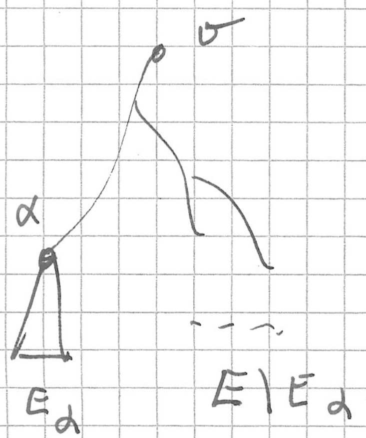
A Variation on Yesterday's Theme.

Let $E \subset T$ be a finite set (to avoid unnecessary technicalities) and let ω be its equilibrium measure. We believe that ω encodes all important potential theoretic information about E.



For instance, we've seen that $\forall \alpha \in T$, the equilibrium measure ω_α of $E_\alpha = E \cap T_\alpha$ from the root α ~~can be~~ is a rescaling of ω:

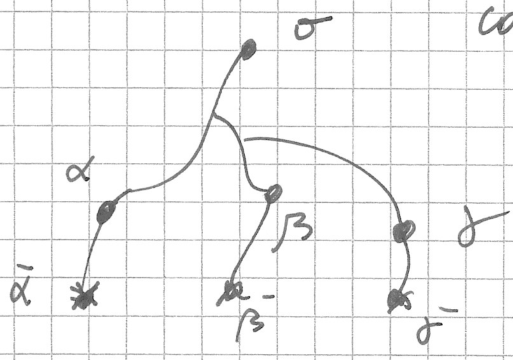
$$(C) \quad \omega_\alpha = \frac{\omega|_{E_\alpha}}{1 - \zeta^{*}\omega(\alpha^{-1})}.$$

Suppose that $\tilde{E} \subset E$ is a subset of E obtained by selecting some of the E_α's, $\tilde{E} = \coprod_{\alpha \in I} E_\alpha$ (where the α's in I ~~are~~ satisfy $\alpha \neq \beta$ in $I \Rightarrow$ neither $\alpha \geq \beta$ nor $\alpha \leq \beta$).

Is there a reasonably efficient way to compute the equilibrium measure $\tilde{\omega}$ of $\tilde{E}$ once we know ω?

MSRI-Theorem. For $\alpha \in I$, let $c(\alpha) = \omega_\alpha(E_\alpha) = \mathrm{Cap}_\alpha(E_\alpha)$.

Consider the ~~tree~~ set I in T and construct a ~~new set~~ $\bar{I} \subset T$ ~~letting~~ choosing $\forall \alpha \in I$ an $\bar{\alpha} \in \bar{I}$ s.t.



(i) $\bar{\alpha} > \alpha$

(ii) $d(\alpha, \bar{\alpha}) = \frac{1}{c(\alpha)} - 1$

(Observe that $c(\alpha) \le 1$. It might be that $1/c(\alpha) \notin \mathbb{N}$; this is not a serious problem).

Compute the equilibrium distribution χ of $\tilde{I}$. Then,

$$(*)\quad \tilde{\omega}|_{E_\alpha} = \frac{\chi(\bar{\alpha})}{c(\alpha)\cdot[1 - \mathcal{J}\mathcal{J}^*\omega(\alpha^{-1})]} \cdot \omega|_{E_\alpha}$$

<u>Obs.</u> The computational effort, then, lies in computing χ.

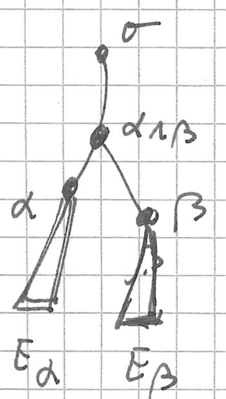
<u>Pf.</u> I write it down for $I = \{\alpha, \beta\}$; to extend it to general I just add notation.

