The Corona Problem in the Dirichlet Space

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Lecture Outlines & Topics Covered

- Motivations for the Problem
- The Corona Problem for $H^\infty(\mathbb{D})$
  - Carleson measures and $\overline{\partial}$-problems;
  - Wolff's proof of the Corona Problem;
  - Jones' constructive solution to $\overline{\partial} b = \mu$;
- The Corona Problem for $M_{\mathcal{D}}$
  - Xiao's Theorem on the Dirichlet space
- The Corona Problem for Multiplier Algebras with the Complete Nevanlinna-Pick Property
  - Reproducing kernel Hilbert function spaces with Complete Nevanlinna-Pick kernel;
  - The Baby Corona Problem & The Corona Problem;
  - Toeplitz Corona Theorem;
- The Corona Problem in Several Variables

Motivations for the Problem

Where Did the Name Come From?

The Beer Problem?

Commutative Banach Algebras

A (commutative) Banach algebra $\mathcal{A}$ is a complex (commutative) algebra $\mathcal{A}$ that is also a Banach space under a norm that satisfies

$$\|fg\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}} \|g\|_{\mathcal{A}} \quad f, g \in \mathcal{A}.$$

We will also assume that there is an identity element $1 \in \mathcal{A}$ and that our algebra is commutative. An element $f \in \mathcal{A}$ is *invertible* if there exists an element $g \in \mathcal{A}$ such that $fg = 1$ and write f^{-1} for g. We let

$$\mathcal{A}^{-1} = \{f \in \mathcal{A} : f^{-1} \text{ exists}\}.$$

Finally, we need to consider the non-zero multiplicative linear functionals on the algebra $\mathcal{A}$. These are simply the complex homomorphisms $m : \mathcal{A} \to \mathbb{C}$. Note that we trivially have that $m(1) = 1$.

Commutative Banach Algebras

Lemma 1

Every complex homomorphism from $\mathcal{A}$ to $\mathbb{C}$ is a continuous linear functional with norm at most 1. Namely,

$$\|m\| = \sup_{f\in\mathcal{A},\|f\|_{\mathcal{A}}\leq 1} |m(f)| \leq 1.$$

Proof: If m is unbounded or if $\|m\| > 1$ then we can find an element $f \in \mathcal{A}$ with $\|f\|_{\mathcal{A}} < 1$ but $m(f) = 1$. Since $\|f\|_{\mathcal{A}} < 1$ we have

$$\left\|\sum_{n=0}^{\infty} f^n\right\|_{\mathcal{A}} \leq \sum_{n=0}^{\infty} \|f^n\|_{\mathcal{A}} \leq \sum_{n=0}^{\infty} \|f\|_{\mathcal{A}}^n = \frac{1}{1-\|f\|_{\mathcal{A}}}.$$

Note that we have $(1-f)\sum_{n=0}^{\infty} f^n = 1$, so $1-f \in \mathcal{A}^{-1}$. However,

$$1 = m(1) = m((1-f)(1-f)^{-1}) = m((1-f)^{-1})(m(1)-m(f)) = 0.$$

Commutative Banach Algebras

Lemma 2

Suppose that M is a maximal ideal of $\mathcal{A}$. Then M is the kernel of a multiplicative linear functional $m : \mathcal{A} \to \mathbb{C}$. Conversely, suppose that $m : \mathcal{A} \to \mathbb{C}$ is a multiplicative linear functional. Then $\ker m$ *is a maximal ideal.*

Proof: For the converse, it is immediate that the $\ker m$ is an ideal. The maximality follows since the codimension of any linear functional is 1. Namely, $\dim(\mathcal{A} \setminus \ker m) = 1$.

In the other direction, one first shows that the maximal ideal M is closed. Then one shows that the quotient algebra $\mathcal{B} = \mathcal{A}/M$ satisfies

$$\mathcal{B} = \mathbb{C}\mathbf{1}$$

where $\mathbf{1} = 1 + M$ denotes the unit in the quotient algebra. The quotient mapping will then define the multiplicative linear functional, and the kernel of this mapping with then be M.

Commutative Banach Algebras

Let $\mathfrak{M}_{\mathcal{A}}$ denote the set of complex homomorphisms of $\mathcal{A}$ (called the *maximal ideal space* of the Banach algebra).
By Lemma 1, we have that $\mathfrak{M}_{\mathcal{A}} \subset \text{Ball}_1(\mathcal{A}^*)$.
Endow $\mathfrak{M}_{\mathcal{A}}$ with the weak-$*$ topology of $\mathcal{A}^*$ (called the *Gelfand topology*).
Namely, the basic neighborhood of a $m_0 \in \mathfrak{M}_{\mathcal{A}}$ is determined by $\epsilon > 0$ and by elements $f_1, \ldots, f_n \in \mathcal{A}$ such that

$$V = \{m \in \mathcal{A}^* : \|m\| \leq 1, |m(f_j) - m_0(f_j)| < \epsilon, \quad 1 \leq j \leq n\}.$$

Because

$$\mathfrak{M}_{\mathcal{A}} = \{m \in \mathcal{A}^* : \|m\| \leq 1, \quad m(fg) = m(f)m(g), f, g \in \mathcal{A}\}$$

we have that $\mathfrak{M}_{\mathcal{A}}$ is a weak-$*$ closed subset of the unit ball of $\mathcal{A}^*$.
The Banach-Alaoglu Theorem gives that the ball of $\mathcal{A}^*$ is weak-$*$ compact, which implies that $\mathfrak{M}_{\mathcal{A}}$ is a compact Hausdorff space.

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Let $\varphi : H^\infty(\mathbb{D}) \to \mathbb{C}$ be a non-zero multiplicative linear functional. Namely,

$$\varphi(fg) = \varphi(f)\varphi(g) \quad \text{and} \quad \varphi(f+g) = \varphi(f) + \varphi(g).$$

As we have seen for any multiplicative linear functional

$$\sup_{f\in H^\infty(\mathbb{D})} |\varphi(f)| \leq \|f\|_{H^\infty(\mathbb{D})}\,.$$

To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$:

$$\varphi_z(f) := f(z) \quad \text{(point evaluation at } z\text{)}.$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Every multiplicative linear functional φ determines a maximal (proper) ideal of $H^\infty(\mathbb{D})$: $\ker\varphi = \{f \in H^\infty(\mathbb{D}) : \varphi(f) = 0\}$.

Conversely, if M is a maximal (proper) ideal of $H^\infty(\mathbb{D})$ then $M = \ker\varphi$ for some multiplicative linear functional.

The maximal ideal space of $H^\infty(\mathbb{D})$, $\mathcal{M}_{H^\infty(\mathbb{D})}$, is the collection of all multiplicative linear functionals φ.

We then have that the maximal ideal space is contained in the unit ball of the dual space $H^\infty(\mathbb{D})$. If we put the weak-$*$ topology on this space then $\mathcal{M}_{H^\infty(\mathbb{D})}$ is a compact Hausdorff space.

The proceeding discussion then shows that $\mathbb{D} \subset \mathcal{M}_{H^\infty(\mathbb{D})}$.

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

One then defines the Corona of $H^\infty(\mathbb{D})$ to be $\mathcal{M}_{H^\infty(\mathbb{D})} \setminus \overline{\mathbb{D}}$.

In 1941, Kakutani asked if there was a Corona in the maximal ideal space $\mathcal{M}_{H^\infty(\mathbb{D})}$ of $H^\infty(\mathbb{D})$, i.e. whether or not the disc $\mathbb{D}$ was dense in $\mathcal{M}_{H^\infty(\mathbb{D})}$?

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Using basic functional analysis, Kakutani's question can be phrased as the following question about analytic functions on the unit disc:

Theorem 1

The open disc $\mathbb{D}$ is dense in $\mathfrak{M}_{H^\infty}$ if and only if the following condition holds: If $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ and if

$$\max_{1\leq j\leq n} |f_j(z)| \geq \delta > 0$$

then there exists $g_1, \cdots, g_n \in H^\infty(\mathbb{D})$ such that

$$f_1 g_1 + \cdots + f_n g_n = 1.$$

Proof of Theorem 1

Proof: Suppose that $\mathbb{D}$ is dense in $\mathfrak{M}_{H^\infty}$. Then, by continuity we have that

$$\max_{1\leq j\leq n} |m(f_j)| \geq \delta$$

for all $m \in \mathfrak{M}_{H^\infty}$. This implies that $\{f_1, \ldots, f_n\}$ is in no proper ideal of $H^\infty(\mathbb{D})$. Hence the ideal generated by $\{f_1, \ldots, f_n\}$ must contain the constant function 1 and so there exists $g_1, \ldots, g_n \in H^\infty(\mathbb{D})$ such that

$$1 = f_1 g_1 + \cdots + f_n g_n.$$

Conversely, suppose that $\mathbb{D}$ is not dense in $\mathfrak{M}_{H^\infty}$. Then for some $m_0 \in \mathfrak{M}_{H^\infty}$ has a neighborhood disjoint from $\mathbb{D}$ and this neighborhood has the form

$$V = \cap_{j=1}^{n} \{m : |m(f_j)| < \delta\}$$

where $\delta > 0$ and $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ with $m_0(f_j) = 0$.

Proof of Theorem 1, Continued

Since $\mathbb{D} \cap V = \emptyset$ we have that

$$\max_{1 \leq j \leq n} |f_j(z)| \geq \delta > 0.$$

But, it is not possible that we have

$$1 = f_1 g_1 + \cdots + f_n g_n$$

since we have that $m_0(f_j) = 0$ for all $1 \leq j \leq n$. $\square$

Exercise 2

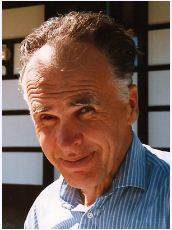
Let $A(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}}) : \sup_{z \in \mathbb{D}} |f(z)| < \infty\}$ *denote the disc algebra. Show that*

$$\mathfrak{M}_{A(\mathbb{D})} = \overline{\mathbb{D}}.$$

Where Did the Name Come From?

The Corona Problem for $H^\infty(\mathbb{D})$

Kakutani's question was settled in 1962 by Carleson: $\overline{\mathbb{D}} = \mathcal{M}_{H^\infty(\mathbb{D})}$.



Lennart Carleson

Theorem 3 (Carleson's Corona Theorem)

Let $\{f_j\}_{j=1}^N \in H^\infty(\mathbb{D})$ *satisfy*

$$0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1, \quad \forall z \in \mathbb{D}.$$

Then there are functions $\{g_j\}_{j=1}^N$ *in* $H^\infty(\mathbb{D})$ *with*

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1 \quad \forall z \in \mathbb{D} \text{ and } \|g_j\|_\infty \leq C_{\delta,N}.$$

Reasons to Care about the Corona Problem

1. Angles between Invariant Subspaces: $K_\theta := H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$
   The angle between K_{θ_1} and K_{θ_2} is positive if and only if there exists ψ_1 and ψ_2 such that
   $$\psi_1\theta_1 + \psi_2\theta_2 = 1.$$

2. Interpolation in $H^\infty(\mathbb{D})$: Let $B(z)$ be a Blaschke product with zeros $\{z_k\} = \mathcal{Z}$, and let $f \in H^\infty(\mathbb{D})$ be such that
   $$|f(z)| + |B(z)| \geq \delta > 0 \quad \forall z \in \mathbb{D}.$$
   Then there exists $p, q \in H^\infty(\mathbb{D})$ such that $pf + qB = 1$ if and only if there is a solution to the interpolation problem
   $$p(z_k) = \frac{1}{f(z_k)} \quad \forall z_k \in \mathcal{Z}.$$

3. Control Theory

Reformulation of the Problem

Let Ω be a domain in $\mathbb{C}^n$.
Let E and E_* be separable complex Hilbert spaces.
$H^\infty_{E_*\to E}(\Omega)$ is the collection of all bounded operator-valued functions.

$$F(z) : E_* \to E \text{ and } \|F\|_{H^\infty_{E_*\to E}(\Omega)} := \sup_{z\in\Omega} \|F(z)\|_{E_*\to E}$$

Question 4 ($H^\infty_{E_*\to E}(\Omega)$-Operator Corona Problem)

Let $F \in H^\infty_{E_*\to E}(\Omega)$. Can we find, preferably local, necessary and sufficient conditions on F so that it has an analytic left inverse? Namely, what conditions imply the existence of a function $G \in H^\infty_{E\to E_*}(\Omega)$ such that

$$G(z)F(z) \equiv I \quad \forall z \in \Omega.$$

A simple necessary condition is: $I \geq F^*(z)F(z) \geq \delta^2 I \quad \forall z \in \Omega.$

Connection to the Usual Corona Problem

Let $\Omega = \mathbb{D}$, the unit disc in the complex plane. Take $F(z) = (f_1(z), \ldots, f_N(z))^T$ in the Operator Corona Problem to recover:

Question 5 (Corona Problem)

Suppose that $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ *with*

$$1 \geq \sum_{j=1}^{N} |f_j(z)|^2 \geq \delta > 0 \quad \forall z \in \mathbb{D}.$$

Do there exist $g_j \in H^\infty(\mathbb{D})$ *such that*

$$\sum_{j=1}^{N} f_j(z) g_j(z) \equiv 1 \quad \forall z \in \mathbb{D}?$$

Known Results for the disc $\mathbb{D}$

- When $E_* = \mathbb{C}$ and $\dim E < \infty$:
  - In 1962 Carleson demonstrated that the simple necessary condition is sufficient;
  - In 1967 Hörmander gave another proof of the result using $\overline{\partial}$-equations;
  - In 1979 Wolff gave a simpler proof of Carleson's result.
- When $E_* = \mathbb{C}, \dim E = \infty$:
  - Rosenblum, Tolokonnikov, and Uchiyama independently gave proofs.
- When $\dim E_* < \infty$ and $\dim E = \infty$: (Matrix Corona Problem)
  - Fuhrmann and Vasyunin independently demonstrated this.
- When $\dim E = \dim E_* = \infty$: (Operator Corona Problem)
  - In 1988 Treil constructed a counter example which indicates that the necessary condition is no longer sufficient.
  - In 2004 he gave another construction which demonstrated the same phenomenon.

Known Results for General Domains and Several Variables

- In 1970 Gamelin showed that the $H^\infty(\Omega)$-Corona problem is true when the domain $\Omega \subset \mathbb{C}$ has "holes."
- More generally, if $\Omega \subset \mathbb{C}$ is a Denjoy domain, then the Corona Theorem is true by a result of Garnett and Jones from 1985.

The Story is Much Different in Several Complex Variables

- Simple cases where the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$. For example, the Ball Algebra $A(\mathbb{B}_n)$ or Polydisc Algebra $A(\mathbb{D}^n)$.
- There are counterexamples to Corona Theorems in several complex variables due to Cole, Sibony and Fornaess, but for domains that are very complicated geometrically.
- When $n \geq 2$ and $\Omega \subset \mathbb{C}^n$ (e.g. $\mathbb{B}_n$ or $\mathbb{D}^n$) the $H^\infty(\Omega)$-Corona Problem is open.

