Introduction to The Dirichlet Space

MSRI Summer Graduate Workshop

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The Disk and Tree

I will

1. Introduce a structured decomposition of the unit disk.
2. Discuss decomposition theorems for the Bergman and Dirichlet space. The decomposition of functions will be related to the decomposition of the disk.
3. Introduce a tree structure associated to the decomposition and function spaces on the tree that are discrete models of the Bergman and Dirichlet space.
4. Describe some results for the discrete model Dirichlet space and indicate how these results are related to the existence of boundary values.
5. Discuss very briefly an instance of using a result from the model to obtain a result on the classical space.

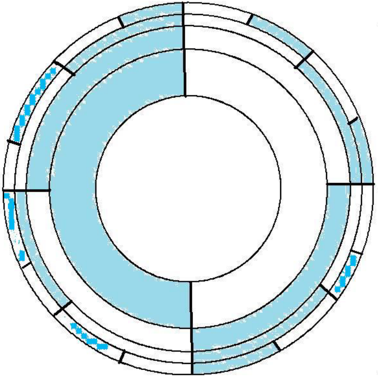
In this talk I hope to present one big idea:

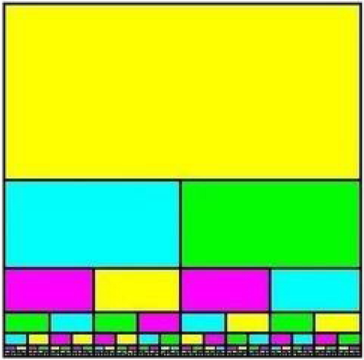
> A tree gives a simplified model for the unit disk. Function spaces on the tree can be useful models for function spaces on the disk.

I will go quickly, use pictures to speed things up, and probably still won't get through the list I just gave.

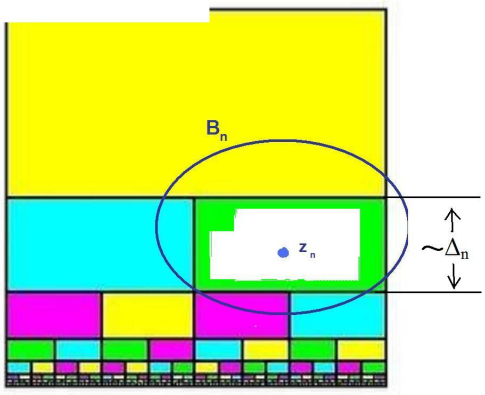
The Decomposition of the Disk

- We divide the disk into rings, narrower as we go to the boundary. We divide the n^{th} ring into 2^n equal pieces. This is shown in the first figure. However the picture rapidly becomes chaotic. The second picture is the standard stylized representation of a part of the disk, for instance the bottom quadrant of the disk, after subdivision; the pieces are called squares.





Art Credit: Zvi Harper

- Let $\{B_n\}_{n=1}^{\infty}$ be an enumeration of the squares and, for each n, select a point $z_n \in B_n$. Set $\Delta_n = 1 - |z_n|^2$. Thus B_n is, roughly, a box with side length Δ_n, Euclidean area Δ_n^2, and hyperbolic area 1. (I am not going to introduce the basics of the hyperbolic geometry of the disk. However if you know that geometry you can see that it is a natural language for this construction.)
- 

The Decomposition Theorem

- Here is an outline of an argument: Suppose we have f in the Bergman space. We write $\kappa_z(w) = (1 - \bar{z}w)^{-2}$ for the Bergman kernel function, $\hat{\kappa}_z$ for the normalized kernel, and κ_n for κ_{z_n}.

$$
\begin{aligned}
f(z) &= \langle f, \kappa_z \rangle_{\mathcal{A}^2} = \int_{\mathbb{D}} f(\zeta) \frac{1}{(1 - z\bar{\zeta})^2} d\xi d\eta \\
&= \sum \int_{B_n} f(\zeta) \frac{1}{(1 - z\bar{\zeta})^2} d\xi d\eta \\
&\sim \sum f(z_n) \frac{1}{(1 - z_n\bar{\zeta})^2} \Delta_n^2 \\
&= \sum f(z_n) \kappa_n \Delta_n^2 \\
&\sim \sum \{ f(z_n) \Delta_n \} \hat{\kappa}_n
\end{aligned}
$$

- Making this approximation scheme quantitative and iterating it leads to a proof of the first statement below, term by term integration then produces the second.

