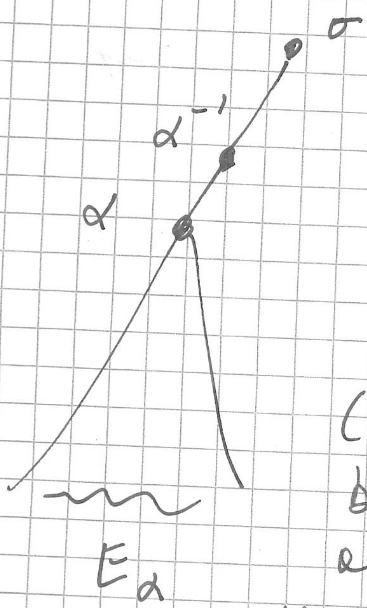
Proof. ($\le$) For $\alpha \in T'$, the subscript α will denote objects related to the tree $T'_\alpha = S(\alpha)$, having root α. In particular, $E_\alpha = E \cap \overline{T'_\alpha}$.
By the general theory, the extremal measure ω_α for E_α and the extremal function $\varphi_\alpha = \mathcal{I}^*_\alpha \omega_\alpha$ satisfy:

$$\mathrm{cap}_\alpha(E_\alpha) = \omega_\alpha(E_\alpha) = \mathcal{E}^\alpha_{T'}(\omega_\alpha) = \|\varphi_\alpha\|^2_{\ell^2(T'_\alpha)}.$$

Claim: ~~$\omega_\alpha =$~~ Let ω be the extremal measure for E in T'. Then:

$$\omega_\alpha = \frac{\omega|E_\alpha}{1 - \mathcal{I}\mathcal{I}^*\omega(\alpha^{-1})} \quad \text{(Rescaling property).}$$



Pf. of Claim. ω_α minimizes $\mathcal{E}_\alpha(\mu)$ over all measures μ supported on E_α s.t.

$$\mathcal{I}_\alpha(\mathcal{I}^*_\alpha \mu)(\xi) \ge 1 \text{ on } E_\alpha$$

(by definition of capacity), but for a null-capacity set at most.

We show now that $\omega|_{E_\alpha}$ minimizes $\mathcal{E}_\alpha(\nu)$ among ν's supported on E_α s.t.

$$(*) \quad \mathcal{I}_\alpha(\mathcal{I}^*_\alpha \nu)(\xi) \ge 1 - \mathcal{I}(\mathcal{I}^*\omega)(\alpha^{-1}). \quad \text{quasi everywhere}$$

Observe first that $\nu = \omega|_{E_\alpha}$ has obviously property (*):

$$\mathcal{I}(\mathcal{I}^*\omega)(\alpha^{-1}) + \mathcal{I}_\alpha(\mathcal{I}^*_\alpha(\omega|_{E_\alpha}))(\xi) = \mathcal{I}(\mathcal{I}^*\omega)(\xi) \ge 1,$$

but for a null-capacity set.

If $\omega|_{E_\alpha}$ did not have the minimizing property, then would exist ν on E_α s.t.

$$I_\alpha(I_\alpha^*\nu)(\xi) \ge 1 - I(I^*\omega)(\alpha^{-1}) \quad \text{q.e. } \xi \in E_\alpha$$

and

$$\mathcal{E}_\alpha^{T'}(\nu) = \sum_{\beta \in T'_\alpha} I^*\nu(\beta)^2 < \sum_{\beta \in T'_\alpha} I^*\omega(\beta)^2 = \mathcal{E}_\alpha^{T'}(\omega|_{E_\alpha}).$$

Define $\Psi : T' \to \mathbb{R}$ by

$$\Psi(\beta) = \begin{cases} I^*\nu(\beta) & \text{if } \beta \in T'_\alpha \\ I^*\omega(\beta) & \text{if } \beta \in T' \setminus T'_\alpha . \end{cases}$$

By hypothesis $I\Psi(\xi) \ge 1$ q.e. on E, ~~and~~ hence

$$\mathrm{Cap}^{T'}(E) \le \|\Psi\|_{\ell^2}^2 = \mathcal{E}_\alpha^{T'}(\nu) + [\mathcal{E}(\omega) - \mathcal{E}_\alpha(\omega)]$$

$< \mathcal{E}(\omega) = \mathrm{Cap}(E)$, a contradiction in terms.

We have shown that $\lambda := \dfrac{\omega|_{E_\alpha}}{1 - I(I^*\omega)(\alpha^{-1})}$ minimizes $\mathcal{E}_\alpha(\mu)$ over the set of measures μ s.t. $I_\alpha(I_\alpha^*\mu) \ge 1$ q.e. on E_α, hence $\lambda = \omega_\alpha$ by uniqueness of extremals, the claim is proved.