Take Away Point

There are ***NO*** *known domains in* $\Omega \subset \mathbb{C}$ *for which the* $H^\infty(\Omega)$*-Corona problem fails.*

There are no ***KNOWN*** *domains in* $\Omega \subset \mathbb{C}^n$ *for which the* $H^\infty(\Omega)$*-Corona problem holds.*

The Corona Problem for $H^\infty(\mathbb{D})$

Carleson Corona Problem

Our goal is now to prove the following important theorem of Carleson

Theorem 6 (Carleson)

Suppose that $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ *and there exists a* $\delta > 0$ *such that*

$$1 \geq \max_{1 \leq j \leq n} \{|f_j(z)|\} \geq \delta > 0.$$

Then there exists $g_1, \ldots, g_n \in H^\infty(\mathbb{D})$ *such that*

$$1 = f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) \quad \forall z \in \mathbb{D}$$

and

$$\|g_j\|_{H^\infty(\mathbb{D})} \leq C_{\delta,n} \quad \forall j = 1, \ldots, n.$$

An obvious remark is that the condition on the functions f is clearly necessary. We can't have all the functions simultaneously vanish if they can generate the function 1.

Motivating the $\overline{\partial}$-problem

First consider the case of two functions so that we can see the connections between this problem and the $\overline{\partial}$-problem we will study. Suppose we have two functions $f_1, f_2 \in H^\infty(\mathbb{D})$ such that

$$\max(|f_1(z)|, |f_2(z)|) \geq \delta.$$

Define the following functions

$$\varphi_1(z) = \frac{\overline{f_1(z)}}{|f_1(z)|^2 + |f_2(z)|^2} \quad \varphi_2(z) = \frac{\overline{f_2(z)}}{|f_1(z)|^2 + |f_2(z)|^2}.$$

The hypotheses on f_1 and f_2 imply that the functions φ_1 and φ_2 are in fact bounded and smooth on $\mathbb{D}$. Note that

$$1 = f_1(z)\varphi_1(z) + f_2(z)\varphi_2(z) \quad \forall z \in \mathbb{D}$$

but the functions φ_1 and φ_2 are in general *not* analytic.

Motivating the $\overline{\partial}$-problem

Now, observe for any function r we have that the functions

$$g_1 = \varphi_1 + rf_2 \quad g_2 = \varphi_2 - rf_1$$

also solve the problem

$$f_1 g_1 + f_2 g_2 = 1.$$

Our goal is to select a good choice of function r so that the resulting choice will make g_1 and g_2 be analytic and bounded. Now, we have that g_1 is analytic if and only if

$$0 = \overline{\partial} g_1 = \overline{\partial}\varphi_1 + f_2 \overline{\partial} r.$$

Similarly, g_2 is analytic if and only if

$$0 = \overline{\partial} g_2 = \overline{\partial}\varphi_2 - f_1 \overline{\partial} r.$$

Using these two equations and the condition that $f_1\varphi_1 + f_2\varphi_2 = 1$ gives that the function r must satisfy the equation

$$\overline{\partial} r = \varphi_1 \overline{\partial}\varphi_2 - \varphi_2 \overline{\partial}\varphi_1.$$

Solving $\overline{\partial}$-Equations

Set $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Recall that a function h is analytic if

$$\overline{\partial} h = 0.$$

We need to solve slightly more general differential equations: $\overline{\partial} F = G$.

Theorem 7

Suppose that G is a smooth compactly supported function in $\mathbb{D}$. Then

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} G(\xi)\frac{1}{z-\xi}\, d\xi \wedge d\overline{\xi} = \frac{1}{\pi}\int_{\mathbb{D}} \frac{G(\xi)}{z-\xi}\, dA(\xi)$$

solves

$$\overline{\partial} F = G.$$

Natural question to ask: Can we solve $\overline{\partial} F = G$ but with some estimates?

Differential Forms and Stokes' Theorem

Recall that Stokes Theorem can (roughly) be stated as

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega.$$

Here Ω is a nice domain, $\partial\Omega$ is the boundary of Ω, ω is a differential form, and d is the exterior differential.

In two variables

$$df = \partial_x dx + \partial_y dy$$

Recall also that $dx \wedge dy = -dy \wedge dx$ and that $dx \wedge dx = dy \wedge dy = 0$.

Writing this in the variables z and $\overline{z}$ we have

$$df = \overline{\partial} f d\overline{z} + \partial f dz$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y) \quad \partial = \frac{1}{2}(\partial_x - i\partial_y)$ and

$$dz = dx + idy \quad d\overline{z} = dx - idy \qquad dz \wedge d\overline{z} = -2idx \wedge dy.$$

Proof of Theorem 7

Fix $\epsilon > 0$ and $z \in \mathbb{D}$ let $\mathbb{D}_\epsilon(z) = \{\xi \in \mathbb{D} : |z - \xi| \geq \epsilon\}$. Note that $\partial\mathbb{D}_\epsilon = \mathbb{T} \cup \{\xi : |\xi - z| = \epsilon\}$. Suppose that φ is a smooth compactly supported function in $\mathbb{D}$. Then, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathbb{D}_\epsilon} \overline{\partial}\varphi(\xi)\frac{1}{\xi - z}d\xi\wedge d\overline{\xi} &= -\frac{1}{2\pi i}\int_{\mathbb{D}_\epsilon}\overline{\partial}\left(\frac{\varphi(\xi)}{\xi - z}\right)d\overline{\xi}\wedge d\xi\\
&= \frac{1}{2\pi i}\int_{|\xi - z| = \epsilon}\frac{\varphi(\xi)}{\xi - z}d\xi + \int_{\mathbb{T}}\frac{\varphi(\xi)}{\xi - z}d\xi\\
&= \frac{1}{2\pi i}\int_{|\xi - z| = \epsilon}\frac{\varphi(\xi)}{\xi - z}d\xi.
\end{aligned}$$

Here we have used the fact that the support of $\varphi \subset \mathbb{D}$ to conclude that the last integral is 0. We have also used the following computation

$$d\left(\frac{\varphi(\xi)}{\xi - z}\right)d\xi = \overline{\partial}\left(\frac{\varphi(\xi)}{\xi - z}\right)d\overline{\xi}\wedge d\xi + \partial\left(\frac{\varphi(\xi)}{\xi - z}\right)d\xi\wedge d\xi.$$

Proof of Theorem 7, Continued

Now note that as $\epsilon \to 0$ we have that

$$\frac{1}{2\pi i}\int_{|\xi - z|=\epsilon}\frac{\varphi(\xi)}{\xi - z}d\xi \to \varphi(z).$$

This says that

$$\frac{1}{2\pi i}\int_{\mathbb{D}}\overline{\partial}\varphi(\xi)\frac{1}{\xi - z}d\xi\wedge d\overline{\xi} = \varphi(z).$$

So, if we have a solution to the problem $\overline{\partial}F = G$ then one solution should be given by

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{D}}G(\xi)\frac{1}{\xi - z}d\xi\wedge d\overline{\xi}.$$

Note that this solution is continuous in the complex plane and smooth in the disc since it is the convolution of a continuous function and a bounded function. We now show that we do indeed have $\overline{\partial}F = G$.

Proof of Theorem 7, Continued

First, note that

$$\begin{aligned}\int_{\mathbb{D}} F\overline{\partial}\varphi dz\wedge d\overline{z}+\int_{\mathbb{D}}\overline{\partial}F\varphi dz\wedge d\overline{z} &= \int_{\mathbb{D}}\overline{\partial}(F\varphi)dz\wedge d\overline{z}\\ &= \int_{\mathbb{T}} F\varphi dz=0.\end{aligned}$$

Here again we have used the support of φ and the analogous computations from above. This then implies

$$\int_{\mathbb{D}} F\overline{\partial}\varphi dz\wedge d\overline{z}=-\int_{\mathbb{D}}\overline{\partial}F\varphi dz\wedge d\overline{z}.$$

Proof of Theorem 7, Continued

Using these computations we see that

$$\begin{aligned}
\int_{\mathbb{D}} \overline{\partial} F \varphi dz \wedge d\overline{z} &= -\int_{\mathbb{D}} F \overline{\partial} \varphi dz \wedge d\overline{z} \\
&= -\int_{\mathbb{D}} \left(\frac{1}{2\pi i} \int_{\mathbb{D}} G(\xi) \frac{1}{\xi - z} d\xi \wedge d\overline{\xi} \right) \overline{\partial} \varphi dz \wedge d\overline{z} \\
&= -\int_{\mathbb{D}} G \left(\frac{1}{2\pi i} \int_{\mathbb{D}} \overline{\partial} \varphi \frac{1}{\xi - z} dz \wedge d\overline{z} \right) d\xi \wedge d\overline{\xi} \\
&= \int_{\mathbb{D}} G \varphi d\xi \wedge d\overline{\xi}.
\end{aligned}$$

Since this is true for all smooth compactly supported φ in $\mathbb{D}$ we have that

$$\overline{\partial} F = G$$

as claimed. □

The above Theorem demonstrates that it is possible to solve equations for the form $\overline{\partial} F = G$.

However, we will want to solve the equation with some norm control, in particular we want to solve the equation and obtain estimates on $\|F\|_\infty$ in terms of information from G.

To accomplish this, we will assume the the function G "generates" Carleson measures for $H^2(\mathbb{D})$. We now prove a result of Wolff that gives the desired estimates.

$\overline{\partial}$-problems and Carleson measures

Theorem 8 (Wolff)

Suppose that $G(z)$ is bounded and smooth on the disc $\mathbb{D}$. Further, assume that the measures

$$\left|G(z)\right|^2 \log\frac{1}{|z|}dA(z) \text{ and } \left|\partial G(z)\right| \log\frac{1}{|z|}dA(z) \in CM(H^2(\mathbb{D}))$$

Then there exists a continuous function $b(z)$ on $\overline{\mathbb{D}}$, smooth on $\mathbb{D}$ such that

$$\overline{\partial} b = G$$

and there exists a constant such that

$$\|b\|_{L^\infty(\mathbb{T})} \lesssim \left\|\left|G(z)\right|^2 \log\frac{1}{|z|}dA(z)\right\|_{CM(H^2)} + \left\|\left|\partial G(z)\right| \log\frac{1}{|z|}dA(z)\right\|_{CM(H^2)}^2.$$

Proof of Theorem 8

By Theorem 7 above, we clearly have one solution to the problem

$$\overline{\partial} b = G.$$

We can obtain lots of solutions by adding functions that are in the kernel of the operator $\overline{\partial}$ and any function h in the disc algebra $A(\mathbb{D})$ allows us to have that $b+h$ also satisfies that $\overline{\partial}(b+h) = G$. The goal is to select a good choice of the function h that allows us to obtain the estimates we seek. A duality argument show that if we take $\|k_j\|_{H^2} \leq 1$ then

$$\inf\left\{\|b\|_\infty : \overline{\partial} b = G\right\} = \sup\left\{\left|\int_{\mathbb{T}} F\overline{k_1 k_2} dm\right| : k_1 \in \overline{H^2(\mathbb{D})}, k_2 \in \overline{H_0^2(\mathbb{D})}\right\}.$$

Here we have that the function F is defined as in the Theorem 7. Since we are supposing that G is bounded and smooth, we have that F is smooth on the $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. A density argument lets us further assume that the functions k_1 and k_2 are smooth across the boundary of $\mathbb{D}$ (just consider dilates of the functions $f_r(z) = f(rz)$ and apply a normal family argument).

Proof of Theorem 8

Now, we apply Green's Theorem to the function $F\overline{k_1 k_2}$. First, we compute the Laplacian of the function $F\overline{k_1 k_2}$ since it will appear in Green's Theorem.

$$\begin{aligned} \Delta(F\overline{k_1 k_2}) &= 4\partial\left(\overline{\partial} F\overline{k_1 k_2} + F\overline{\partial}(\overline{k_1 k_2})\right) \\ &= 4\left(\partial G\overline{k_1 k_2} + G\left(\overline{k_1' k_2 + k_1 k_2'}\right)\right). \end{aligned}$$

Here we have used that $\overline{\partial} F = G$ and that $k_1 k_2$ is anti-holomorphic. Substituting in this information we find:

$$\begin{aligned} \int_{\mathbb{T}} F\overline{k_1 k_2}\,dm &= F(0)\overline{k_1(0)k_2(0)} + \int_{\mathbb{D}} \Delta(F\overline{k_1 k_2})\log\frac{1}{|z|}dA \\ &= 2\int_{\mathbb{D}} \partial G\overline{k_1 k_2}\log\frac{1}{|z|}dA(z) \\ &+ 2\int_{\mathbb{D}} G\left(\overline{k_1' k_2 + k_1 k_2'}\right)\log\frac{1}{|z|}dA(z) \\ &= I + II. \end{aligned}$$

Proof of Theorem 8

We estimate each of these integrals separately. First, consider the integral corresponding to I. Making obvious estimates, we have

$$\begin{aligned}
\left|\int_{\mathbb{D}} \partial G \overline{k_1 k_2} \log \frac{1}{|z|} dA(z)\right| &\leq \int_{\mathbb{D}} |k_1|\,|k_2|\,|\partial G| \log \frac{1}{|z|} dA(z) \\
&\leq \left(\int_{\mathbb{D}} |k_1|^2 \,|\partial G| \log \frac{1}{|z|} dA(z)\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{D}} |k_2|^2 \,|\partial G| \log \frac{1}{|z|} dA(z)\right)^{\frac{1}{2}} \\
&\leq \left\| |\partial G(z)| \log \frac{1}{|z|} dA(z) \right\|_{CM(H^2)}^2 \|k_1\|_{H^2} \|k_2\|_{H^2} .
\end{aligned}$$

Proof of Theorem 8

Turning to term II, it suffices to handle the term $k_1'k_2$ since the other will follow by symmetry.

$$\begin{aligned}\left|\int_{\mathbb{D}} G\overline{k_1'k_2}\log\frac{1}{|z|}dA(z)\right| &\leq \int_{\mathbb{D}}\left|G\overline{k_1'k_2}\right|\log\frac{1}{|z|}dA(z)\\ &\leq \left(\int_{\mathbb{D}}|k_1'|^2\log\frac{1}{|z|}dA(z)\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{\mathbb{D}}|k_2|^2|G|^2\log\frac{1}{|z|}dA(z)\right)^{\frac{1}{2}}\\ &\leq \left\||G(z)|^2\log\frac{1}{|z|}\right\|_{CM(H^2)}\|k_1\|_{H^2}\|k_2\|_{H^2}\end{aligned}$$

Thus, we see that we have the following estimate for term II

$$\int_{\mathbb{D}} G\left(\overline{k_1'k_2+k_1k_2'}\right)\log\frac{1}{|z|}dA(z)\lesssim\left\||G(z)|^2\log\frac{1}{|z|}\right\|_{CM(H^2)}\|k_1\|_{H^2}\|k_2\|_{H^2}.$$

Proof of Theorem 8

Putting the estimates for terms I and II together gives us that

$$\left|\int_{\mathbb{T}} F\overline{k_1 k_2}dm\right| \lesssim \left\| |G(z)|^2 \log\frac{1}{|z|}\right\|_{CM(H^2(\mathbb{D}))} \|k_1\|_{H^2(\mathbb{D})} \|k_2\|_{H^2(\mathbb{D})} + \left\| |\partial G(z)| \log\frac{1}{|z|} dA(z)\right\|_{CM(H^2(\mathbb{D}))}^2 \|k_1\|_{H^2(\mathbb{D})} \|k_2\|_{H^2(\mathbb{D})}.$$

This last estimate then proves that

$$\sup\left\{\left|\int_{\mathbb{T}} F\overline{k_1 k_2}dm\right| : k_1 \in \overline{H^2(\mathbb{D})}, k_2 \in \overline{H^2_0(\mathbb{D})}\right\} \lesssim \left\| |G(z)|^2 \log\tfrac{1}{|z|}\right\|_{CM(H^2(\mathbb{D}))} + \left\| |\partial G(z)| \log\tfrac{1}{|z|} dA(z)\right\|_{CM(H^2(\mathbb{D}))}^2.$$

□

Wolff's Proof of the Corona Problem

Theorem 9 (Carleson)

Suppose that $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ *and there exists a* $\delta > 0$ *such that*

$$1 \geq \max_{1 \leq j \leq n} \{|f_j(z)|\} \geq \delta > 0.$$

Then there exists $g_1, \ldots, g_n \in H^\infty(\mathbb{D})$ *such that*

$$1 = f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) \quad \forall z \in \mathbb{D}$$

and

$$\|g_j\|_{H^\infty(\mathbb{D})} \leq C_{\delta,n} \quad \forall j = 1, \ldots, n.$$

We will now use Wolff's $\overline{\partial}$-problem with estimates to prove this Theorem.

