{H} = \mathcal{D}$ proved above is similar in many respects to the result of Maz'ya and Verbitsky on Schrödinger forms on the Sobolev space $\mathcal{H} = W^{1,2}$, not involving function theory at all: Let Q be a complex-valued distribution on $\mathbb{R}^n$, $n \geq 3$.Then

$$\left| \int_{\mathbb{R}^n} u(x)\, v(x)\, \overline{Q(x)} dx \right| \lesssim \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},$$

  holds *if and only if* $Q = \operatorname{div} \Gamma$ where

$$\int_{\mathbb{R}^n} |u(x)|^2 |\Gamma(x)|^2 dx \lesssim \|\nabla u\|_{L^2}^2.$$

- It is fascinating that although there is a great deal of variety in the techniques used in the two proofs, there is a surprising similarity in the answers obtained. The answer, quite generally, is that for some differential operator $\mathfrak{D}$, $|\mathfrak{D} b|^2$ can be used to define a Carleson measure for $\mathcal{H}$.

An open problem

- The theorem for the Hilbert space $\mathcal{H} = \mathcal{D}$ proved above is similar in many respects to the result of Maz'ya and Verbitsky on Schrödinger forms on the Sobolev space $\mathcal{H} = W^{1,2}$, not involving function theory at all: Let Q be a complex-valued distribution on $\mathbb{R}^n$, $n \geq 3$.Then

$$\left| \int_{\mathbb{R}^n} u(x)\, v(x)\, \overline{Q(x)} dx \right| \lesssim \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},$$

  holds *if and only if* $Q = \operatorname{div} \Gamma$ where

$$\int_{\mathbb{R}^n} |u(x)|^2 |\Gamma(x)|^2 dx \lesssim \|\nabla u\|_{L^2}^2 .$$

- It is fascinating that although there is a great deal of variety in the techniques used in the two proofs, there is a surprising similarity in the answers obtained. The answer, quite generally, is that for some differential operator $\mathfrak{D}$, $|\mathfrak{D} b|^2$ can be used to define a Carleson measure for $\mathcal{H}$.
- What specific connections are there?

THE

END