Using homogeneity of $\mathcal{E}_\alpha$:

$$\mathcal{E}_\alpha(\omega|_{E_\alpha}) = [1 - I(I^*\omega)(\alpha^{-1})]^2 \cdot \mathcal{E}_\alpha(\omega_\alpha)$$

$$= [1 - I(I^*\omega)(\alpha^{-1})]^2 \cdot \omega_\alpha(E_\alpha) \quad \text{(extremality)}$$

$$= [1 - I(I^*\omega)(\alpha^{-1})] \cdot \omega(E_\alpha), \quad \text{hence}$$

$$\frac{I_\alpha^*(I_\alpha^*\omega)^2(\alpha)}{\omega(S(\alpha))} = \frac{\mathcal{E}(\omega|_{E_\alpha})}{\omega(E_\alpha)} = [1 - I(I^*\omega)(\alpha^{-1})] \le 1,$$

with equality $\iff \alpha = \sigma$.

Hence, $\|\omega\|_{CM} = 1$ and

$$\mathrm{Cap}^{T}(E) = \omega(E) = \frac{\omega(E)}{\|\omega\|_{CM}},$$

showing that $\displaystyle\sup_{\mathrm{supp}(\mu)\subseteq E} \frac{\mu(E)}{\|\mu\|_{CM(T)}} \geq \frac{\omega(E)}{\|\omega\|_{CM}} = \mathrm{Cap}^{T}(E)$.

($\geq$) By def. of $\|\cdot\|_{CM}$ (at the root), if μ is supported on E, then:

$$\frac{\mu(E)}{\|\mu\|_{CM}} \leq \frac{\mu(E)}{\left(\frac{\mathcal{E}(\mu)}{\mu(E)}\right)} = \frac{\mu(E)^2}{\mathcal{E}(\mu)} \leq \mathrm{Cap}^{T}(E).$$

The proof gives more than the statement.

Let ω be the equilibrium measure of a set $E \subseteq T$. Then:

(1) $\|\omega\|_{CM(T)} = 1$; moreover:

(2) $\|\omega|_{E_\alpha}\|_{CM(T_\alpha)} = 1 - I(I^*\omega)(\alpha^{-1})$, equivalently:

$$\|\omega_\alpha\|_{CM(T_\alpha)} = 1 \quad \forall \alpha.$$

<u>Question</u> (I don't know the answer): does property (2) characterize equilibrium measures?

Theorem 2. Let $\sigma_\mu := (\mathcal{I}^*\mu)^2 . \,\mathrm{d}\pi$. Let $\lambda : \bar{\pi} \to [0,1]$ be measurable. If $\mathcal{I}^*\sigma_\mu \leq \mathcal{I}^*\mu$ on $\bar{\pi}$, then
$$\mathcal{I}^*(\sigma_{\lambda\mu}) \leq 2 . \, \mathcal{I}^*(\lambda\mu). \qquad \text{(Monotonicity)}.$$

Corollary: $\nu \leq \mu \Rightarrow \|\nu\|_{CM(\bar{\pi})} \leq 2 . \|\mu\|_{CM(\bar{\pi})}$.

Pf. Since $\mathcal{I}^*$ "looks forward", it suffices to check the statement at the root (and rescaling). Let

$$\mathcal{M}_\mu \lambda(\alpha) := \max_{\sigma \leq \gamma \leq \alpha} \frac{\mathcal{I}^*(\lambda\mu)(\gamma)}{\mathcal{I}^*\mu(\gamma)}$$ (an ad hoc MAX. FUNCT., already seen).

$$\mathcal{I}^*\sigma_{\lambda\mu}(\sigma) = \sum_\alpha \left[\frac{\mathcal{I}^*(\lambda\mu)(\alpha)}{\mathcal{I}^*\mu(\alpha)}\right]^2 \mathcal{I}^*\mu(\alpha)^2$$

$$\leq \sum_\alpha \left[\mathcal{M}_\mu \lambda(\alpha)\right]^2 . \, \sigma_\mu(\alpha) =$$

$$= 2 . \int_0^1 t . \, \sigma_\mu(\zeta \in \bar{\pi} : \mathcal{M}_\mu \lambda(\zeta) > t) \, dt$$

Indeed, $\{\zeta : \mathcal{M}_\mu \lambda(\zeta) > t\} = \coprod S(\alpha_j)$ is the disjoint union of successor sets, hence: $t . \sigma_\mu(\zeta : \mathcal{M}_\mu \lambda(\zeta) > t)$

$$= \sum_j t \, \sigma_\mu(S(\alpha_j)) \underset{\text{by hypothesis}}{\leq} \sum_j t . \, \mathcal{I}^*\mu(\alpha_j)$$