Observe that, by the fact ⓒ recalled before, which applies to both ω and $\tilde{\omega}$, $\forall \alpha \in I$ there are constants K_α, H_α s.t.

$$K_\alpha \cdot \omega|_{E_\alpha} = \omega_\alpha = H_\alpha \cdot \tilde{\omega}|_{E_\alpha},$$

hence there is λ_α s.t. $\tilde{\omega}|_{E_\alpha} = \lambda_\alpha \cdot \omega|_{E_\alpha}$ $\left(\lambda_\alpha = \frac{K_\alpha}{H_\alpha}\right)$.

We have to find the λ_α's.



Since $\tilde{\omega}$ is the equilibrium ⓜ of $E_\alpha \cup E_\beta$, $\mathcal{J}\mathcal{J}^*\tilde{\omega} = 1$ on $E_\alpha \cup E_\beta$.

For $\xi \in E_\alpha$, this translates into:

$$1 = \mathcal{J}\mathcal{J}^*\tilde{\omega}(\xi) = d(o, \alpha\wedge\beta)[\lambda_\alpha \omega(E_\alpha) + \lambda_\beta \omega(E_\beta)] + d(\alpha\wedge\beta, \alpha)\lambda_\alpha \omega(E_\alpha) + \lambda_\alpha \cdot \sum_{\gamma=\alpha}^{\xi} \mathcal{J}^*\omega(\beta)$$

$$= d(o, \alpha\wedge\beta)\cdot[\lambda_\alpha \omega(E_\alpha) + \lambda_\beta \omega(E_\beta)] + d(\alpha\wedge\beta, \alpha)\lambda_\alpha \omega(E_\alpha) + \lambda_\alpha \cdot [1 - \mathcal{J}\mathcal{J}^*\omega(\alpha^{-1})] =$$

Set $\mu_{\alpha/\beta} = \lambda_{\alpha/\beta} \cdot [1 - \mathcal{J}\mathcal{J}^*\omega(\alpha/\beta^{-1})]$ and use ⓒ

$$= \mathfrak{A}(\sigma, \alpha\wedge\beta)\,[\mu_\alpha\, \omega_\alpha(E_\alpha) + \mu_\beta\, \omega_\beta(E_\beta)]$$

$$+ \mathfrak{A}(\alpha\wedge\beta, \alpha)\, \mu_\alpha\, \omega_\alpha(E_\alpha) + \mu_\alpha$$

$$= \mathfrak{A}(\sigma, \alpha\wedge\beta)\,[\mu_\alpha\, c(\alpha) + \mu_\beta\, c(\beta)] + \mathfrak{A}(\alpha\wedge\beta, \alpha)\, \mu_\alpha\, c(\alpha) + \mu_\alpha$$

Interchanging the rôles of α and β and rewriting things slightly differently, using also the notation $\nu_\alpha = \mu_\alpha\, c(\alpha)$:

$$\left\{\begin{aligned} 1 &= \mathfrak{A}(\sigma, \alpha\wedge\beta)(\nu_\alpha + \nu_\beta) + \left\{\left[\mathfrak{A}(\alpha\wedge\beta, \alpha) + \frac{1}{c(\alpha)} - 1\right] + 1\right\} \nu_\alpha \\ 1 &= \mathfrak{A}(\sigma, \alpha\wedge\beta)(\nu_\alpha + \nu_\beta) + \left\{\left[\mathfrak{A}(\alpha\wedge\beta, \beta) + \frac{1}{c(\beta)} - 1\right] + 1\right\} \cdot \nu_\beta \end{aligned}\right\} (*)$$

or

$$\left\{\begin{aligned} 1 &= \mathfrak{A}(\sigma, \bar\alpha\wedge\bar\beta)(\nu_\alpha + \nu_\beta) + [\mathfrak{A}(\bar\alpha\wedge\bar\beta, \bar\alpha) + 1]\, \nu_\alpha \\ 1 &= \mathfrak{A}(\sigma, \bar\alpha\wedge\bar\beta)(\nu_\alpha + \nu_\beta) + [\mathfrak{A}(\bar\alpha\wedge\bar\beta, \bar\beta) + 1]\, \nu_\beta \end{aligned}\right\} (**)$$

by our definition of $\bar\alpha$, $\bar\beta$.