Proof of Theorem 9

Without loss of generality, we may assume that the functions are analytic in a neighborhood of the closed disc $\overline{\mathbb{D}}$ (Reduce to a normal families argument). Define the functions

$$\varphi_j(z) = \frac{\overline{f_j(z)}}{\sum_{j=1}^n |f_j(z)|^2} \quad \forall z \in \mathbb{D}.$$

The hypotheses on the functions f_j clearly give that $|\varphi_j(z)| \leq C_{n,\delta}$. These functions are in general not analytic, and so we must correct them. We set

$$g_j(z) = \varphi_j(z) + \sum_{k=1}^n a_{j,k}(z) f_k(z)$$

where the functions $a_{j,k}(z)$ are to be determined, but we will require that $a_{j,k}(z) = -a_{k,j}(z)$. Note this alternating condition implies that

$$\sum_{j=1}^n f_j(z) g_j(z) = \sum_{j=1}^n f_j(z)\varphi_j(z) + \sum_{j=1}^n \sum_{k=1}^n a_{j,k}(z) f_j(z) f_k(z) = 1.$$

Proof of Theorem 9, Continued

To have the alternating characteristic of $a_{j,k}$ we set

$$a_{j,k}(z) = b_{j,k}(z) - b_{k,j}(z)$$

for some yet to be determined functions. We will chose the functions $b_{j,k}$ to be solutions to the follow $\overline{\partial}$ problem:

$$\overline{\partial} b_{j,k} = \varphi_j \overline{\partial} \varphi_k := G_{j,k}.$$

Using this, we see that

$$\begin{aligned} \overline{\partial} g_j &= \overline{\partial}\varphi_j + \sum_{k=1}^n f_k \overline{\partial} a_{j,k} \\ &= \overline{\partial}\varphi_j + \sum_{k=1}^n f_k \left(\overline{\partial} b_{j,k} - \overline{\partial} b_{k,j} \right) \\ &= \overline{\partial}\varphi_j + \sum_{k=1}^n f_k \left(\varphi_j \overline{\partial} \varphi_k - \varphi_k \overline{\partial} \varphi_j \right) \end{aligned}$$

Proof of Theorem 9, Continued

$$
\begin{aligned}
&= \overline{\partial}\varphi_j + \varphi_j \overline{\partial}\left(\sum_{k=1}^{n} f_k \varphi_k\right) - \overline{\partial}\varphi_j \sum_{k=1}^{n} f_k \varphi_k \\
&= \overline{\partial}\varphi_j + \varphi_j \overline{\partial} 1 - \overline{\partial}\varphi_j 1 = 0.
\end{aligned}
$$

So the functions g_j are analytic. To prove that $|g_j(z)| \leq C_{n,\delta}$ are bounded, it suffices to prove the functions $b_{j,k}$ are bounded by $C_{n,\delta}$. With this in mind, and having the result of Wolff at our disposal, we must show that the measures

$$
|G_{j,k}(z)|^2 \log \frac{1}{|z|} dA(z) \text{ and } |\partial G_{j,k}(z)| \log \frac{1}{|z|} dA(z)
$$

are $H^2(\mathbb{D})$-Carleson measures.

Proof of Theorem 9, Continued

Claim 10

Let $f_j \in H^\infty(\mathbb{D})$ and $G_{j,k}$ be defined as above. Then the measures

$$\left|G_{j,k}(z)\right|^2 \log\frac{1}{|z|}dA(z) \text{ and } \left|\partial G_{j,k}(z)\right| \log\frac{1}{|z|}dA(z)$$

are dominated by (up to a constant $C_{n,\delta}$) by the following measure

$$\sum_{j=1}^{n}\left|f_j'(z)\right|^2 \log\frac{1}{|z|}dA(z).$$

Exercise 11

Show that for $f \in H^\infty(\mathbb{D})$ that we have $\left|f'(z)\right|^2 \log\frac{1}{|z|}dA(z)$ is a $H^2(\mathbb{D})$-Carleson measure. Hint: Use the alternate norm for $H^2(\mathbb{D})$ and think about the product rule for derivatives.

Proof of Theorem 9, Continued

Consider the expression $\left|G_{j,k}\right|^2$. Note that by the hypotheses on f_j we have that $|\varphi_j| \leq C_{n,\delta}$ and so,

$$\left|G_{j,k}\right|^2 \leq C_{n,\delta}\left|\overline{\partial}\varphi_k\right|^2.$$

Now, if we compute we see that

$$\begin{aligned}\overline{\partial}\varphi_k &= \frac{\overline{f_k'}}{\sum_{j=1}^n |f_j|^2} - \frac{\overline{f_k}\sum_{j=1}^n f_j\overline{f_j'}}{\left(\sum_{j=1}^n |f_j|^2\right)^2} \\ &= \frac{\sum_{j=1}^n f_j(\overline{f_j}\,\overline{f_k'} - \overline{f_k}\,\overline{f_j'})}{\left(\sum_{j=1}^n |f_j|^2\right)^2}.\end{aligned}$$

Using this, we see that

$$\left|G_{j,k}\right|^2 \leq C_{n,\delta}\left|\overline{\partial}\varphi_k\right|^2 \leq C_{n,\delta}\frac{\sum_{j=1}^n |f_j|^2\sum_{j=1}^n \left|f_j'\right|^2}{\left(\sum_{j=1}^n |f_j|^2\right)^2} \leq C_{n,\delta}\sum_{j=1}^n\left|f_j'\right|^2.$$

Proof of Theorem 9, Continued

Next consider $|\partial G_{j,k}|$. First, observe that

$$\partial G_{j,k} = \partial\varphi_j \overline{\partial}\varphi_k + \varphi_j \partial\overline{\partial}\varphi_k$$

By the computations above, we have that

$$\overline{\partial}\varphi_k = \frac{\sum_{j=1}^{n} f_j(\overline{f_j}\overline{f_k'} - \overline{f_k}\overline{f_j'})}{\left(\sum_{j=1}^{n} |f_j|^2\right)^2}.$$

Direct computation also gives,

$$\partial\varphi_j = -\frac{\overline{f_j}\sum_{l=1}^{n} f_l'\overline{f_l}}{\left(\sum_{j=1}^{n} |f_j|^2\right)^2}.$$

Proof of Theorem 9, Continued

Finally, we have that

$$\partial\overline{\partial}\varphi_k = \frac{\sum_{j=1}^n f_j'(\overline{f_j}\overline{f_k'} - \overline{f_k}\overline{f_j'})}{\left(\sum_{j=1}^n |f_j|^2\right)^2} - 2\frac{\left(\sum_{l=1}^n f_l'\overline{f_l}\right)\left(\sum_{l=1}^n \left(\overline{f_l}\overline{f_k'} - \overline{f_k}\overline{f_l'}\right)\right)}{\left(\sum_{j=1}^n |f_j|^2\right)^3}.$$

Now consider the term $\left|\partial\varphi_j\overline{\partial}\varphi_k\right|$. It is obvious that we can dominate this expression by

$$C_{n,\delta}\sum_{j,k}\left|f_j'\right|\left|f_k'\right| \leq C_{n,\delta}\sum_{k=1}^n\left|f_k'\right|^2.$$

Similarly, we have that $\left|\varphi_j\partial\overline{\partial}\varphi_k\right|$ can be dominated by an identical expression. This then proves the claim. An application of Wolff's Theorem, Theorem 8, then gives the bounded solution we seek. □

Here is an exercise to see if you really understand the proof of the Corona Theorem just given

Exercise 12

Suppose that $f_1, \ldots, f_n, g \in H^\infty(\mathbb{D})$ *such that*

$$|g(z)| \leq \sum_{j=1}^{n} |f_j(z)| \, .$$

Show that there exists $g_1, \ldots, g_n \in H^\infty(\mathbb{D})$ *such that*

$$g^3 = \sum_{j=1}^{n} f_j g_j .$$

Hint: Mimic Wolff's proof of the Corona Theorem but start with $\psi_j = g\varphi_j$, *with* φ_j *as defined above.*

Suggested Method for Proving Corona Problems

Step 1 (Find "trivial" Solutions): *Use the condition* $0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1$ *to construct smooth and bounded solutions to the problem.*

Step 2 (Use Algebra): *Write any solution to the problem as*

$$g_j = \varphi_j + \sum_{k=1}^{N} a_{k,j} f_k$$

where the φ_j *are the smooth solutions from Step 1 and the* $a_{k,j}$ *are some yet to be determined functions.*

Step 3 (Find Bounds on the Solutions): $\overline{\partial}$*-equation, Nehari's Theorem*

Constructive Proofs of $\overline{\partial}$-problems

We now give a more constructive method to solve the $\overline{\partial}$ problem. We have seen from above, that to solve the equation $\overline{\partial} F = G$ we can set

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} G(\xi)\frac{1}{z-\xi} d\xi \wedge d\overline{\xi}.$$

However, we can also use another kernel to accomplish this. Choose a function $K(z,\xi)$ that is analytic in z, $K(z,z) = 1$ and that is smooth, then:

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} K(z,\xi) G(\xi)\frac{1}{z-\xi} d\xi \wedge d\overline{\xi}.$$

solves $\overline{\partial} F = G$.

Exercise 13

Show that this is true. Hint: $K(z,\xi) = 1 + (z-\xi)\tilde{K}(z,\xi)$.

Constructive Proofs of $\overline{\partial}$-problems

Theorem 14 (Jones)

Let $\mu \in CM(H^2)$ *and* $\sigma = \frac{|\mu|}{\|\mu\|_{CM(H^2)}}$. *Set* $S(\mu)(z) = \int_{\mathbb{D}} K(\sigma, z, \zeta)\, d\mu(\zeta)$ *with*

$$K(\sigma, z, \zeta) \equiv \frac{2i}{\pi} \frac{1 - |\zeta|^2}{(z - \zeta)\left(1 - \overline{\zeta} z\right)} \times \exp\left\{ \int_{|\omega| \geq |\zeta|} \left(-\frac{1 + \overline{\omega} z}{1 - \overline{\omega} z} + \frac{1 + \overline{\omega}\zeta}{1 - \overline{\omega}\zeta} \right) d\sigma(\omega) \right\},$$

we have that:

1. $S(\mu) \in L^1_{loc}(\mathbb{D})$, $\overline{\partial} S(\mu) = \mu$ *in the sense of distributions;*
2. $\int_{\mathbb{D}} |K(\sigma, x, \zeta)|\, d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)}$ *for all* $x \in \mathbb{T} = \partial\mathbb{D}$, *so* $\|S(\mu)\|_{L^\infty(\mathbb{T})} \lesssim \|\mu\|_{CM(H^2)}$.

Constructive Proofs of $\overline{\partial}$-problems

Note that the kernel $K(\sigma, z, \xi)$ is analytic in z,

$$K(\sigma, z, \xi) = \frac{2i}{\pi} \frac{1}{z - \xi} \tilde{K}(\sigma, z, \xi).$$

The kernel $\tilde{K}(\sigma, z, \xi)$ is smooth with

$$\tilde{K}(\sigma, z, z) = \frac{1 - |z|^2}{1 - |z|^2} \exp\left\{ \int_{|\omega| \geq |z|} \left(-\frac{1 + \overline{\omega} z}{1 - \overline{\omega} z} + \frac{1 + \overline{\omega} z}{1 - \overline{\omega} z} \right) d\sigma(\omega) \right\} = 1.$$

So $\overline{\partial} S(\mu) = \mu$ follows from the exercise above.

Remains to show that we have the desired estimates.

Proof of Theorem 14

Proof: Observe that if we prove

$$\int_{\mathbb{D}} |K(\sigma, x, \zeta)| \, d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)} \quad \forall x \in \mathbb{T} = \partial\mathbb{D}$$

then

$$|S(\mu)(z)| \leq \int_{\mathbb{D}} |K(\sigma, z, \zeta)| \, d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)} .$$

Note that for the measure σ we have

$$\begin{aligned} \operatorname{Re}\left(\int_{|w|\geq|\zeta|}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}\right)d\sigma(w)\right) &= \int_{|w|\geq|\zeta|}\operatorname{Re}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}\right)d\sigma(w) \\ &\leq 2\int_{\mathbb{D}}\frac{1-|\zeta|^2}{|1-\overline{w}\zeta|^2}d\sigma(w) \\ &\leq 2\left\|\tilde{k}_\zeta\right\|^2_{H^2(\mathbb{D})} = 2, \end{aligned}$$

where $\tilde{k}_\zeta(z) = \frac{(1-|\zeta|^2)^{\frac{1}{2}}}{1-\overline{\zeta}z}$ is the normalized reproducing kernel for $H^2(\mathbb{D})$.

Proof of Theorem 14, Continued

It will suffice to control the boundary values of the function $S(\mu)$, and so we can take $z \in \mathbb{T}$.

