~~Proof~~. Before giving the proof, observe that we have ~~three~~ four interesting characterizations of ~~capacity~~ energy on the tree ($\mu \geq 0$ on ∂T):

$$\mathcal{E}_{T}(\mu) := \sum_{\alpha} I^{*}\mu(\alpha)^{2} \qquad \text{(A)}$$

$$= \int_{\partial T} d\mu(\zeta) \int_{\partial T} d\mu(\xi)\, d(\zeta, \xi) \qquad \text{(B)}$$

$$\approx \int_{\partial T} \left[\int_{\partial T} \delta_{T}(\eta, \zeta)^{-1/2} d\mu(\zeta) \right]^{2} d\mu(\eta). \qquad \text{(C)}$$

$$\text{(see (*) below)} \approx \int_{\partial T} d\zeta \cdot \sum_{\alpha \in P_{G}(\zeta)} \left(\frac{\mu(S(\alpha))}{2^{d(\alpha)/2}} \right)^{2} \qquad \text{(D)}$$

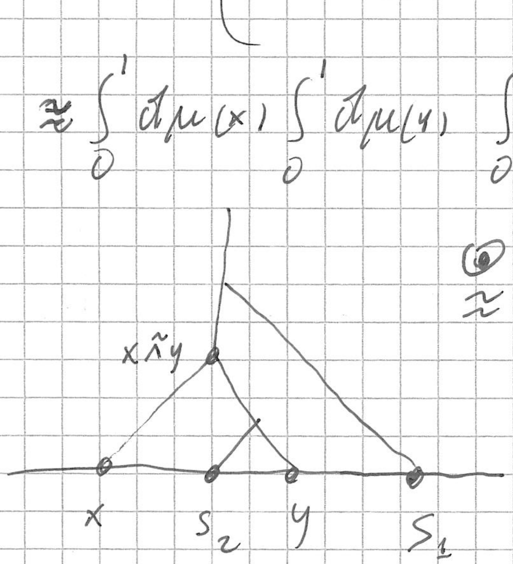
We find similar expressions for the Bessel capacity on $[0,1] \equiv \partial\Delta$. Let $\mu \geq 0$ be a Borel measure on $\partial\Delta$.

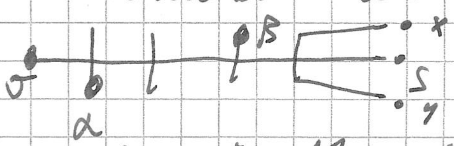
$$\mathcal{E}_{2,1/2}(\mu) = \int_0^1 \left(\int_0^1 \frac{d\mu(t)}{|s-t|^{1/2}} \right)^2 ds \qquad \text{(C')}$$

$$\approx \int_0^1 \left(\sum_{\zeta \in \Lambda^{-1}(s)} \sum_{\alpha \in P(\zeta)} \frac{\mu(S(\alpha))}{|Q_\alpha|^{1/2}} \right)^2 ds$$

$$\approx \int_0^1 d\mu(x) \int_0^1 d\mu(y) \int_0^1 ds \sum_{\alpha, \beta :\, x \in \alpha;\, y \in \beta;\, \alpha, \beta \in P_G(s)} |Q(\alpha)|^{-1/2} \cdot |Q(\beta)|^{-1/2}$$

$$\overset{(*)}{\approx} \int_0^1 d\mu(x) \int_0^1 d\mu(y) \cdot \log \frac{4}{|x-y|} \qquad \text{(B')}$$

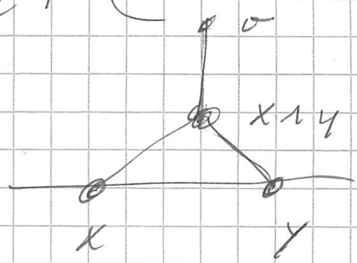


Proof of (*): $x \in \alpha$, $y \in \beta$ and $\alpha, \beta \in P_G(s)$ means that α and β refer to boxes situated near a G-fattened geodesic: 

Either β comes after α (as in the picture) and both (enlarged by a factor 2) contain x and y, and this gives a ~~sum~~ contribution: (a)