$\leq \sum_j \mathcal{I}^*(\lambda\mu)(\alpha_j)$ (minimality of the α_j's)

$\leq \mathcal{I}^*(\lambda\mu)(\sigma)$ (additivity) $\Rightarrow \mathcal{I}^*\sigma_{\lambda\mu}(\sigma) \leq 2 . \, \mathcal{I}^*(\lambda\mu)(\sigma)$ ∎

Theorem 3 (Stegenga). Let $\mu \geq 0$ be a measure on T. Then, $\mu \in CM(\overline{T}) \iff$ for all sets $E = \coprod_j \overline{S(\alpha_j)}$ we have

$$\mu\Big(\bigcup_j \overline{S(\alpha_j)}\Big) \leq C \cdot \mathrm{Cap}_T\Big(\bigcup_j \overline{S(\alpha_j)}\Big)$$

(Testing over finite unions suffices).

Obs. that $\mathrm{Cap}_T\Big(\bigcup_j \overline{S(\alpha_j)}\Big) = \mathrm{Cap}_T(\{\alpha_j\})$.

Pf. Suppose that $\mu \in CM(T)$. ~~WLOG~~ (~~cont~~. Rescaling, we can assume that

$$\sup_{\alpha \in T} \frac{I^*(I^*\mu)^2(\alpha)}{I^*\mu(\alpha)} \leq 1$$

Let ~~$\mu_E = 1_E \mu$~~ E as above:

$\mu|_E \leq \mu \Rightarrow \|\mu|_E\|_{CM(T)} \leq 2$ by Thm. 2.

Hence, $\mathrm{Cap}_T(E) = \sup_{\mathrm{supp}(\nu) \subseteq E} \frac{\nu(E)}{\|\nu\|_{CM(T)}}$ by Thm. 1

$$\geq \frac{\mu(E)}{\|\mu|_E\|_{CM(T)}} \geq \frac{\mu(E)}{2},$$ as wished.

That the capacitary condition implies the testing condition is known and it will not be discussed here. ∎

To have Stegenga's Thm., however, we need the capacity of boundary sets.

No problem.

Lemma. $\mathrm{cap}_{T'}\left(\bigcup S(\alpha_j)\right) \le 4 \cdot \mathrm{cap}_{T'}\left(\bigcup_j \partial S(\alpha_j)\right)$

Pf. Let φ be extremal for $\mathrm{cap}_{T'}(E)$, $E = \bigcup_j \partial S(\alpha_j)$

$I\varphi \ge 1$ on E and $\mathrm{cap}_{T'}(E) = \|\varphi\|_{\ell^2}^2$

We show that φ is "near extremal" for $S(E) = \coprod_j S(\alpha_j)$.

We know that $\varphi = I^* \omega$ (ω extremal) and ω is constant on each $\partial S(\alpha_j)$ by symmetry:

$\exists\, \Gamma_j > 0$: $\omega(\partial S(\beta)) = \Gamma_j\, 2^{-d(\beta)}$ $\forall \beta \ge \alpha_j$.

Then, $\varphi(\beta) = \Gamma_j\, 2^{-d(\beta)}$ $\forall \beta \ge \alpha_j$.

$\forall \zeta \in \partial S(\alpha_j)$: $1 - I\varphi(\alpha_j) = I\varphi(\zeta) - I\varphi(\alpha_j)$

$$= \sum_{\alpha_j < \beta \le \zeta} \varphi(\beta) = \Gamma_j \sum_{\alpha_j < \beta < \zeta} 2^{-d(\beta)} = \Gamma_j \cdot 2^{-d(\alpha_j)}$$

$= \varphi(\alpha_j) \Rightarrow 1 - \varphi(\alpha_j) = I\varphi(\alpha_j)$.

Since φ is decreasing ~~with~~ on T',

$\varphi(\alpha_j) \le 1/d(\alpha_j) \Rightarrow I\varphi(\alpha_j) \ge 1 - \frac{1}{d(\alpha_j)} \ge \frac{1}{2}$.

This means that 2φ is admissible for the set $\{\alpha_j\}$, hence that

$$\mathrm{cap}_{T'}(\{\alpha_1, \dots, \alpha_n\}) \le \|2\varphi\|_{\ell^2}^2 = 4 \cdot \mathrm{cap}_{T'}(E)$$

$$\|$$

$\mathrm{cap}_{T'}(S(E))$ ∎