Using the estimate from above and that $z \in \mathbb{T}$ we have

$$\begin{aligned}
|K(\sigma, z, \xi)| &= \frac{2}{\pi} \frac{1-|\xi|^2}{\left|1-\bar{\xi} z\right|^2} \left| \exp\left\{ \int_{|\omega| \geq |\xi|} \left(-\frac{1+\overline{\omega} z}{1-\overline{\omega} z} + \frac{1+\overline{\omega}\xi}{1-\overline{\omega}\xi} \right) d\sigma(\omega) \right\} \right| \\
&= \frac{2}{\pi} \frac{1-|\xi|^2}{\left|1-\bar{\xi} z\right|^2} \exp\left\{ \operatorname{Re} \int_{|\omega| \geq |\xi|} -\frac{1+\overline{\omega} z}{1-\overline{\omega} z} d\sigma(\omega) \right\} \\
&\quad \times \exp\left\{ \operatorname{Re} \int_{|\omega| \geq |\xi|} \frac{1+\overline{\omega} z}{1-\overline{\omega} z} d\sigma(\omega) \right\} \\
&\leq \frac{2}{\pi} e^2 \frac{1-|\xi|^2}{\left|1-\bar{\xi} z\right|^2} \exp\left\{ -\int_{|\omega| \geq |\xi|} \frac{1-|\omega|^2}{|1-\overline{\omega} z|^2} d\sigma(\omega) \right\}.
\end{aligned}$$

Proof of Theorem 14, Continued

We can then use this estimate on the kernel $K(\sigma, z, \zeta)$ to give

$$\begin{aligned}
|S(\mu)(z)| &= \left| \int_{\mathbb{D}} K(\sigma, z, \zeta)\, d\mu(\zeta) \right| \\
&\leq \|\mu\|_{CM(H^2)} \int_{\mathbb{D}} |K(\sigma, z, \zeta)|\, d\sigma(\zeta) \\
&\leq e^2 \|\mu\|_{CM(H^2)} \frac{2}{\pi} \int_{\mathbb{D}} \frac{1-|\zeta|^2}{\left|1-\overline{\zeta} z\right|^2} \\
&\quad \times \exp\left\{ -\int_{|\omega| \geq |\zeta|} \frac{1-|\omega|^2}{|1-\overline{\omega} z|^2} d\sigma(\omega) \right\} d\sigma(\zeta).
\end{aligned}$$

It remains to show that

$$\int_{\mathbb{D}} \frac{1-|\zeta|^2}{\left|1-\overline{\zeta} z\right|^2} \exp\left\{ -\int_{|\omega| \geq |\zeta|} \frac{1-|\omega|^2}{|1-\overline{\omega} z|^2} d\sigma(\omega) \right\} d\sigma(\zeta) \leq 1.$$

Proof of Theorem 14, Continued

First, suppose that we have $d\sigma = \sum_{j=1}^{N} a_j \delta_{\zeta_j}$ with $|\zeta_j| \leq |\zeta_{j+1}|$. Further, set $\beta_j = a_j \frac{1-|\zeta_j|^2}{|1-\overline{\zeta_j} z|^2}$ and

$$t_j = \sum_{k=j}^{N} \beta_k, \text{ and so } \beta_j = t_j - t_{j-1}.$$

If we evaluate the above integral for this measure and use the resulting notation, we see that the integral becomes

$$\sum_{j=1}^{N} (t_j - t_{j-1}) e^{-t_j} \leq \int_0^\infty e^{-t} dt = 1.$$

A standard measure theory argument then finishes the proof that

$$\int_{\mathbb{D}} \frac{1-|\zeta|^2}{\left|1-\overline{\zeta} z\right|^2} \exp\left\{ -\int_{|\omega|\geq|\zeta|} \frac{1-|\omega|^2}{|1-\overline{\omega} z|^2} d\sigma(\omega) \right\} d\sigma(\zeta) \leq 1. \qquad \square$$

Corona Problem for $M_{\mathcal{D}}$

Refreshers about the Dirichlet Space

The Dirichlet space $\mathcal{D}$ is the Hilbert space of functions $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \, dxdy < \infty.$$

The multiplier algebra of a space $\mathcal{D}$ is the collection of functions such that

$$\mathcal{M}_{\mathcal{D}} := \{\varphi \in \mathrm{Hol}(\mathbb{D}) : \|\varphi f\|_{\mathcal{D}} \leq C \|f\|_{\mathcal{D}} \quad \forall f \in \mathcal{D}\}$$

Which is normed by $\|\varphi\|_{\mathcal{M}_{\mathcal{D}}} := \inf\{C : \|\varphi f\|_{\mathcal{D}} \leq C \|f\|_{\mathcal{D}} \quad \forall f \in \mathcal{D}\}$.

Proposition 15

$$\mathcal{M}_{\mathcal{D}} \subsetneq H^{\infty}(\mathbb{D}) \text{ with } \|\varphi\|_{H^{\infty}(\mathbb{D})} \leq \|\varphi\|_{\mathcal{M}_{\mathcal{D}}}.$$

While the observation that $\mathcal{M}_{\mathcal{D}} \subsetneq H^{\infty}(\mathbb{D})$ is useful, it would be better to have a full characterization of this algebra in terms of objects that we already have seen.

Refreshers about the Dirichlet Space

Define the space $\mathcal{X}$ to be the collection of functions such that $|b'|^2\, dA(z)$ is a Carleson measure for the Dirichlet space $\mathcal{D}$. Namely, a function $b \in \mathcal{X}$ if and only if

$$\int_{\mathbb{D}} |f(z)|^2 \left|b'(z)\right|^2 dA(z) \leq C \left\|f\right\|_{\mathcal{D}}^2 \quad \forall f \in \mathcal{D}.$$

We can norm this space $\mathcal{X}$ by setting $d\mu_b := |b'(z)|^2\, dA(z)$ and

$$\left\|b\right\|_{\mathcal{X}}^2 = |b(0)|^2 + \inf\left\{C : \int_{\mathbb{D}} |f(z)|^2\, d\mu_b(z) \leq C\left\|f\right\|_{\mathcal{D}}^2\right\}.$$

Proposition 16

$$\mathcal{M}_{\mathcal{D}} = H^\infty(\mathbb{D}) \cap \mathcal{X}$$

Moreover,

$$\left\|b\right\|_{\mathcal{M}_{\mathcal{D}}} \approx \left\|b\right\|_\infty + \left\|b\right\|_{\mathcal{X}}$$

$\overline{\partial}$-problem on $M_{\mathcal{D}}$

Since we need a boundary estimate, we will work with another more convenient norm for the Dirichlet space and its multiplier algebra.

$$\|f\|^2_{\mathcal{W}^{\frac{1}{2}}(\mathbb{T})} = \|f\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{f(z)-f(w)}{z-w}\right|^2 dm(z)dm(w)$$

We then have that $\varphi \in M_{\mathcal{D}}$ if and only if $f \in H^\infty(\mathbb{D})$ and

$$\int_E\int_{\mathbb{T}} \left|\frac{f(z)-f(w)}{z-w}\right|^2 dm(z)dm(w) \leq C\,\mathrm{Cap}(E) \quad E \subset \mathbb{T}.$$

Theorem 17 (Xiao)

If $|g(z)|^2\, dA(z) \in CM(\mathcal{D})$ then there is a function f such that $\overline{\partial} f = g$ and

$$\|f\|_{\mathcal{M}_{\mathcal{W}^{\frac{1}{2}}(\mathbb{T})}} \lesssim \left\| |g(z)|^2\, dA(z)\right\|_{CM(\mathcal{D})}.$$

Sketch of Proof of Theorem 17

- First note that $|g(z)|\, dA(z) \in CM(H^2(\mathbb{D}))$ (straightforward computation);
- Apply Jones' Theorem, Theorem 14 to see that $\overline{\partial} f = g$ with $\|g\|_{L^\infty(\mathbb{T})} \lesssim 1$;
- It suffices to show that for any collection of intervals $\{I_k\} \subset \mathbb{T}$ that we have
$$\int_{\cup_k T(I_k)} |\nabla f(z)|^2\, dA(z) \leq C\, \textrm{Cap}\,(\cup_k I_k)\,.$$
- Simple computations then give that
$$|\nabla f(z)| \lesssim \int_{\mathbb{D}} \frac{|g(w)|}{|1-\overline{w}z|^2} dA(w).$$

Sketch of Proof of Theorem 17, Continued

Lemma 18 (Rochberg & Wu)

Let $\alpha \in (-\infty, \frac{1}{2}]$ *and* $\beta > \max\{-1, -1-2\alpha\}$ *and* $b > \max\{\frac{\beta+3}{2}, \frac{\beta+3}{2} - \alpha\}$. *Also, let*

$$Tf(z) = \int_{\mathbb{D}} \frac{f(w)}{|1-\overline{w}z|^{b}}(1-|w|^2)^{b-2}dA(w).$$

If $|f(z)|^2(1-|z|^2)^{\beta}dA(z)$ *is an* $\mathcal{D}_\alpha$*-Carleson measure, then* $|Tf(z)|^2(1-|z|^2)^{\beta}dA(z)$ *is also an* $\mathcal{D}_\alpha$*-Carleson measure.*

This lemma shows that it is possible to produce new Carleson measures from old Carleson measures via "spreading them out". This finishes the proof of Theorem 17.

There is an alternate way to conclude the estimates at this point using singular integral theory (Beurling operator).

The Corona Theorem in $M_{\mathcal{D}}$

Theorem 19 (Tolokonnikov, Xiao, Trent)

Suppose that $f_1, \ldots, f_n \in M_{\mathcal{D}}$ *are such that*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{D}.$$

Then there exists $g_1, \ldots, g_N \in M_{\mathcal{D}}$ *such that*

- $\sum_{j=1}^{N} f_j(z) g_j(z) = 1$ *for all* $z \in \mathbb{D}$*;*
- $\sum_{j=1}^{N} \|g_j\|_{M_{\mathcal{D}}} \leq C_{\delta,N}$.

There are three different proofs of this theorem now. The order above is chronological.

The condition that f_j do not simultaneously vanish is clearly necessary.

Proof of Theorem 19

Proof: For $j = 1, \ldots, N$, define

$$\varphi_j(z) = \frac{\overline{f_j(z)}}{\sum_{j=1}^{N} |f_j(z)|^2}.$$

Then we have that

$$1 = \sum_{j=1}^{N} f_j(z)\varphi_j(z).$$

Suppose that we can find functions b_{jk} such that

(i) $\overline{\partial} b_{jk} = \varphi_j \overline{\partial} \varphi_k$;

(ii) $\|b_{jk}\|_{M_{\mathcal{D}}} \lesssim 1$.

Then we define

$$g_j(z) = \varphi_j(z) + \sum_{k=1}^{N} \left(b_{jk}(z) - b_{kj}(z)\right) f_k(z).$$

Proof of Theorem 19, Continued

These functions are analytic since,

$$\begin{aligned}
\overline{\partial} g_j &= \overline{\partial}\varphi_j + \sum_{k=1}^{N} \left(\overline{\partial} b_{jk} - \overline{\partial} b_{kj}\right) f_k \\
&= \overline{\partial}\varphi_j + \sum_{k=1}^{N} \left(\varphi_j \overline{\partial}\varphi_k - \varphi_k \overline{\partial}\varphi_j\right) f_k \\
&= \overline{\partial}\varphi_j - \left(\sum_{k=1}^{N} \varphi_k f_k\right) \overline{\partial}\varphi_j + \varphi_j \overline{\partial} \left(\sum_{k=1}^{N} f_k \varphi_k\right) \\
&= \overline{\partial}\varphi_j - \overline{\partial}\varphi_j + \varphi_j \overline{\partial} 1 = 0.
\end{aligned}$$

Moreover, because of the estimate on b_{jk} it is clear that $g_j \in M_{\mathcal{D}}$. Finally, with these functions we have

$$\sum_{j=1}^{N} f_j(z) g_j(z) = 1$$

Proof of Theorem 19, Continued

The only issue that remains is to demonstrate that it is in fact possible to find such solutions b_{jk}, and to do so, it suffices to demonstrate that the functions $\varphi_j \overline{\partial} \varphi_k$ are in fact Carleson measures for the Dirichlet space $\mathcal{D}$. Computations from a previous lecture give

$$\varphi_j \overline{\partial} \varphi_k = \frac{\overline{f_j}}{\sum_j |f_j(z)|^2} \frac{\sum_l f_l \left(\overline{f_l f_k' - f_k f_l'} \right)}{\left(\sum_j |f_j(z)|^2 \right)^2}.$$

This then implies that

$$\left| \varphi_j \overline{\partial} \varphi_k \right|^2 \lesssim \frac{1}{\left(\sum_j |f_j(z)|^2 \right)^3} \sum_{l=1}^{N} |f_l'|^2 \lesssim \sum_{l=1}^{N} |f_l'|^2.$$

The last estimate follows since the functions $f_l \in M_{\mathcal{D}}$. An application of Theorem 17 provides the solutions and estimates we seek.

Corona Problem for Multiplier Algebras with Complete Nevanlinna-Pick Kernels

Abstract Setup

- $\Omega \subset \mathbb{C}^n$ open set
- $\mathcal{H}$ an irreducible complete Nevanlinna-Pick reproducing kernel Hilbert function space on Ω:
  - For each $\lambda \in \Omega$ there exists a function $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H};$$

  - The Nevanlinna-Pick problem is solvable for scalars and matrices;
  - The kernel $k_\lambda(\cdot)$ is irreducible if it doesn't vanish, i.e., $k_\lambda(\mu) \neq 0$;
- $M_{\mathcal{H}}$ will denote the multiplier algebra of $\mathcal{H}$, i.e. $\varphi \in M_{\mathcal{H}}$ if $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$ and

$$\|\varphi\|_{M_{\mathcal{H}}} = \|M_\varphi\|_{\mathcal{H} \to \mathcal{H}}$$

We now formulate two different Corona questions for the space $\mathcal{H}$ and $M_{\mathcal{H}}$ that turn out to guide the study of questions in this area.

Baby Corona Problem for $\mathcal{H}$

Question 20 (Baby Corona Problem for $\mathcal{H}$)

Given $f_1, \ldots, f_N \in M_{\mathcal{H}}$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \Omega$$

and $h \in \mathcal{H}$. *Does there exist a constant* $C_{n,N,\delta}$ *and functions* $k_1, \ldots, k_N \in \mathcal{H}$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \|k_j\|_{\mathcal{H}}^2 &\leq C_{n,N,\delta} \|h\|_{\mathcal{H}}^2, \\ \sum_{j=1}^{N} k_j(z) f_j(z) &= h(z) \quad \forall z \in \Omega? \end{aligned}$$

Corona Problem for $M_{\mathcal{H}}$

Question 21 (Corona Problem for Multiplier Algebras $M_{\mathcal{H}}$)

Given $f_1, \ldots, f_N \in M_{\mathcal{H}}$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1 \quad \forall z \in \Omega.$$

Are there functions $g_1, \ldots, g_N \in M_{\mathcal{H}}$ *and a constant* $C_{n,N,\delta}$ *such that:*

$$\begin{aligned} \sum_{j=1}^{N} \|g_j\|_{M_{\mathcal{H}}} &\leq C_{n,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1 \quad \forall z \in \Omega? \end{aligned}$$

Density in Maximal Ideal Space

Recall that we showed the equivalence between the Corona problem for $H^\infty(\mathbb{D})$ and the density of $\mathbb{D}$ in the maximal ideal space.

Repeating the argument given from a previous lecture we have the following theorem for the multiplier algebra $M_{\mathcal{H}}$ of a reproducing kernel Hilbert space $\mathcal{H}$.