Equations (**) in the unknowns ν_α, ν_β are exactly those giving $\chi(\bar\alpha) = \nu_\alpha$ and $\chi(\bar\beta) = \nu_\beta$.

[Why? They are the equations (*) when $c(\alpha) = c(\beta) = 1$, i.e. when $E_\alpha = \{\alpha\}$ and $E_\beta = \{\beta\}$, in which case it is obvious that their solutions $(\nu(\alpha), \nu(\beta))$ is the equilibrium measure for for $\{\alpha, \beta\}$ in T'].

Unravelling definitions:

$$\tilde\omega|_{E_\alpha} = \lambda_\alpha \cdot \omega|_{E_\alpha} = [1 - JJ^*\omega(\alpha^{-1})]^{-1} \mu_\alpha \cdot \omega|_{E_\alpha}$$

$$= c(\alpha)^{-1}[1 - JJ^*\omega(\alpha^{-1})]^{-1}\, \nu_\alpha \cdot \omega|_{E_\alpha} =$$

$$= c(\alpha)^{-1} \cdot [1 - JJ^*\omega(\alpha^{-1})]^{-1}\, \chi(\bar\alpha) \cdot \omega|_{E_\alpha} \quad \blacksquare$$

Corollary: $\mathrm{Cap}\left(\bigvee_{\alpha\in I} E_\alpha\right) = \mathrm{Cap}(\tilde{I})$.

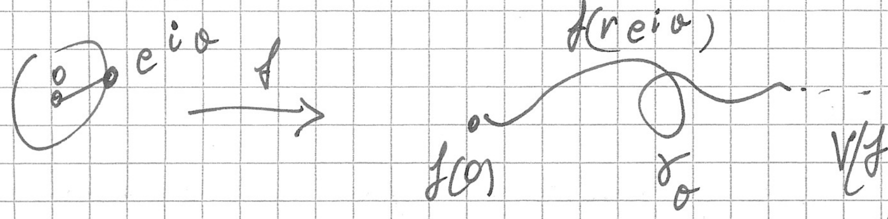
Pf. Use (E): $\mathrm{Cap}\left(\bigvee_{\alpha\in I} E_\alpha\right) = \tilde{\omega}\left(\bigvee_{\alpha} E_\alpha\right) =$

$$= \sum_{\alpha\in I} \frac{\chi(\tilde{\alpha})\,\omega(E_\alpha)}{c(\alpha)\cdot[1-\zeta\,\zeta^{\alpha}\omega(\alpha-1)]} = \sum_{\alpha\in I} \chi(\tilde{\alpha}) = \mathrm{Cap}(\tilde{I})$$ ∎

Beurling's Thm. For $f\in \mathrm{Hol}(\Delta)$ and $e^{i\theta}\in\partial\Delta$, let

$$V(f)(e^{i\theta}) = \int_0^1 |f'(re^{i\theta})|\,dr$$

be the radial variation of f at $e^{i\theta}$;



$V(f)(e^{i\theta}) =$
$= \mathrm{length}(\gamma_\theta)$.

Proposition. Let $\mu\geq 0$ be a Borel measure on $\partial\Delta$. Then $\mu\in CM(\mathcal{D})$ iff

(○) $$\int_{\partial\Delta} V(f)(e^{i\theta})^2\,d\mu(\theta) \le C(\mu)\cdot\|f\|_{\mathcal{D}}^2.$$

Pf. ($\Leftarrow$) If (○) holds, since $V(f)(e^{i\theta}) \geq |f(e^{i\theta})-f(0)|$ for all θ, we have that $\mu\in CM(\mathcal{D})$.

