Exercises for harmonically weighted Dirichlet spaces

1. Provide the missing details of the proof of the Wold decomposition theorem for 2-isometries:
Let $T \in \mathcal{B}(\mathcal{H})$ be a 2-isometry.

Then

$$T = S \oplus U \quad \text{with respect to } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where U is unitary, $\mathcal{H}_2 = \bigcap_n T^n \mathcal{H}$
and S is an analytic 2-isometry

2. Show that if T is a 2-isometry, then $\sigma(T) \subseteq \overline{\mathbb{D}}$.

3. Show that the following identity holds for all polynomials f and all $\zeta \in \mathbb{T}$

$$D_\zeta(f) = \int_{|z|=1} \frac{|f(z) - f(\zeta)|^2}{|z-\zeta|^2} \frac{|dz|}{2\pi} = \int_{|z|<1} |f'(z)|^2 \frac{1-|z|^2}{|z-\zeta|^2} \frac{dA(z)}{\pi}$$

4. For $|\lambda| < 1$ and $f \in H^2$ define

$$D_\lambda(f) = \int_{|z|=1} \frac{|f(z) - f(\lambda)|^2}{|z-\lambda|^2} \frac{|dz|}{2\pi}.$$

Show that

$$D_\zeta(f) = \text{ntl-} \lim_{\lambda \to \zeta} D_\lambda(f)$$

holds for all $f \in H^2$ and for all $\zeta \in \mathbb{T}$.

Suggestion: If $D_\zeta(f) < \infty$, then $g(z) = \frac{f(z)-f(\zeta)}{z-\zeta} \in H^2$ and $f(z) = f(\zeta) + (z-\zeta)g(z)$. Express $D_\lambda(f)$ in terms of g and use that $\frac{|\lambda-\zeta|}{1-|\lambda|^2}$ is bounded for λ in a nontangential approach region. Handle $D_\zeta(f) = \infty$ separately.

5. Let $f \in H^2$, $|\zeta| = 1$.

(a) Show that there exists $c > 0$ such that

$$|f(\zeta)|^2 \le c(\|f\|^2_{H^2} + D_\zeta(f)) \quad \text{whenever } D_\zeta(f) < \infty.$$

(b) Let $\mu \in M_+(\mathbb{T})$ and $f \in H^2$. Show that $f \in D(\mu)$ if and only if $zf \in D(\mu)$ and

$$\|zf\|^2_{D(\mu)} = \|f\|^2_{D(\mu)} + \int |f(\zeta)|^2 d\mu(\zeta).$$

In particular, $(M_z, D(\mu))$ is bounded and is an analytic 2-isometry.

6. Let φ be an inner function, $f \in H^2$ and $\lambda \in \overline{\mathbb{D}}$. Show that

$$D_\lambda(\varphi f) = D_\lambda(\varphi)|f(\lambda)|^2 + D_\lambda(f).$$

Here we define $0 \cdot \infty = 0$. Suggestion: Start with $|\lambda| < 1$ and use Problem 4.

7. Let $k_\lambda(z) = \frac{1}{1-c\bar{\lambda}z(1+\bar{\lambda}z)}$, where $0 < c \leq 1/2$. Then clearly k is a CNP kernel for some Hilbert space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$. Let $\mathcal{M} = \{f \in \mathcal{H} : f(0) = 0\}$. Then $\mathcal{M}$ is a multiplier invariant subspace. Calculate the reproducing kernel for $\mathcal{M}$ and show that it is not a CNP kernel.

Hint: If $\sum_{n\geq 0} a_n \overline{\lambda}^n z^n$ is positive definite then all $a_n \geq 0$.

8. If k is the RK for $D(\mu)$, then $k_\lambda(z) \neq 0$ for all $\lambda, z \in \mathbb{D}$.

Hint: If $k_\lambda(\alpha) = 0$, then $g = \frac{k_\lambda}{\varphi_\alpha} \in D(\mu)$ with $\|g\| \leq \|k_\lambda\|$ (here $\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$ and use Problem 5). Then $|g(\lambda)| > \|g\|\|k_\lambda\|$ unless $\|k_\lambda\| = 0$.