Theorem 22 (Density in Maximal Ideal Space)

The set Ω is dense in $\mathfrak{M}_{\mathcal{H}}$ if and only if the following condition holds: If $f_1, \ldots, f_n \in M_{\mathcal{H}}$ and if

$$\max_{1 \leq j \leq n} |f_j(z)| \geq \delta > 0$$

then there exists $g_1, \cdots, g_n \in M_{\mathcal{H}}$ such that

$$f_1 g_1 + \cdots + f_n g_n = 1.$$

Corona implies Baby Corona

Indeed, if the Corona Problem is true and we take $h \in \mathcal{H}$, we can see that the Baby Corona Problem follows. Suppose $f_1, \ldots, f_N \in M_{\mathcal{H}}$, and there exists $g_1, \ldots, g_N \in M_{\mathcal{H}}$ such that

$$\sum_{j=1}^{N} \|g_j\|_{M_{\mathcal{H}}} \leq C_{n,N,\delta} \qquad \sum_{j=1}^{N} g_j(z) f_j(z) = 1 \quad \forall z \in \Omega.$$

Multiplying the second equation by h, we find

$$h(z) = \sum_{j=1}^{N} g_j(z) f_j(z) h(z) = \sum_{j=1}^{N} k_j(z) f_j(z) \quad \forall z \in \Omega.$$

Since $g_1, \ldots, g_N \in M_{\mathcal{H}}$ we then have that $k_j := g_j h \in \mathcal{H}$ with $\|k_j\|_{\mathcal{H}} \leq \|g_j\|_{M_{\mathcal{H}}} \|h\|_{\mathcal{H}}$, so the claimed estimates follow as well.

Baby Corona implies Corona?

It turns out that in certain situations the Baby Corona Problem is equivalent to the Corona Problem.

Recall that the Baby Corona Problem is the following question: Given $f_1, \ldots, f_N \in M_{\mathcal{H}}$ satisfying

$$\left|f_1(z)\right|^2 + \cdots + \left|f_N(z)\right|^2 \geq \delta > 0, \qquad z \in \Omega, \tag{1}$$

and $h \in \mathcal{H}$. Are there functions $k_1, \ldots, k_N \in \mathcal{H}$ such that

$$\begin{aligned} \|k_1\|_{\mathcal{H}}^2 + \cdots + \|k_N\|_{\mathcal{H}}^2 &\leq \frac{1}{\delta}\|h\|_{\mathcal{H}}^2, \qquad (2) \\ k_1(z) f_1(z) + \cdots + k_N(z) f_N(z) &= h(z), \quad \forall z \in \Omega? \end{aligned}$$

We now will rephrase the Baby Corona Problem in operator theory language.

Baby Corona implies Corona?

More succinctly, (2) is equivalent to the operator lower bound

$$\mathcal{M}_f \mathcal{M}_f^* - \delta I_{\mathcal{H}} \geq 0, \tag{3}$$

where $f \equiv (f_1, \ldots, f_N)$, $\mathcal{M}_f : \oplus^N \mathcal{H} \to \mathcal{H}$ by $\mathcal{M}_f g = \sum_{\alpha=1}^N g_j f_j$, and $\mathcal{M}_f^* h = \left(\mathcal{M}_{f_j}^* h\right)_{j=1}^N$.

For $f = (f_j)_{j=1}^N \in \oplus^N \mathcal{H}$ and $h \in \mathcal{H}$, define $\mathbb{M}_f h = (f_j h)_{j=1}^N$ and

$$\|f\|_{Mult(\mathcal{H}, \oplus^N \mathcal{H})} = \|\mathbb{M}_f\|_{\mathcal{H} \to \oplus^N \mathcal{H}} = \sup_{\|h\|_{\mathcal{H}} \leq 1} \|\mathbb{M}_f h\|_{\oplus^N \mathcal{H}}.$$

Note that $\max_{1 \leq j \leq N} \left\|\mathcal{M}_{f_j}\right\|_{M_{\mathcal{H}}} \leq \|f\|_{Mult(\mathcal{H}, \oplus^N \mathcal{H})} \leq \sqrt{\sum_{j=1}^N \left\|\mathcal{M}_{f_j}\right\|_{M_{\mathcal{H}}}^2}$.

Equivalence of (2) and (3)

To see this note that (3) is equivalent to

$$\delta \langle h, h\rangle_{\mathcal{H}} \leq \langle h, \mathcal{M}_f\mathcal{M}_f^* h\rangle_{\mathcal{H}} = \langle \mathcal{M}_f^* h, \mathcal{M}_f^* h\rangle_{\oplus^N\mathcal{H}} . \tag{4}$$

From functional analysis, we obtain that the bounded map $\mathcal{M}_f : \oplus^N\mathcal{H} \to \mathcal{H}$ is onto. If $\mathcal{N} = \ker \mathcal{M}_f$, then $\widehat{\mathcal{M}_f} : \mathcal{N}^\perp \to \mathcal{H}$ is invertible. Now (4) implies that $\widehat{\mathcal{M}_f}^* : \mathcal{H} \to \mathcal{N}^\perp$ is invertible and that $\left\|\left(\widehat{\mathcal{M}_f}^*\right)^{-1}\right\| \leq \frac{1}{\sqrt{\delta}}$. By duality we then have $\left\|\left(\widehat{\mathcal{M}_f}\right)^{-1}\right\| \leq \frac{1}{\sqrt{\delta}}$. Thus given $h \in \mathcal{H}$, there is $k \in \mathcal{N}^\perp$ satisfying $\mathcal{M}_f k = h$ and

$$\|k\|^2_{\oplus^N\mathcal{H}} = \left\|\left(\widehat{\mathcal{M}_f}\right)^{-1} h\right\|^2_{\oplus^N\mathcal{H}} \leq \frac{1}{\delta} \|h\|^2_{\mathcal{H}} ,$$

which is (2).

Equivalence of (2) and (3), Continued

Conversely, using (2) we compute that

$$\begin{aligned}\left\|\mathcal{M}_f^* h\right\|_{\oplus^N \mathcal{H}} &= \sup_{\|p\|_{\oplus^N \mathcal{H}} \leq 1} \left|\left\langle p, \mathcal{M}_f^* h\right\rangle_{\oplus^N \mathcal{H}}\right| = \sup_{\|p\|_{\oplus^N \mathcal{H}} \leq 1} \left|\left\langle \mathcal{M}_f p, h\right\rangle_{\mathcal{H}}\right| \\ &\geq \left|\left\langle \mathcal{M}_f \frac{k}{\|k\|_{\oplus^N \mathcal{H}}}, h\right\rangle_{\mathcal{H}}\right| = \frac{\|h\|_{\mathcal{H}}^2}{\|k\|_{\oplus^N \mathcal{H}}} \geq \sqrt{\delta}\, \|h\|_{\mathcal{H}},\end{aligned}$$

which is (4), and hence (3). Next we note that (1) is necessary for (3) as can be seen by testing (4) on reproducing kernels k_z:

$$\delta \left\langle k_z, k_z\right\rangle_{\mathcal{H}} \leq \left\langle \mathcal{M}_f^* k_z, \mathcal{M}_f^* k_z\right\rangle_{\oplus^N \mathcal{H}} = \left|f(z)\right|^2 \left\langle k_z, k_z\right\rangle_{\mathcal{H}}$$

since $\mathcal{M}_f^* k_z = \left(\overline{f_\alpha(z)} k_z\right)_{\alpha=1}^N$.

Baby Corona implies Corona?

Toeplitz Corona Theorem

Theorem 23 (Toeplitz Corona Theorem, (Agler and McCarthy))

Let $\mathcal{H}$ be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\epsilon > 0$ and let $f_1, \ldots, f_N \in M_{\mathcal{H}}$. Then the following are equivalent:

(i) *There exists $g_1, \ldots, g_N \in M_{\mathcal{H}}$ such that $\sum_{j=1}^N f_j g_j = 1$ and $\|g\|_{Mult(\mathcal{H}, \oplus^N \mathcal{H})} \leq \frac{1}{\epsilon}$;*

(ii) *For any $h \in \mathcal{H}$, there exists $k_1, \ldots, k_N \in \mathcal{H}$ such that $h = \sum_{j=1}^N k_j f_j$ and $\sum_{j=1}^N \|k_j\|_{\mathcal{H}}^2 \leq \frac{1}{\epsilon^2} \|h\|_{\mathcal{H}}^2$.*

Moral: If the Hilbert space has a reproducing kernel with enough structure, then the Corona Problem and the Baby Corona Problem are the same question.

Suggested Method for Proving Corona Problems

Step 1 (Baby Corona Problem): *Show that for functions* $f_1, \ldots, f_N \in M_{\mathcal{H}}$ *such that* $0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1$ *for all* $z \in \Omega$ *and* $h \in \mathcal{H}$ *that it is possible to find* $k_1, \ldots, k_N \in \mathcal{H}$ *such that*

(i) $\sum_{j=1}^{N} f_j(z) k_j(z) = h(z)$ *for all* $z \in \Omega$;

(ii) $\sum_{j=1}^{N} \|k_j\|_{\mathcal{H}}^2 \leq C_{\delta,N,n} \|h\|_{\mathcal{H}}^2$.

Step 2 (Toeplitz Corona Problem): *Suppose that* $f_1, \ldots, f_N \in M_{\mathcal{H}}$ *are such that* $\epsilon^2 I \leq M_f M_f^* \leq I$. *Then find* $g_1, \ldots, g_N \in M_{\mathcal{H}}$ *such that*

(i) $\sum_{j=1}^{N} f_j(z) g_j(z) = 1$ *for all* $z \in \Omega$;

(ii) $\|g\|_{Mult(\mathcal{H}, \oplus^N \mathcal{H})} \leq \frac{1}{\epsilon^2}$.

Step 1 is generally "easy". Step 2 is generally hard and requires some magic.

Proof of Theorem 23

(i) implies (ii)

Multiply $\mathcal{M}_f g = 1$ by h to get $\mathcal{M}_f gh = h$. Set $k = gh = (g_1 h, \ldots, g_N h)$ and then

$$\begin{aligned} \|k\|^2_{\oplus^N \mathcal{H}} &= \|gh\|^2_{\oplus^N \mathcal{H}} \\ &= \|g_1 h\|^2_{\mathcal{H}} + \cdots + \|g_N h\|^2_{\mathcal{H}} \\ &= \|\mathbb{M}_g h\|_{\oplus^N \mathcal{H}} \\ &\leq \|g\|^2_{Mult(\mathcal{H}, \oplus^N \mathcal{H})} \|h\|^2_{\mathcal{H}} \\ &\leq \frac{1}{\epsilon^2} \|h\|^2_{\mathcal{H}}. \end{aligned}$$

The other direction is a little more difficult.

Proof of Theorem 23

(ii) implies (i): Preliminaries

An important point to notice is that we can transform operator bounds into bounds on the kernel functions. For example, $\mathcal{M}_f \mathcal{M}_f^* - \epsilon I_{\mathcal{H}} \geq 0$ for $\mathcal{H}$ in (3), can be recast in terms of kernel functions, namely

$$\left\{ \langle f(\zeta), f(\lambda) \rangle_{\mathbb{C}^N} - \epsilon \right\} k(\zeta, \lambda) \geq 0. \tag{5}$$

Indeed, if we let $h = \sum_{i=1}^J \xi_i k_{x_i}$ in (4) we obtain

$$\begin{aligned} \epsilon \sum_{i,j=1}^J \xi_i \overline{\xi_j} k(x_j, x_i) &= \epsilon \sum_{i,j=1}^J \xi_i \overline{\xi_j} \left\langle k_{x_i}, k_{x_j} \right\rangle_{\mathcal{H}} \\ &\leq \sum_{\alpha=1}^N \left\langle \sum_{i=1}^J \xi_i \overline{f_\alpha(x_i)} k_{x_i}, \sum_{j=1}^J \xi_j \overline{f_\alpha(x_j)} k_{x_j} \right\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^J \xi_i \overline{\xi_j} \left\{ \sum_{\alpha=1}^N \overline{f_\alpha(x_i)} f_\alpha(x_j) \right\} k(x_j, x_i). \end{aligned}$$

Proof of Theorem 23

(ii) implies (i): Preliminaries

A similar calculation shows that the operator upper bound

$$I_{\mathcal{H}} - \mathcal{M}_f \mathcal{M}_f^* \geq 0$$

which is equivalent to $\|\mathcal{M}_f\|_{\oplus^N \mathcal{H} \to \mathcal{H}} \leq 1$, can be recast in terms of kernel functions as

$$\left\{1 - \langle f(\zeta), f(\lambda)\rangle_{\mathbb{C}^N}\right\} k(\zeta, \lambda) \geq 0. \tag{6}$$

We need to understand $\|f\|_{Multi(\mathcal{H}, \oplus^N \mathcal{H})} = \|\mathbb{M}_f\|_{\mathcal{H} \to \oplus^N \mathcal{H}}$ in terms of kernel functions and must consider $N \times N$ matrix kernel functions. Recall that $\mathbb{M}_f : \mathcal{H} \to \oplus^N \mathcal{H}$ by $\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N$. Then for $g \in \oplus^N M_{\mathcal{H}}$,

$$\langle \mathbb{M}_f h, g\rangle_{\oplus^N \mathcal{H}} = \sum_{\alpha=1}^N \langle f_\alpha h, g_\alpha\rangle_{\mathcal{H}} = \left\langle h, \sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha \right\rangle_{\mathcal{H}},$$

and so $\mathbb{M}_f^* g = \sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha$.

Proof of Theorem 23

(ii) implies (i): Preliminaries

We can then see that $\|\mathbb{M}_f\|^2_{\mathcal{H}\to\oplus^N\mathcal{H}} \leq 1$, which is the same as $\|\mathbb{M}_f^*\|^2_{\oplus^N\mathcal{H}\to\mathcal{H}} \leq 1$, is equivalent to

$$\begin{aligned} 0 &\leq \sum_{\alpha=1}^N \|g_\alpha\|^2_{\mathcal{H}} - \left\|\sum_{\alpha=1}^N \mathcal{M}^*_{f_\alpha} g_\alpha\right\|^2_{\mathcal{H}} = \langle g, g\rangle_{\oplus^N\mathcal{H}} - \langle \mathbb{M}_f^* g, \mathbb{M}_f^* g\rangle_{\mathcal{H}} \\ &= \left\langle \left(I_{\oplus^N\mathcal{H}} - \mathbb{M}_f\mathbb{M}_f^*\right) g, g\right\rangle_{\oplus^N\mathcal{H}}. \end{aligned} \tag{7}$$

Which is the operator bound

$$I_{\oplus^N\mathcal{H}} - \mathbb{M}_f\mathbb{M}_f^* \geq 0. \tag{8}$$

Proof of Theorem 23

(ii) implies (i): Preliminaries

To obtain an equivalent kernel estimate, let
$g=(g_\alpha)_{\alpha=1}^N=\left(\sum_{i=1}^J \xi_i^\alpha k_{x_i^\alpha}\right)_{\alpha=1}^N$ so that

$$\mathbb{M}_f^* g=\sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha=\sum_{\alpha=1}^N\sum_{i=1}^J \xi_i^\alpha \overline{f_\alpha\left(x_i^\alpha\right)} k_{x_i^\alpha}.$$

If we substitute this in (7) we obtain

$$\begin{aligned}\sum_{\alpha,\beta=1}^N\sum_{i,j=1}^J \xi_i^\alpha\overline{\xi_j^\beta}\,\overline{f_\alpha\left(x_i^\alpha\right)} f_\beta\left(x_j^\beta\right) k\left(x_j^\beta,x_i^\alpha\right) &= \langle \mathbb{M}_f^* g,\mathbb{M}_f^* g\rangle_{\mathcal{H}}\\ &\leq \langle g,g\rangle_{\oplus^N\mathcal{H}}\\ &= \sum_{\alpha,\beta=1}^N \delta_\beta^\alpha\sum_{i,j=1}^J \xi_i^\alpha\overline{\xi_j^\beta} k\left(x_j^\beta,x_i^\alpha\right).\end{aligned}$$

Proof of Theorem 23

(ii) implies (i): Preliminaries

This can be rearranged to give

$$\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{J}\xi_i^{\alpha}\overline{\xi_j^{\beta}}\left[\left\{\delta_{\beta}^{\alpha}-\overline{f_{\alpha}\left(x_i^{\alpha}\right)}f_{\beta}\left(x_j^{\beta}\right)\right\}k\left(x_j^{\beta},x_i^{\alpha}\right)\right]\geq 0.$$

If we view $f(\zeta)\in\mathcal{B}\left(\mathbb{C},\mathbb{C}^N\right)$ we can rewrite this last expression as

$$\left\{I_{\mathbb{C}^N}-f(\zeta)f(\lambda)^*\right\}k(\zeta,\lambda)\geq 0. \tag{9}$$

This is the required matrix-valued kernel equivalence of the multiplier bound appearing in the Toeplitz Corona Theorem.