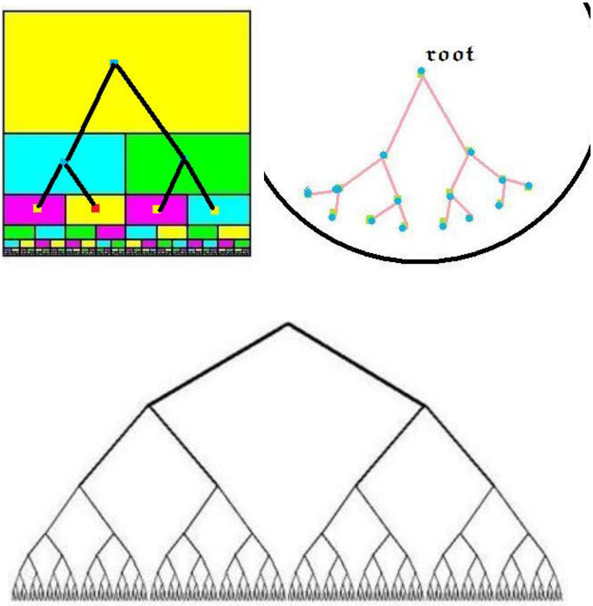
Theorem

1. *Given* $f \in \mathcal{A}^2$ *there are scalars* $\{\lambda_n\} \in \ell^2$ *with* $\|\{\lambda_n\}\|_{\ell^2} \sim \|f\|_{\mathcal{A}^2}$ *so that*
$$f(z) = \sum \lambda_n \hat{\kappa}_n$$
This can be done so that $\lambda_n \sim f(z_n)\Delta_n$. *Conversely....*
2. *Given* $b \in \mathcal{D}$ *there are scalars* $\{\beta_n\} \in \ell^2$ *with* $\|\{\beta_n\}\|_{\ell^2} \sim \|b\|_{\mathcal{D}}$ *so that*
$$b(z) = \sum \beta_n (\Delta_n \hat{\kappa}_n)$$
This can be done so that $\beta_n \sim b'(z_n)$ *and conversely....*

Note: The $\hat{\kappa}_n$ are normalized in $\mathcal{A}^2$, the functions $\Delta_n \hat{\kappa}_n$ are normalized in $\mathcal{D}$: $\|\Delta_n \hat{\kappa}_n\|_{\mathcal{D}} \sim 1$.

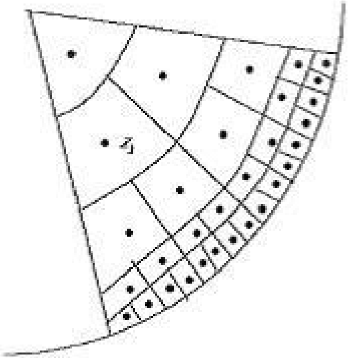
The Tree and "Integration"

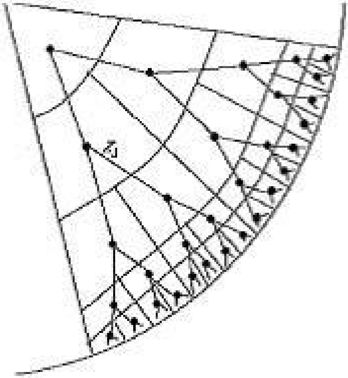
- We form a tree $\mathcal{T}$ (connected loopless graph) using the set of boxes as nodes and and connecting each node ($=$ box) with the two nodes from the boxes directly below.
- There is one special node, $\mathfrak{root}$, the root of the tree. It corresponds to origin of the disk.
- The three pictures show the first few steps in building the tree, a stylized presentation of the corresponding points of the disk, and the standard representation of the tree.
- Notice in the last picture that, informally at least, there seems to be a notion of "path going to the boundary" and perhaps even of an ideal boundary.



Art Credit: Zen Harper

Think of T as sitting inside $\mathbb{D}$ with the root o at 0.
Think of ∂T as being the same as $\mathbb{T}$.





- We define a linear operator $\mathcal{I}$ mapping functions on $\mathcal{T}$ to functions on $\mathcal{T}$. For h defined on $\mathcal{T}$ define $H = \mathcal{I}h$ by

$$H(B_\alpha) = \mathcal{I}h(B_\alpha) = \sum_{\mathfrak{root} \to B_\alpha} h(B_\beta).$$

  The summation over the nodes on the natural (geodesic) path from the root to B_a. $\mathcal{I}$ is our discrete model for integration along a segment from 0 to a point ζ in the disk.
- The inverse, discrete differentiation is given by

$$(\Delta H)(B_\alpha) = H(B_{\alpha^-}) - H(B_\alpha) = h(B_\alpha)$$

The Tree Bergman and Dirichlet Spaces

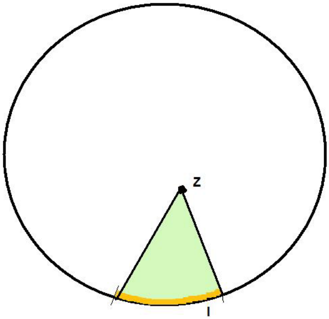
- The decomposition theorem establishes a close relationship between functions $f \in \mathcal{A}^2$ and sequences $\{\lambda_\alpha\} \in \ell^2(\mathcal{T})$; at a very informal level we think of $\lambda_n \sim f(z_n)\Delta_n$ and of $\{\hat{\kappa}_n\}$ as a variation on the idea of an orthonormal basis. With this as background we define the tree Bergman space $\mathcal{A}^2(\mathcal{T})$ by $\mathcal{A}^2(\mathcal{T}) = \ell^2(\mathcal{T})$.
- The functions in the Dirichlet space are indefinite integrals of Bergman space functions. With that as a guide we define tree Dirichlet space $\mathcal{D}(\mathcal{T})$ by