(A) Proof that (D) $\approx$ (B)

$$(D) = \int_{\partial T} d\varsigma \cdot \sum_{\alpha \in P(\varsigma)} 2^{d(\alpha)} \mu(S(\alpha))^2 = \int_{\partial T} d\varsigma \sum_{\alpha \in P(\varsigma)} 2^{d(\alpha)} \left(\int \chi(\alpha \in P(x))\, d\mu(x) \right)^2$$



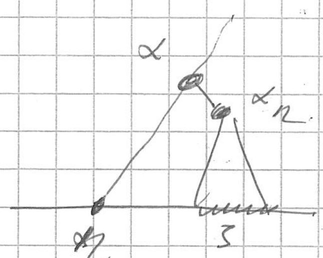
$$= \int d\mu(x) \int d\mu(y) \cdot \int \sum_{\alpha \in P(\varsigma) \cap P(x) \cap P(y)} 2^{d(\alpha)}\, d\varsigma$$

$$= \int d\mu(x) \int d\mu(y) \cdot \sum_{\alpha = o}^{x \wedge y} 2^{d(\alpha)} \cdot |\{\varsigma \in S(\alpha)\}|$$

$$= \int d\mu(x) \int d\mu(y)\; d(x \wedge y) = (B)$$

Observe that (D) $\leq$ (C) *a priori*:

$$(C) = \int_{\partial T} \left(\int_{\partial T} \delta_T(\eta, \varsigma)^{-1/2} d\mu(\varsigma) \right)^2 d\eta =$$



$$= \int_{\partial T} \left(\int_{\partial T} 2^{d(\eta \wedge \varsigma)/2} d\mu(\varsigma) \right)^2 d\eta$$

$$= \int_{\partial T} \left(\sum_{\alpha \in P(\eta)} 2^{d(\alpha)/2} \mu(S(\alpha_\eta)) \right)^2 d\eta$$

$$\approx \int_{\partial T} \left(\sum_{\alpha \in P(\eta)} 2^{d(\alpha)/2} \mu(S(\alpha)) \right)^2 d\eta \qquad \text{(Exercise!)}$$

$$\geq \int_{\partial T} \sum_{\alpha \in P(\eta)} \left(2^{d(\alpha)/2} \mu(S(\alpha)) \right)^2 d\eta, \text{ since } \left(\sum a_n\right)^2 \geq \sum a_n^2.$$

$$\| \;(D)$$

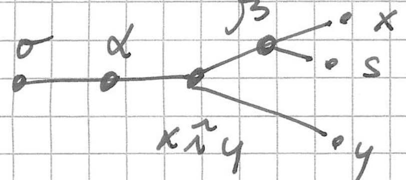
The fact that (D) $\approx$ (C) means that the imbedding $\ell^1 \subseteq \ell^2$ can be inverted "on average". [For $2 \neq p \in (1, \infty]$, this is the Muckenhoupt–Wheeden / Wolff inequality].

End of (A)

$$(C) \approx \sum_{\beta = o}^{x \tilde{\wedge} y} |Q(\beta)|^{-1/2} |Q(\beta)| \cdot \sum_{\alpha = o}^{\beta} |Q(\alpha)|^{-1/2}$$

$$\approx \sum_{\beta = o}^{x \tilde{\wedge} y} |Q(\beta)|^{1/2} \cdot |Q(\beta)|^{-1/2} \approx d(o, x \tilde{\wedge} y) \approx$$

$$\approx 1 + \left|\log |x-y|^{-1}\right| \approx \log \frac{4}{|x-y|}, \text{ where}$$

or:  $\sigma \le \alpha \le x \tilde{\wedge} y$ and

$x \tilde{\wedge} y \le \beta < x$,

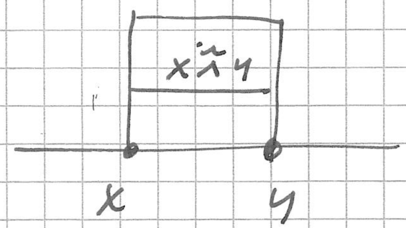
giving contribution: 2 $\approx$

$$\approx \sum_{\alpha=\sigma}^{x\tilde{\wedge} y} |Q(\alpha)|^{-1/2} \cdot \sum_{\beta=x\tilde{\wedge} y}^{x} |Q(\beta)|^{-1/2} \cdot |Q(\beta)|$$

$$\approx \sum_{\alpha=\sigma}^{x\tilde{\wedge} y} |Q(\alpha)|^{-1/2} \cdot |Q(x\tilde{\wedge} y)|^{1/2} \approx 1$$

Thus 1 + 2 $\approx 1 + \log \frac{4}{|x-y|}$.