($\Rightarrow$) The local estimate used in the proof of the Discretization Theorem ~~[illegible]~~ (Lecture I) can be trivially be extended to

(*) $$V(f)(\Delta(\zeta)) \le 3\varphi(\zeta) \quad (\forall \zeta\in\partial T)$$

where $$\varphi(\alpha) = \left(\int_{\bigcup \alpha_n:\ d_G(n,\alpha)\le 2} |f'(w)|^2\,du\,dv\right)^{1/2}$$

$\mu\in CM(\mathcal{D}) \Rightarrow \mu\in CM(T)$, and together with (*) and the proof of ($\Leftarrow$) in the Discretization Theorem gives (○) ∎

Corollary (Beurling's Thm e).
Let $f \in \mathcal{D}$. Then,
$\mathrm{Cap}(\{e^{i\theta} : V(f)(e^{i\theta}) = +\infty\}) = 0$.

Pf. Let $\mu \in CM(\mathcal{D})$, $\mathrm{supp}(\mu) \subseteq \partial\Delta$, ~~Then~~ and let $E = \{e^{i\theta} : V(f)(e^{i\theta}) = +\infty\}$.
Then, $\forall f \in \mathcal{D}$:

$$\int_{\partial\Delta} V(f)^2 \, d\mu \le C \cdot \|f\|_{\mathcal{D}} < +\infty$$
$$\vee\!\!\;\backslash$$
$$\int_E V(f)^2 \, d\mu = \mu(E) \cdot (+\infty)$$
$$\Rightarrow \mu(E) = 0 \qquad (\forall \mu)$$
$$\Rightarrow \mathrm{Cap}(E) = 0 \quad \blacksquare$$

On an exercise given yesterday. ~~[illegible]~~

(1) $\mu, \nu \ge 0$ on $\partial\mathbb{T} \Rightarrow \dfrac{\mathcal{E}(\mu+\nu)}{\|\mu+\nu\|} \le \dfrac{\mathcal{E}(\mu)}{\|\mu\|} + \dfrac{\mathcal{E}(\nu)}{\|\nu\|}$

where $\|\mu\| := \int_{\partial\mathbb{T}} d\mu$.

Pf. We want to show $(\le)$ in
$$\mathcal{E}(\mu+\nu) = \mathcal{E}(\mu) + 2\,\mathcal{E}(\mu,\nu) + \mathcal{E}(\nu) \le$$
$$\le \|\mu+\nu\|\left(\frac{\mathcal{E}(\mu)}{\|\mu\|} + \frac{\mathcal{E}(\nu)}{\|\nu\|}\right) = (\|\mu\| + \|\nu\|)\left(\frac{\mathcal{E}(\mu)}{\|\mu\|} + \frac{\mathcal{E}(\nu)}{\|\nu\|}\right)$$
$$= \mathcal{E}(\mu) + \mathcal{E}(\nu) + p\,\mathcal{E}(\mu) + \frac{1}{p}\mathcal{E}(\nu), \text{ where}$$
$$p = \frac{\|\nu\|}{\|\mu\|} \text{ and } \mathcal{E}(\mu,\nu) = I^*(I^*\mu \cdot I^*\nu).$$

~~It~~ It suffices to show that $2 I^*\mu \cdot I^*\nu \le p\, I^*\mu^2 + \frac{1}{p} I^*\nu^2$,
but this follows from $2xy = 2\dfrac{x}{\sqrt{p}} \cdot \sqrt{p}\,y \le \dfrac{x^2}{p} + p\,y^2$.

(2) It follows that
$$\|\mu+\nu\|_{CM} \le \|\mu\|_{CM} + \|\nu\|_{CM}\ ; \text{ where } \|\mu\|_{CM} = \sup_\alpha \frac{\mathcal{E}_\alpha(\mu)}{\mu(S(\alpha))}.$$