Proof of Theorem 23

(ii) implies (i)

Since k is an irreducible complete Nevanlinna-Pick kernel, we can find a Hilbert space $\mathcal{K}$ and a map $b : \Omega \to \mathcal{K}$ with $b(\lambda_0) = 0$ and such that

$$k(\zeta, \lambda) = \frac{1}{1 - \langle b(\zeta), b(\lambda) \rangle_{\mathcal{K}}}. \tag{10}$$

From (3) we now obtain (5):

$$K(\zeta, \lambda) \equiv \{\langle f(\zeta), f(\lambda) \rangle_{\mathbb{C}^N} - \epsilon\} \, k(\zeta, \lambda) \geq 0.$$

By a kernel-valued version of the Lax-Milgram Theorem, we can factor the left hand side $K(\zeta, \lambda)$ as $\langle G(\zeta), G(\lambda) \rangle_{\mathcal{E}}$ where $G : \Omega \to \mathcal{E}$ for some auxiliary space $\mathcal{E}$. Indeed, define $F : \Omega \to \mathcal{E}$ by $F(\zeta) = K_\zeta$ so that

$$K(\zeta, \lambda) = \langle K_\lambda, K_\zeta \rangle_{\mathcal{E}} = \langle F(\lambda), F(\zeta) \rangle_{\mathcal{E}}.$$

Proof of Theorem 23, Continued

(ii) implies (i)

Now fix an orthonormal basis $\{\mathbf{e}_\alpha\}_\alpha$ for $\mathcal{K}$ and define a conjugate linear operator Γ by

$$\Gamma\left(\sum_\alpha c_\alpha \mathbf{e}_\alpha\right) = \sum_\alpha \overline{c_\alpha}\mathbf{e}_\alpha.$$

Then $G = \Gamma \circ F$ satisfies

$$K(\zeta,\lambda) = \langle F(\lambda), F(\zeta)\rangle_{\mathcal{E}} = \langle \Gamma\circ F(\zeta), \Gamma\circ F(\lambda)\rangle_{\mathcal{E}} = \langle G(\zeta), G(\lambda)\rangle_{\mathcal{E}}.$$

Hence

$$\langle f(\zeta), f(\lambda)\rangle_{\mathbb{C}^N} - \epsilon = \left[1 - \langle b(\zeta), b(\lambda)\rangle_{\mathcal{K}}\right] \langle G(\zeta), G(\lambda)\rangle_{\mathcal{E}},$$

or equivalently,

$$\langle f(\zeta), f(\lambda)\rangle_{\mathbb{C}^N} + \langle b(\zeta), b(\lambda)\rangle_{\mathcal{K}} \langle G(\zeta), G(\lambda)\rangle_{\mathcal{E}} = \epsilon + \langle G(\zeta), G(\lambda)\rangle_{\mathcal{E}}. \tag{11}$$

Proof of Theorem 23, Continued

(ii) implies (i)

Define the space $\mathcal{N}_1$ and $\mathcal{N}_2$ by:

$$\begin{aligned}
\mathcal{N}_1 &= Span\left\{\begin{pmatrix} f(\lambda) \\ b(\lambda)\otimes G(\lambda)\end{pmatrix} u : u\in\mathbb{C}, \lambda\in\Omega\right\}\subset\mathbb{C}^N\oplus(\mathcal{K}\otimes\mathcal{E}),\\
\mathcal{N}_2 &= Span\left\{\begin{pmatrix} \sqrt{\epsilon} \\ G(\lambda)\end{pmatrix} u : u\in\mathbb{C}, \lambda\in\Omega\right\}\subset\mathbb{C}\oplus\mathcal{E}.
\end{aligned}$$

We rewrite (11) in terms of inner products of direct sums of Hilbert spaces,

$$\begin{aligned}
&\langle f(\zeta), f(\lambda)\rangle_{\mathbb{C}^N} + \langle b(\zeta)\otimes G(\zeta), b(\lambda)\otimes G(\lambda)\rangle_{\mathcal{K}\otimes\mathcal{E}}\\
&= \langle\sqrt{\epsilon},\sqrt{\epsilon}\rangle_{\mathbb{C}} + \langle G(\zeta), G(\lambda)\rangle_{\mathcal{E}},
\end{aligned}$$

Then we can interpret this as saying that the map from $\mathcal{N}_1$ to $\mathcal{N}_2$ is an *isometry*!

Proof of Theorem 23, Continued

(ii) implies (i)

Using (ii) we have obtained (11) that defines a linear isometry V' from the linear span $\mathcal{N}_1$ of the elements $f(\lambda)\, u \oplus (b(\lambda) \otimes G(\lambda))\, u$ onto the subspace $\mathcal{N}_2$.

Extend this isometry V' to an isometry V from all of $\mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{E})$ *onto* $\mathbb{C} \oplus \mathcal{E}$, where we add an infinite-dimensional summand to $\mathcal{E}$ if necessary.

Decompose the extended isometry V as a block matrix

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^N \\ \mathcal{K} \otimes \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathcal{E} \end{bmatrix}. \tag{12}$$

Since V is an onto isometry we obtain the formulas,

$$\begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{13}$$

$$= V^*V = I_{\mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{E})} = \begin{bmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I_{\mathcal{K} \otimes \mathcal{E}} \end{bmatrix}.$$

Proof of Theorem 23, Continued

(ii) implies (i)

Then (12) on the subspace $\mathcal{N}_1$ becomes

$$\begin{aligned} Af(\lambda) + B\left[b(\lambda) \otimes G(\lambda)\right] &= \sqrt{\epsilon}, \\ Cf(\lambda) + D\left[b(\lambda) \otimes G(\lambda)\right] &= G(\lambda). \end{aligned} \tag{14}$$

Now define $g : \Omega \to \mathcal{B}\left(\mathbb{C}, \mathbb{C}^N\right)$ by

$$\overline{g(\lambda)}^* = A + B\left\{ b(\lambda) \otimes \left(I - DE_{b(\lambda)}\right)^{-1} C \right\}, \tag{15}$$

where E_b is the map $E_b : \mathcal{E} \to \mathcal{K} \otimes \mathcal{E}$ given by

$$E_b v = b \otimes v, \qquad v \in \mathcal{E}. \tag{16}$$

Observe that $E_b^*(c \otimes w) = \langle c, b \rangle_{\mathcal{K}} w$, so that

$$E_b^* E_c = \langle c, b \rangle_{\mathcal{K}} I_{\mathcal{E}}. \tag{17}$$

Proof of Theorem 23, Continued

(ii) implies (i)

From this we conclude that $I - DE_{b(\lambda)}$ is invertible. Indeed, (10) shows that $\langle b, b\rangle_{\mathcal{K}} < 1$ and (17) then implies that $E_{b(\lambda)}$ is a *strict* contraction. From the equation $B^*B + D^*D = I_{\mathcal{K}\otimes\mathcal{E}}$ in (13) we see that D is a contraction, which altogether implies $\left\| DE_{b(\lambda)} \right\| < 1$.

Thus $\overline{g(\lambda)}^*$ satisfies

$$\begin{aligned}
\overline{g(\lambda)}^* f(\lambda) &= Af(\lambda) + B\left[b(\lambda) \otimes \left(I - DE_{b(\lambda)}\right)^{-1} Cf(\lambda)\right] \qquad (18)\\
&= Af(\lambda) + B\left[b(\lambda) \otimes G(\lambda)\right]\\
&= \sqrt{\epsilon},
\end{aligned}$$

Which is a restatement of (part) of (i).

To see the estimate in (i) holds, we must show that $g(\lambda)$ is a contractive multiplier, i.e. that (9) holds:

$$\left\{I_{\mathbb{C}^N} - g(\zeta)\, g(\lambda)^*\right\} k(\zeta, \lambda) \geq 0. \qquad (19)$$

Proof of Theorem 23, Continued

(ii) implies (i)

We then compute

$$\begin{aligned}\overline{g(\lambda)}^{*} &= A + BE_{b(\lambda)}\left(I - DE_{b(\lambda)}\right)^{-1} C, \\ \overline{g(\zeta)} &= A^{*} + C^{*}\left(I - E^{*}_{b(\zeta)}D^{*}\right)^{-1} E^{*}_{b(\zeta)}B^{*}.\end{aligned}$$

and then using (13) we obtain

$$I_{\mathbb{C}^N} - \overline{g(\zeta)\, g(\lambda)^{*}} =$$
$$\left(1 - \overline{\langle b(\zeta), b(\lambda)\rangle_{\mathcal{K}}}\right) C^{*}\left(I - E^{*}_{b(\zeta)}D^{*}\right)^{-1}\left(I - DE_{b(\lambda)}\right)^{-1} C.$$

This is then enough to conclude that $I_{\mathbb{C}^N} - g(\zeta)\, g(\lambda)^{*} \geq 0$.

Baby Corona for $H^\infty(\mathbb{B}_n)$ versus Corona for $H^\infty(\mathbb{B}_n)$

We know that the Corona Problem always implies the Baby Corona Problem. By the Toeplitz Corona Theorem, we know that, under certain conditions on the reproducing kernel, these problems are in fact equivalent. But, what happens if we don't have these conditions?

Theorem 3 (Equivalence between Corona and Baby Corona, (Amar 2003))

Let $\{g_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$. Then there exists $\{f_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$ with

$$\sum_{j=1}^N f_j(z)g_j(z) = 1 \quad \forall z \in \mathbb{B}_n \quad \text{and} \quad \sup_{z\in\mathbb{B}_n} \sum_{j=1}^N |g_j(z)|^2 \leq \frac{1}{\epsilon^2}$$

if and only if

$$\mathcal{M}_g^\mu(\mathcal{M}_g^\mu)^* \geq \epsilon^2 I_\mu$$

for all probability measures μ on $\partial\mathbb{B}_n$.

Baby Corona for $H^\infty(\mathbb{B}_n)$ versus Corona for $H^\infty(\mathbb{B}_n)$

This is a great theorem since it suggests how to attack the Corona Problem for $H^\infty(\mathbb{B}_n)$. But, the difficulty is that one must solve the Baby Corona Problem for **every** probability measure on $\partial\mathbb{B}_n$.

Instead, it is possible to reduce this to a class of probability measures for which the methods of harmonic analysis and operator theory are more amenable.

Theorem 4 (Trent, BDW (2008))

Assume that $\mathcal{M}_g^H \mathcal{M}_g^{H*} \geq \epsilon^2 I_w$ *for all* $w \in \mathcal{W}$. *Then there exists a* $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$, *so that*

$$\sum_{j=1}^N f_j(z) g_j(z) = 1 \ \forall z \in \mathbb{B}_n \quad \text{and} \quad \sup_{z \in \mathbb{B}_n} \sum_{j=1}^N |f_j(z)|^2 \leq \frac{1}{\epsilon^2}.$$

This reduces the $H^\infty(\mathbb{B}_n)$ Corona Problem to a certain “weighted” Baby Corona Problem.

The Corona Problem in Several Variables

Besov-Sobolev Spaces on the Unit Ball

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n$ such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}} < \infty,$$

  where $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m + \sigma > \frac{n}{2}$.
- Various choices of σ give important examples of classical function spaces:
  - $\sigma = 0$: Corresponds to the Dirichlet Space;
  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space;
  - $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces.
- Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}$$

- It isn't too difficult to compute (or show) that the kernel function k_λ is given by

$$k_\lambda(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$
  - $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^n}$
  - $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Multiplier Algebras of Besov-Sobolev Spaces $M_{B_2^\sigma}(\mathbb{B}_n)$

- We are interested in the multiplier algebras, $M_{B_2^\sigma}(\mathbb{B}_n)$, for $B_2^\sigma(\mathbb{B}_n)$. A function φ belongs to $M_{B_2^\sigma}(\mathbb{B}_n)$ if

$$\|\varphi f\|_{B_2^\sigma(\mathbb{B}_n)} \leq C\,\|f\|_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f \in B_2^\sigma(\mathbb{B}_n)$$

$$\|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} = \inf\{C : \text{above inequality holds}\}.$$

- It is easy to see that $M_{B_2^\sigma}(\mathbb{B}_n) = H^\infty(\mathbb{B}_n) \cap \mathcal{X}_2^\sigma(\mathbb{B}_n)$. Where $\mathcal{X}_2^\sigma(\mathbb{B}_n)$ is the collection of functions φ such that for all $f \in B_2^\sigma(\mathbb{B}_n)$:

$$\int_{\mathbb{B}_n} |f(z)|^2 \left|\left(1-|z|^2\right)^{m+\sigma} \varphi^{(m)}(z)\right|^2 d\lambda_n(z) \leq C\,\|f\|^2_{B_2^\sigma(\mathbb{B}_n)}, \qquad (\ddagger)$$

with $\|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)} = \inf\{C : (\ddagger) \text{ holds}\}$.
Thus, we have

$$\|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \approx \|\varphi\|_{H^\infty(\mathbb{B}_n)} + \|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)}.$$

The Corona Problem for $M_{B_2^\sigma}(\mathbb{B}_n)$

We wish to study a generalization of Carleson's Corona Theorem to higher dimensions and additional function spaces.

Question 24 (Corona Problem)

Given $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$*. Does there exist a constant* $C_{n,\sigma,N,\delta}$ *and functions* $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|g_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^N g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n? \end{aligned}$$

The Baby Corona Problem

It is easy to see that the Corona Problem for $M_{B_2^\sigma}(\mathbb{B}_n)$ implies a "simpler" question that one can consider.

Question 25 (Baby Corona Problem)

Given $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$ *and* $h \in B_2^\sigma(\mathbb{B}_n)$. *Does there exist a constant* $C_{n,\sigma,N,\delta}$ *and functions* $k_1, \ldots, k_N \in B_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|k_j\|^2_{B_2^\sigma(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \|h\|^2_{B_2^\sigma(\mathbb{B}_n)}, \\ \sum_{j=1}^N k_j(z) f_j(z) &= h(z), \qquad z \in \mathbb{B}_n? \end{aligned}$$

Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

Theorem 26 (Ş. Costea, E. Sawyer, BDW)

Let $0 \leq \sigma$ *and* $1 < p < \infty$. *Given* $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,N,p,\delta}$ *such that for each* $h \in B_p^\sigma(\mathbb{B}_n)$ *there are* $k_1, \ldots, k_N \in B_p^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \|k_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \\ \sum_{j=1}^{N} g_j(z) k_j(z) &= h(z), \qquad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Corollary 27 (Ş. Costea, E. Sawyer, BDW)

Let $0 \leq \sigma \leq \frac{1}{2}$. *Given* $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,N,\delta}$ *and there are functions* $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$\sum_{j=1}^{N} \|f_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \leq C_{n,\sigma,N,\delta}$$

$$\sum_{j=1}^{N} g_j(z) f_j(z) = 1, \qquad z \in \mathbb{B}_n.$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

The proof of this Corollary follows from the main Theorem very easily.