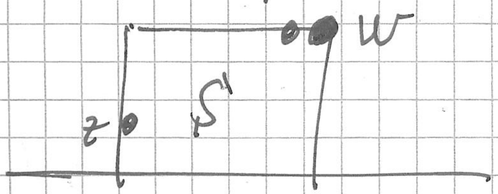
Note: $x\tilde{\wedge}y$ is the top-half box of the smallest Carleson box containing both x and y:



$\partial\Delta$

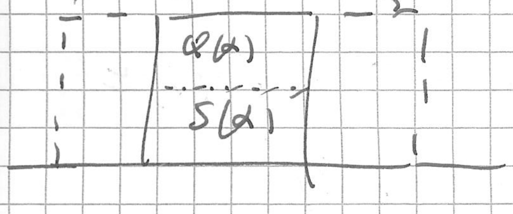
~~It corresponds~~ It is contained in boundedly many boxes $Q(\beta)$ at level $\approx$ ~~slog~~ $\log \frac{4}{|x-y|}$.

Remark: the same meaning is attached to the important quantity $\delta = |1 - z\bar{w}|$, with $z, w \in \Delta$:



$\delta \approx$ Euclidean diameter of S', the smallest Carleson box containing z and w.

Similarly, we have that, denoting by $\tilde{S}(\alpha)$ the enlargement by a factor 2 of $S(\alpha)$ (in Euclidean terms):



(A') $\displaystyle\sum_\alpha \mu(\tilde{S}(\alpha))^2 = \int_{\partial\Delta} d\mu(\zeta) \int_{\partial\Delta} d\mu(\xi)\, K(\zeta,\xi),$

where $\displaystyle K(\zeta,\xi) = \sum_{\zeta,\xi\in\tilde{S}(\alpha)} 1 \approx d(\zeta \wedge \xi)$

$\displaystyle\approx \log\frac{4}{|\zeta-\xi|},$

so that $\displaystyle\sum_\alpha \mu(\tilde{S}(\alpha))^2 \approx$ (B').

Exercise: show that (C') $\approx$ (D'), where

(D') $\displaystyle = \int_{\partial\Delta} dx \sum_{\alpha\in P_G(x)} \left(\frac{\mu(\tilde{S}(\alpha))}{|Q(\alpha)|^{1/2}}\right)^{1/2}.$

Remark. For a measure μ on $\partial\Delta$,

(A') $\displaystyle = \sum_\alpha \mu(\tilde{S}(\alpha))^2 \approx \sum_\alpha \mu(S(\alpha))^2 =$ (A''),

which is like (A), if we identify somehow the measure μ with a measure on $\partial T'$.

Proof of Thm. (#1). If $\mu \geq 0$ is a Borel measure on $\partial T'$ having an atom, $\mathcal{E}_{T'}(\mu) = +\infty$, we can then assume our measures on $\partial T'$ to be atomless; and the same holds for measures on $\partial\Delta$.