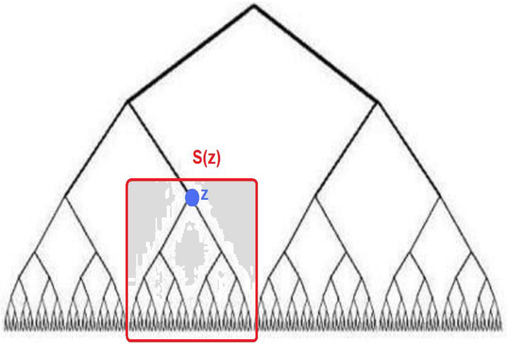
$$\mathcal{D}(\mathcal{T}) = \{\mathcal{I}h : h \in \mathcal{A}^2(\mathcal{T})\}$$

and we norm $\mathcal{D}(\mathcal{T})$ so this is an isometry.

- There is an informal coherence between the two definitions: If we start with $F \in \mathcal{D}$ (with $F(0) = 0$) then we have $F' \in \mathcal{A}^2$. Thus $\{\mathcal{F}_n'\} = \{F'(z_n)\Delta_n\}$ is in $\ell^2(\mathcal{T}) = \mathcal{A}^2(\mathcal{T})$. Now we form $\mathcal{F}$ in $\mathcal{D}(\mathcal{T})$ using our discrete model integration operator; $\mathcal{F} = \mathcal{I}\mathcal{F}$ '.
- We have

$$\begin{aligned} \mathcal{F}(z_n) &= \sum \mathcal{F}\prime(z_k) \\ &= \sum F'(z_n)\Delta_n \\ &\sim \int_0^{z_n} F'(z)dz = F(z_n) - F(0) \\ &= F(z_n). \end{aligned}$$

- Although it is not clear if it will be productive, we can now import many of the definitions and ideas from $\mathcal{D}$ to $\mathcal{D}(\mathcal{T})$: reproducing kernel, multiplier, Carleson measure, HSIS, etc. etc. The translation process is generally mechanical. The picture below is an example of a matching of structures.
- 



A Glance at Function Theory on the Tree

- The analogy between the function theories on $\mathcal{D}(\mathcal{T})$ and on $\mathcal{D}$ turns out to be deep and broad. Sometimes work on the tree model suggests what might be true in $\mathcal{D}$ and what proofs might work in $\mathcal{D}$. Also, some results in the dyadic space can be pulled back to $\mathcal{D}$. We will hear more about this later; for now I will just mention a few facts that indicate the flavor.
- Some particular results:
  - Carleson measures for $\mathcal{D}(\mathcal{T})$ are characterized by the testing condition. That is, the ideas and formulas for $\mathcal{D}(\mathcal{T})$ are essentially the same as for $\mathcal{D}$; the pictures are slightly different. There is real analytical work required for the proof that the testing condition is sufficient.
  - There is also a (tree) capacity characterization of Carleson measures.
  - A set in the "boundary of the tree" has capacity zero if and only if it is a null set for all Carleson measures. (This result comes out of the direct proof that the testing condition and the capacity condition are equivalent.)

Consequences for "Boundary Values"

- Every $F \in \mathcal{D}(\mathcal{T})$ has "boundary values" off a set of capacity zero.
  - Proof discussion: if $F = \mathcal{I}(\cdot F)$ then $F^* = \mathcal{I}(|\Delta F|)$ is a majorant for the variation of F along any path. Hence it suffices to show that $F^* < \infty$ off an appropriate exceptional set. We have $F^* \in \mathcal{D}(\mathcal{T})$ and the "partial sums" of F^* are positive and monotonic. With these facts one can show that $\int |F^*|^2 \, d\mu < \infty$ for any $\mathcal{D}(\mathcal{T})$ Carleson measure μ on the "boundary" of $\mathcal{T}$.
  - Hence the set on which $|F^*| = \infty$ is a $\mu-$null set. μ was an arbitrary Carleson measure and hence the exceptional set has capacity zero.
  - Variations of this argument give a range of results between this "radial convergence" result and the NRS theorem, larger convergence regions played off against larger exceptional sets. The variations use more complicated choices for the majorant and, sometimes, generalizations of $\mathcal{D}(\mathcal{T})$.
  - There are then mechanical ways to use the tree results to get the analogous disk results.

Gains and Losses

- The construction of the majorant F^* made central use of the fact that the map from $\{a_n\}$ to $\{|a_n|\}$ is an isometry on the tree Bergman space. That fact, and other similar ones, are great conveniences when working with the model spaces and there are no easy analogs for the spaces of holomorphic functions.
- Analysis on the tree space can be used to give all the other classical boundary convergence results for $\mathcal{D}$ except the NRS result. It is not clear if there is a fundamental obstacle to proving the NRS theorem that way. (One difference between the NRS result and the others is that the NRS result does not include related results on variation as one approaches the boundary.)

Back to the Disk

- To prove that the testing condition characterizes Carleson measures $\mathcal{D}$ one writes the disk as a union of boxes, replaces the integrand by a well selected constant majorant on each box, and observes that the question has been reduced to knowing if the tree testing condition characterizes tree Carleson measures.