- When $0 \leq \sigma \leq \frac{1}{2}$ the spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces with a complete Nevanlinna-Pick kernel.
- By the Toeplitz Corona Theorem, we then have that the Baby Corona Problem is equivalent to the full Corona Problem. The result then follows.

An additional corollary of the above result is the following:

Corollary 28

For $0 \leq \sigma \leq \frac{1}{2}$, the unit ball $\mathbb{B}_n$ is dense in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$.

This is because the density of the the unit ball $\mathbb{B}_n$ in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$ is equivalent to the Corona Theorem above.

Sketch of Proof of the Baby Corona Theorem

Given $g_1, \ldots, g_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ satisfying

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

- Set $\varphi_j(z) = \frac{\overline{g_j(z)}}{\sum_j |g_j(z)|^2} h(z)$. We have that $\sum_{j=1}^{N} g_j(z)\varphi_j(z) = h(z)$.
  - This solution is smooth and satisfies the correct estimates, but is far from analytic.
- In order to have an analytic solution, we will need to solve a sequence of $\overline{\partial}$-equations:
  - For η a $\overline{\partial}$-closed $(0,q)$ form, we want to solve the equation

    $$\overline{\partial}\psi = \eta$$

    for ψ a $(0, q-1)$ form.
  - To accomplish this, we will use the Koszul complex. This gives an algorithmic way of solving the $\overline{\partial}$-equations for each $(0,q)$ with $1 \leq q \leq n$ after starting with a $(0,n)$ form.

Sketch of Proof of the Baby Corona Theorem

- This produces a correction to the initial guess of φ_j, call it ξ_j, and set $f_j = \varphi_j - \xi_j$. By the Koszul complex we will have that each f_j is in fact analytic.
  Algebraic properties of the Koszul complex give that $\sum_j f_j g_j = h$.
  However, now the estimates that we seek are in doubt.

- To guarantee the estimates, we have to look closer at the solution operator to the $\overline{\partial}$-equation on $\overline{\partial}$-closed $(0, q)$ forms. Following the work of Øvrelid and Charpentier, one can compute that the solution operator is an integral operator that that takes $(0, q)$ forms to $(0, q-1)$ forms with integral kernel:

$$\frac{(1 - w\overline{z})^{n-q}\left(1 - |w|^2\right)^{q-1}}{\triangle (w, z)^n}(\overline{w}_j - \overline{z}_j) \quad \forall\, 1 \leq q \leq n.$$

Here $\triangle (w, z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.

Sketch of Proof of the Baby Corona Theorem

- One then needs to show that these solution operators map the Besov-Sobolev spaces $B_\sigma^p(\mathbb{B}_n)$ to themselves. This is accomplished by a couple of key facts:
  - The Besov-Sobolev spaces are very "flexible" in terms of the norm that one can use. One need only take the parameter m sufficiently high.
  - We show that these operators are very well behaved on "real variable" versions of the space $B_\sigma^p(\mathbb{B}_n)$. These, of course, contain the space that we are interested in.
  - To show that the solution operators are bounded on $L^p(\mathbb{B}_n; dV)$ the original proof uses the Schur Test. To handle the boundedness on $B_\sigma^p(\mathbb{B}_n)$, we can also use the Schur test but this requires a little more work to handle the derivative.
- Finally, key to this approach, properties of the kernel and the unit ball are exploited to achieve the desired estimates.
- We now explain some (but not all!) of the ingredients behind this sketch.

The Koszul Complex

- Define

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \qquad \frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$$

where $z_k = x_k + iy_k$ and $dz_k = dx_k + idy_k$, $d\overline{z}_k = dx_k - idy_k$. Let

$$\overline{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z_k}} d\overline{z_k}.$$

- Given a $(0,1)$-form $\eta(z) = \eta_1(z)\, d\overline{z_1} + \cdots + \eta_n(z)\, d\overline{z_n}$ in the ball $\mathbb{B}_n$, the $\overline{\partial}$-equation for η is

$$\overline{\partial} f = \eta \quad \text{in the ball } \mathbb{B}_n. \tag{20}$$

- More generally, we can let $\eta = \sum_{|I|=p,|J|=q+1} \eta_{I,J}(z)\, dz^I \wedge d\overline{z}^J$ be a $(p, q+1)$-form in the ball and ask for a (p,q)-form f to satisfy (20).

The Koszul Complex

- If $f = (f_j)_{j=1}^N$ satisfies $|f|^2 = \sum_{j=1}^N |f_j|^2 \geq 1$, let

$$\Omega_0^1 = \frac{\overline{f}}{|f|^2} = \left(\frac{\overline{f_j}}{|f|^2}\right)_{j=1}^N = \left(\Omega_0^1(j)\right)_{j=1}^N,$$

which we view as a 1-tensor (in $\mathbb{C}^N$) of $(0,0)$-forms with components $\Omega_0^1(j) = \frac{\overline{f_j}}{|f|^2}$.

- Then $g = \Omega_0^1 h$ satisfies $f \cdot g = h$, but in general fails to be analytic.
- The Koszul complex provides a scheme when f, h are holomorphic for solving a sequence of $\overline{\partial}$ equations that result in a correction term $\Lambda_f \Gamma_0^2$ that when subtracted from f above yields an *analytic* solution to $f \cdot g = h$.

Lifting of Forms

- The 1-tensor of $(0,1)$-forms $\overline{\partial}\Omega_0 = \left(\overline{\partial}\frac{\overline{f_j}}{|f|^2}\right)_{j=1}^N = \left(\overline{\partial}\Omega_0^1(j)\right)_{j=1}^N$ is given by

$$\overline{\partial}\Omega_0^1(j) = \overline{\partial}\frac{\overline{f_j}}{|f|^2} = \frac{1}{|f|^4}\sum_{k=1}^N f_k\overline{\{f_k\partial f_j - \partial f_k f_j\}}.$$

- A key fact is that this 1-tensor of $(0,1)$-forms can be written as

$$\overline{\partial}\Omega_0^1 = \Lambda_f\Omega_1^2 \equiv \left[\sum_{k=1}^N \Omega_1^2(j,k)\, f_k\right]_{j=1}^N,$$

where the 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2(j,k)\right]_{j,k=1}^N = \left[\frac{\overline{\{f_k\partial f_j - \partial f_k f_j\}}}{|f|^4}\right]_{j,k=1}^N.$$

- The form $\overline{\partial}\Omega_0^1$ has been factored as $\Lambda_f\Omega_1^2$ where Ω_1^2 is *antisymmetric*.

Solving the complex ...

- We can repeat this process and by induction we have

$$\overline{\partial}\Omega_q^{q+1} = \Lambda_f \Omega_{q+1}^{q+2}, \qquad 0 \leq q \leq n,$$

  where Ω_q^{q+1} is an *alternating* $(q+1)$-tensor of $(0,q)$-forms.
- Recall that h is holomorphic. When $q = n$ we have that $\Omega_n^{n+1} h$ is $\overline{\partial}$-closed since every $(0,n)$-form is $\overline{\partial}$-closed.
- This allows us to begin solving a chain of $\overline{\partial}$ equations

$$\overline{\partial}\Gamma_{q-2}^q = \Omega_{q-1}^q h - \Lambda_f \Gamma_{q-1}^{q+1},$$

...using that the forms are closed

- Since $\Omega_n^{n+1}h$ is $\overline{\partial}$-closed and alternating, there is an alternating $(n+1)$-tensor Γ_{n-1}^{n+1} of $(0, n-1)$-forms satisfying

$$\overline{\partial}\Gamma_{n-1}^{n+1} = \Omega_n^{n+1}h.$$

- Now note that the n-tensor $\Omega_{n-1}^n h - \Lambda_f \Gamma_{n-1}^{n+1}$ of $(0, n-1)$-forms is $\overline{\partial}$-closed:

$$\overline{\partial}\left(\Omega_{n-1}^n h - \Lambda_f \Gamma_{n-1}^{n+1}\right) = \overline{\partial}\Omega_{n-1}^n h - \overline{\partial}\Lambda_f \Gamma_{n-1}^{n+1} = \Lambda_f \Omega_n^{n+1} h - \Lambda_f \Omega_n^{n+1} h = 0.$$

- Thus there is an alternating n-tensor Γ_{n-2}^n of $(0, n-2)$-forms satisfying

$$\overline{\partial}\Gamma_{n-2}^n = \Omega_{n-1}^n h - \Lambda_f \Gamma_{n-1}^{n+1}.$$

The Bezout equation

- With the convention that $\Gamma_n^{n+2} \equiv 0$, induction shows that there are alternating $(q+2)$-tensors Γ_q^{q+2} of $(0,q)$-forms for $0 \le q \le n$ satisfying

$$\begin{aligned} \overline{\partial}\left(\Omega_q^{q+1} h - \Lambda_f \Gamma_q^{q+2}\right) &= 0, \qquad 0 \le q \le n, \\ \overline{\partial}\Gamma_{q-1}^{q+1} &= \Omega_q^{q+1} h - \Lambda_f \Gamma_q^{q+2}, \qquad 1 \le q \le n. \end{aligned} \tag{21}$$

- Now

$$g \equiv \Omega_0^1 h - \Lambda_f \Gamma_0^2$$

is holomorphic by the first line in (21) with $q = 0$, and since Γ_0^2 is antisymmetric, we compute that $\Lambda_f \Gamma_0^2 \cdot f = \Gamma_0^2(f,f) = 0$ and

$$f \cdot g = \Omega_0^1 h \cdot g - \Lambda_f \Gamma_0^2 \cdot f = h - 0 = h.$$

- Thus $g = (g_1, g_2, \ldots, g_N)$ is an N-vector of holomorphic functions satisfying $f \cdot g = h$.

Charpentier's solution kernels

We begin with some notation. Denote by $\triangle : \mathbb{C}^n \times \mathbb{C}^n \to [0,\infty)$ the map:

$$\begin{aligned} \triangle(w,z) &= |1-w\overline{z}|^2 - \left(1-|w|^2\right)\left(1-|z|^2\right) \qquad (22) \\ &= \left(1-|z|^2\right)|w-z|^2 + |\overline{z}(w-z)|^2 \\ &= \left(1-|w|^2\right)|w-z|^2 + |\overline{w}(w-z)|^2 \\ &= |1-w\overline{z}|^2 \, |\varphi_w(z)|^2 \\ &= |1-w\overline{z}|^2 \, |\varphi_z(w)|^2 \end{aligned}$$

Here

$$|\varphi_z(w)| = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-w\overline{z}|^2}.$$

The Cauchy-Leray Form

The Cauchy-Leray form

$$\mu(\xi, w, z) \equiv \frac{1}{(\xi(w-z))^n} \sum_{i=1}^{n} (-1)^{i-1} \xi_i \left[\wedge_{j\neq i} d\xi_j\right] \wedge_{i=1}^{n} d(w_i - z_i),$$

is a closed form on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$. One then lifts the form μ via a section $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ to give a closed form on $\mathbb{C}^n \times \mathbb{C}^n$:

$$s^*\mu\,(w,z) \equiv \frac{1}{(s\,(w,z)\,(w-z))^n} \sum_{i=1}^{n} (-1)^{i-1} s_i\,(w,z)\,[\wedge_{j\neq i} ds_j] \wedge_{i=1}^{n} d\,(w_i - z_i)$$

Fix s to be the following section used by Charpentier:

$$s(w,z) \equiv \overline{w}(1 - w\overline{z}) - \overline{z}(1 - |w|^2).$$

We compute that $s(w,z)(w-z) = \triangle\,(w,z)$ by (22).

Charpentier's Forms

Define the Cauchy Kernel on $\mathbb{B}_n \times \mathbb{B}_n$ by $\mathcal{C}_n(w,z) \equiv s^*\mu(w,z)$ where s is Charpentier's section.

Definition 29

For $0 \leq p \leq n$ and $0 \leq q \leq n-1$ we let $\mathcal{C}_n^{p,q}$ be the component of $\mathcal{C}_n(w,z)$ that has bidegree (p,q) in z and bidegree $(n-p, n-q-1)$ in w.

Thus if η is a $(p,q+1)$-form in w, then $\mathcal{C}_n^{p,q} \wedge \eta$ is a (p,q)-form in z and a multiple of the volume form in w. Let $\omega_n(z) = \bigwedge_{j=1}^n dz_j$. For n a

positive integer and $0 \leq q \leq n-1$ let P_n^q denote the collection of all permutations ν on $\{1,\ldots,n\}$ that map to $\{i_\nu, J_\nu, L_\nu\}$ where J_ν is an increasing multi-index with $\operatorname{card}(J_\nu) = n-q-1$ and $\operatorname{card}(L_\nu) = q$. Let $\epsilon_\nu \equiv \operatorname{sgn}(\nu) \in \{-1,1\}$ denote the signature of the permutation ν.

Explicit Formulas for Charpentier Kernels

Theorem 30 (Charpentier)

Let n be a positive integer and suppose that $0 \leq q \leq n-1$. Then

$$\mathcal{C}_n^{0,q}(w,z) = \sum_{\nu \in P_n^q} (-1)^q \, \Phi_n^q(w,z) \, sgn(\nu) \left(\overline{w_{i_\nu}} - \overline{z_{i_\nu}}\right) \times \bigwedge_{j \in J_\nu} d\overline{w_j} \bigwedge_{l \in L_\nu} d\overline{z_l} \bigwedge \omega_n(w). \tag{23}$$

where $\Phi_n^q(w,z) \equiv \frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}$ *for* $0 \leq q \leq n-1$.

We can rewrite the formula for $\mathcal{C}_n^{0,q}(w,z)$ in (23) as

$$\mathcal{C}_n^{0,q}(w,z) = \Phi_n^q(w,z) \sum_{|J|=q} \sum_{k \notin J} (-1)^{\mu(k,J)} \left(\overline{z_k} - \overline{w_k}\right) d\overline{z}^J \wedge d\overline{w}^{(J\cup\{k\})^c} \wedge \omega_n(w)$$

Ameliorated Kernels

We now wish to define right inverses with improved behaviour at the boundary. We consider the case when the right side f of the $\overline{\partial}$ equation is a $(p, q+1)$-form in $\mathbb{B}_n$.

As usual for a positive integer $s > n$ we will "project" the formula $\overline{\partial}\mathcal{C}_s^{p,q} f = f$ in $\mathbb{B}_s$ for a $\overline{\partial}$-closed form f in $\mathbb{B}_s$ to a formula $\overline{\partial}\mathcal{C}_{n,s}^{p,q} f = f$ in $\mathbb{B}_n$ for a $\overline{\partial}$-closed form f in $\mathbb{B}_n$.