Let $\omega \geq 0$ be a measure (without atoms) on $\partial T'$. Then, $(\Lambda_* \omega)(\tilde{S}(\alpha)) := \omega(\Lambda^{-1}(\tilde{S}(\alpha))) = \omega(S(\alpha))$, hence

$$(1)\quad \mathcal{E}_{\partial\Delta}(\Lambda_*\omega) \approx \sum_\alpha (\Lambda_*\omega)(\tilde{S}(\alpha))^2 \approx \sum_\alpha (\Lambda_*\omega)(S(\alpha))^2$$

$$= \sum_\alpha \omega(S(\alpha))^2 = \mathcal{E}_{\partial T'}(\omega).$$

Let now $\mu \geq 0$ be an atomless measure on $\partial\Delta$. Then, unravelling definitions,

$$\Lambda^*\mu(A) = \mu(\Lambda(A)) \quad \text{for all } A \subseteq \partial T, \text{ measurable},$$

and since (abuse of language, as before):

$$\cancel{\Lambda(S(\alpha)) = S} \quad \Lambda(S(\alpha) \cap \partial T) = S(\alpha) \cap \partial\Delta,$$

$$(2) \quad \mathcal{E}_{\partial\Delta}(\mu) \approx \sum_\alpha \mu(S(\alpha))^2 = \sum_\alpha \Lambda^*\mu(S(\alpha))^2 \approx \mathcal{E}_{\partial T}(\Lambda^*\mu)$$

The energies of the corresponding measures are equivalent.

Let now $F \subseteq \partial T$ be closed and let $\omega \geq 0$ be a measure supported on $\Lambda(F)$, atomless. Then (exercise in measure theory) $\mathrm{supp}(\Lambda^*\omega) \subseteq F$.

Since $\|\Lambda^*\omega\|_1 = \|\omega\|_1$, by (2) we have

$$\mathrm{Cap}_{\partial\Delta}(\Lambda(F)) = \sup\left\{\frac{\|\omega\|_1^2}{\mathcal{E}_{\partial\Delta}(\omega)} : \mathrm{supp}(\omega) \subseteq \Lambda(F)\right\}$$

$$\lesssim \sup\left\{\frac{\|\mu\|_1^2}{\mathcal{E}_{\partial T}(\mu)} : \mathrm{supp}(\mu) \subseteq F\right\} = \mathrm{Cap}_{\partial T}(F).$$

In the other direction, let $\mathrm{supp}(\nu) \subseteq F$. Then $\mathrm{supp}(\Lambda_*\nu) \subseteq \Lambda(F)$, hence, by (1),

$$\mathrm{Cap}_{\partial T}(F) = \sup\left\{\frac{\|\nu\|_1^2}{\mathcal{E}_{\partial T}(\nu)} : \mathrm{supp}(\nu) \subseteq F\right\}$$

$$\lesssim \sup\left\{\frac{\|\omega\|_1^2}{\mathcal{E}_{\partial\Delta}(\omega)} : \mathrm{supp}(\omega) \subseteq \Lambda(F)\right\} = \mathrm{Cap}_{\partial\Delta}(\Lambda(F)).$$

This shows (1) in Thm. (A). (2) follows easily from (1).

CAPACITY AND CARLESON MEASURES.

We are going to discuss the relation between capacity and Carleson measures (STEGENGA, '80). Having discretized both concepts, we might (and will) work directly on the tree, translating the results into the Dirichlet space language afterwards. We start from a simple observation.

Let $\mu \geq 0$ be a Borel measure on $\overline{T}$ and define

$$\|\mu\|_{CM(T)} := \sup_{\alpha \in T} \frac{I^*(I^*\mu)^2(\alpha)}{I^*\mu(\alpha)}$$

By the characterization of $CM(\mathcal{D})$,

$$\|\mu\|_{CM(T)} \approx [\mu]_{CM(\mathcal{D})}$$

The numerator of the quantity defining $\|\mu\|_{CM(T)}$ is the energy $\mathcal{E}^{\alpha}_{T}(\mu)$ of the measure $\mu|_{S(\alpha)}$ on the tree $S(\alpha)$ having root $\alpha \in T$.

<u>Theorem 1.</u> Let $E \subseteq \overline{T}$ be compact. Then

$$\mathrm{Cap}_T(E) = \sup_{\mathrm{supp}(\mu) \subseteq E} \frac{\mu(E)}{\|\mu\|_{CM(T)}}$$

(I use $CM(T) = CM(\mathcal{T})$ to denote the space of Carleson measures).

i.e. the capacity of a set is known once we know which Carleson measures it supports.

<u>Corollary</u>: $\mathrm{Cap}_T(E) = 0 \iff CM(E) = 0$.