To accomplish this we define *ameliorated* operators $\mathcal{C}_{n,s}^{p,q}$ by

$$\mathcal{C}_{n,s}^{p,q} = \mathrm{R}_n \mathcal{C}_s^{p,q} \mathrm{E}_s,$$

where for $n < s$, E_s (R_n) is the extension (restriction) operator that takes forms $\Omega = \sum \eta_{I,J} dw^I \wedge d\overline{w}^J$ in $\mathbb{B}_n$ ($\mathbb{B}_s$) and extends (restricts) them to $\mathbb{B}_s$ ($\mathbb{B}_n$) by

$$\begin{aligned} \mathrm{E}_s \left(\sum \eta_{I,J} dw^I \wedge d\overline{w}^J \right) &\equiv \sum \left(\eta_{I,J} \circ R \right) dw^I \wedge d\overline{w}^J, \\ \mathrm{R}_n \left(\sum \eta_{I,J} dw^I \wedge d\overline{w}^J \right) &\equiv \sum_{I,J \subset \{1,2,\ldots,n\}} \left(\eta_{I,J} \circ E \right) dw^I \wedge d\overline{w}^J. \end{aligned}$$

Ameliorated Kernels, Continued

Here R is the natural orthogonal projection from $\mathbb{C}^s$ to $\mathbb{C}^n$ and E is the natural embedding of $\mathbb{C}^n$ into $\mathbb{C}^s$. In other words, we extend a form by

taking the coefficients to be constant in the extra variables, and we restrict a form by discarding all wedge products of differentials involving the extra variables and restricting the coefficients accordingly.
For $s > n$ we observe that the operator $\mathcal{C}_{n,s}^{p,q}$ has integral kernel

$$\mathcal{C}_{n,s}^{p,q}(w,z) \equiv \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \mathcal{C}_s^{p,q}\left(\left(w,w'\right),(z,0)\right) dV\left(w'\right), \qquad z,w \in \mathbb{B}_n,$$

where $\mathbb{B}_{s-n}$ denotes the unit ball in $\mathbb{C}^{s-n}$ with respect to the orthogonal decomposition $\mathbb{C}^s = \mathbb{C}^n \oplus \mathbb{C}^{s-n}$, and dV denotes Lebesgue measure.

Ameliorated Kernels, Continued

If $f(w)$ is a $\overline{\partial}$-closed form on $\mathbb{B}_n$ then $f(w, w') = f(w)$ is a $\overline{\partial}$-closed form on $\mathbb{B}_s$ and we have for $z \in \mathbb{B}_n$,

$$\begin{aligned} f(z) &= f(z,0) = \overline{\partial} \int_{\mathbb{B}_s} \mathcal{C}_s^{p,q}\left((w,w'),(z,0)\right) f(w)\, dV(w)\, dV(w') \\ &= \overline{\partial} \int_{\mathbb{B}_n} \left\{ \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \mathcal{C}_s^{p,q}\left((w,w'),(z,0)\right) dV(w') \right\} f(w)\, dV(w) \\ &= \overline{\partial} \int_{\mathbb{B}_n} \mathcal{C}_{n,s}^{p,q}(w,z)\, f(w)\, dV(w). \end{aligned}$$

We have proved the following:

Theorem 31

For all $s > n$ and $\overline{\partial}$-closed forms f in $\mathbb{B}_n$, we have

$$\overline{\partial}\mathcal{C}_{n,s}^{p,q} f = f \text{ in } \mathbb{B}_n.$$

Ameliorated Kernels, Continued

Theorem 32

Suppose that $s > n$ and $0 \leq q \leq n-1$. Then we have $\mathcal{C}_{n,s}^{0,q}(w,z)$ is given by

$$\mathcal{C}_{n}^{0,q}(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right)^{j}$$

Note that the numerator and denominator are *balanced* in the sense that the sum of the exponents in the denominator minus the corresponding sum in the numerator (counting $\triangle(w,z)$ double) is $s+n+j-(s+j-1)=n+1$, the exponent of the invariant measure of the ball $\mathbb{B}_n$.

Invariant Derivatives

- Define

$$\nabla_z = \left(\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\right) \text{ and } \overline{\nabla_z} = \left(\frac{\partial}{\partial \overline{z_1}}, ..., \frac{\partial}{\partial \overline{z_n}}\right)$$

- Recall that the gradient with invariant length given by

$$\begin{aligned} \widetilde{\nabla} f(a) &= (f \circ \varphi_a)'(0) = f'(a)\,\varphi_a'(0) \\ &= -f'(a)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\}. \end{aligned}$$

- We want an analogue of this operator on the tree $\mathcal{T}_n$. We define for $z \in \mathbb{B}_n$,

$$\begin{aligned} D_a f(z) &= f'(z)\,\varphi_a'(0) \\ &= -f'(z)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\}. \end{aligned}$$

A Tree Seminorm

Lemma 33

Let $a, b \in \mathbb{B}_n$ satisfy $\beta(a,b) \leq C$. There is a positive constant C_m depending only on C and m such that

$$C_m^{-1} \left|D_b^m f(z)\right| \leq \left|D_a^m f(z)\right| \leq C_m \left|D_b^m f(z)\right|,$$

for all $f \in \mathrm{Hol}(\mathbb{B}_n)$.

Definition 34

Suppose $\sigma \geq 0$, $1 < p < \infty$ and $m \geq 1$. We define a "tree semi-norm" $\|\cdot\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)}$ by

$$\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} = \left(\sum_{\alpha \in \mathcal{T}_n} \int_{B_d(c_\alpha, C_2)} \left|\left(1-|z|^2\right)^\sigma D^m_{c_\alpha} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}.$$

Pointwise Multipliers

Lemma 35

We have

$$\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \approx \|f\|_{B^\sigma_{p,m}(\mathbb{B}_n)}.$$

Let $\varphi \in H^\infty(\mathbb{B}_n) \cap B^\sigma_p(\mathbb{B}_n)$. *If* $m > \frac{n}{p} - \sigma$ *and* $0 \leq \sigma < \infty$, *then* φ *is a pointwise multiplier on* $B^\sigma_p(\mathbb{B}_n)$ *if and only if*

$$\left|\left(1-|z|^2\right)^{m+\sigma} \nabla^m \varphi(z)\right|^p d\lambda_n(z) \tag{24}$$

is a $B^\sigma_p(\mathbb{B}_n)$*-Carleson measure on* $\mathbb{B}_n$. *If* $m > 2\left(\frac{n}{p} - \sigma\right)$ *and* $0 \leq \sigma < \frac{n}{p} + 1$, *then* (24) *can be replaced by*

$$\left|\left(1-|z|^2\right)^{\sigma} D^m \varphi(z)\right|^p d\lambda_n(z).$$

The Real Variable Besov Space

Definition 36

We denote by $\mathcal{X}^m$ the vector of all differential operators of the form $X_1 X_2 \cdots X_m$ where each X_i is either the identity operator I, the operator $\overline{D}$, or the operator $\left(1-|z|^2\right) R$. We calculate the products $X_1 X_2 \cdots X_m$ by composing $\overline{D_a}$ and $\left(1-|a|^2\right) R$ and then setting $a=z$ at the end. Note that $\overline{D_a}$ and $\left(1-|a|^2\right) R$ commute since the first is an antiholomorphic derivative and the coefficient z in $R = z \cdot \nabla$ is holomorphic.

Definition 37

We define the norm $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n)}$ for f smooth on the ball $\mathbb{B}_n$ by

$$\|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n)} \equiv \left(\int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^\sigma \mathcal{X}^m f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}}.$$

Estimates for the Differentiation Operators

Lemma 38

$$\left|\left(\overline{z-w}\right)^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}F(w)\right|\leq C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^m\left|\overline{D}^m F(w)\right|\qquad m=|\alpha|$$

$$\begin{aligned}|D_z\triangle(w,z)| &\leq C\left\{\left(1-|z|^2\right)\triangle(w,z)^{\frac{1}{2}}+\triangle(w,z)\right\}\\ \left|\left(1-|z|^2\right)R\triangle(w,z)\right| &\leq C\left(1-|z|^2\right)\sqrt{\triangle(w,z)}\end{aligned}$$

$$\begin{aligned}\left|D_z^m\left\{(1-\overline{w}z)^k\right\}\right| &\leq C\left|1-\overline{w}z\right|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{\frac{m}{2}}\\ \left|\left(1-|z|^2\right)^m R^m\left\{(1-\overline{w}z)^k\right\}\right| &\leq C\left|1-\overline{w}z\right|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^m\end{aligned}$$

The Main Estimates

From the Koszul complex we must show that $g \in B_p^\sigma(\mathbb{B}_n)$ where

$$
\begin{aligned}
g &= \Omega_0^1 h - \Lambda_f \Gamma_0^2 \\
&= \Omega_0^1 h - \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\left(\Omega_1^2 h - \Lambda_f \Gamma_1^3\right) \\
&= \Omega_0^1 h - \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Omega_1^2 h + \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Lambda_f \Gamma_1^3 \\
&= \Omega_0^1 h - \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Omega_1^2 h + \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Lambda_f\left(\mathcal{C}_{n,s_2}^{0,1}\Omega_2^3 h - \Lambda_f \Gamma_2^4\right) \\
&\vdots \\
&= \Omega_0^1 h - \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Omega_1^2 h + \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Lambda_f \mathcal{C}_{n,s_2}^{0,1}\Omega_2^3 h - \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\Lambda_f \mathcal{C}_{n,s_2}^{0,1}\Lambda_f \mathcal{C}_{n,s_3}^{0,2}\Omega_3^4 h - \\
&\quad + (-1)^n \Lambda_f \mathcal{C}_{n,s_1}^{0,0}\cdots\Lambda_f \mathcal{C}_{n,s_n}^{0,n-1}\Omega_n^{n+1} h \\
&\equiv \mathcal{F}^0 + \mathcal{F}^1 + \cdots + \mathcal{F}^n \\
&= \sum_{\mu=0}^{n} \mathcal{F}^\mu
\end{aligned}
$$

The Main Estimates

The goal is to establish

$$\|g\|_{B_p^\sigma(\mathbb{B}_n)} \le C_{\delta,n,N,p} \|h\|_{\Lambda_p^\sigma(\mathbb{B}_n)},$$

which we accomplish by showing that

$$\|\mathcal{F}^\mu\|_{B^\sigma_{p,m_1}(\mathbb{B}_n)} \le C_{\delta,n,N,p} \|h\|_{\Lambda^\sigma_{p,m_\mu}(\mathbb{B}_n)}, \qquad 0 \le \mu \le n,$$

for a choice of integers m_μ satisfying

$$\frac{n}{p} - \sigma < m_1 < m_2 < \cdots < m_\ell < \cdots < m_n.$$

Recall that we defined both of the norms $\|F\|_{B^\sigma_{p,m_\mu}(\mathbb{B}_n)}$ and $\|F\|_{\Lambda^\sigma_{p,m_\mu}(\mathbb{B}_n)}$ for smooth functions F in the ball $\mathbb{B}_n$.

The Main Estimates

The norms $\left\|\cdot\right\|_{\Lambda^{\sigma}_{p,m}(\mathbb{B}_n)}$ will now be used to estimate the composition of Charpentier solution operators in each function

$$\mathcal{F}^{\mu} = \Lambda_f \mathcal{C}^{0,0}_{n,s_1} \cdots \Lambda_f \mathcal{C}^{0,\mu-1}_{n,s_\mu} \Omega^{\mu+1}_{\mu} h$$

as follows. We will use the facts that

$$\begin{aligned} \left\|h\right\|^p_{B^{\sigma}_p(\mathbb{B}_n)} &\approx \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{\sigma} \mathcal{X}^m h(z)\right|^p d\lambda_n(z), \\ \left\|g\right\|^p_{M_{B^{\sigma}_p(\mathbb{B}_n)}} &\approx \left\|g\right\|^p_{\infty} + \left\|\left|\left(1-|z|^2\right)^{\sigma} \mathcal{X}^m g(z)\right|^p d\lambda_n(z)\right\|_{B^{\sigma}_p(\mathbb{B}_n)-Carleson}, \end{aligned}$$

for $0 \leq \sigma < \frac{n}{p} + 1$ and $m > 2\left(\frac{n}{p} - \sigma\right)$.

Schur's Lemma on Besov-Sobolev Spaces

Lemma 39

Let $a, b, c, t \in \mathbb{R}$. Then the operator

$$T_{a,b,c}f(z) = \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^a \left(1-|w|^2\right)^b \left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\overline{z}|^{n+1+a+b+c}} f(w)\, dV(w)$$

is bounded on $L^p\left(\mathbb{B}_n; \left(1-|w|^2\right)^t dV(w)\right)$ if and only if $c > -2n$ and

$$-pa < t+1 < p(b+1).$$

We will use this Lemma for appropriate choices of a, b, c. This, plus some more computations, shows that the it is possible to obtain estimates to $\overline{\partial}$ in the space $\Lambda^{\sigma}_{p,m}(\mathbb{B}_n)$.

Obtaining the Estimates

We will accomplish this by showing

$$\left\|\mathcal{F}^{\mu}\right\|_{B_p^{\sigma}(\mathbb{B}_n)} \leq C_{n,\sigma,p,\delta} \left\|h\right\|_{B_p^{\sigma}(\mathbb{B}_n)}, \qquad 0 \leq \mu \leq n.$$

Then we can then estimate by

$$\left\|\mathcal{F}^{\mu}\right\|_{B_p^{\sigma}(\mathbb{B}_n)} \leq \left\|T_1 \circ T_2 \circ \cdots \circ T_{\mu}\Omega_{\mu}^{\mu+1}h\right\|_{B_p^{\sigma}(\mathbb{B}_n)}.$$

Here $T_{\mu} = T_{a_{\mu},b_{\mu},c_{\mu}}$ for appropriate $a_{\mu}, b_{\mu}, c_{\mu}$. So, we have

$$\left\|\mathcal{F}^{\mu}\right\|_{B_p^{\sigma}(\mathbb{B}_n)} \leq C_{n,\sigma,p,\delta} \left\|\Omega_{\mu}^{\mu+1}h\right\|_{B_p^{\sigma}(\mathbb{B}_n)}.$$

Finally, show that for $\Omega_{\mu}^{\mu+1}$ arising from the Koszul complex, then

$$\left\|\Omega_{\mu}^{\mu+1}h\right\|_{B_p^{\sigma}(\mathbb{B}_n)} \leq C_{n,\sigma,p,\delta} \left\|h\right\|_{B_p^{\sigma}(\mathbb{B}_n)}.$$

Multilinear Estimates and Embeddings

To control this last term we use the following lemma:

Lemma 40

Suppose that $1 < p < \infty$, $0 \leq \sigma < \infty$, $M \geq 1$, $m > 2\left(\frac{n}{p} - \sigma\right)$ *and* $\alpha = (\alpha_0, \ldots, \alpha_M) \in \mathbb{Z}_+^{M+1}$ *with* $|\alpha| = m$. *For* $g_1, \ldots, g_M \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n)}$ *and* $h \in B_p^\sigma(\mathbb{B}_n)$ *we have*

$$\int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{p\sigma} |(\mathcal{X}^{\alpha_1} g_1)(z)|^p \cdots |(\mathcal{X}^{\alpha_M} g_M)(z)|^p \, |(\mathcal{X}^{\alpha_0} h)(z)|^p \, d\lambda_n(z)$$

$$\leq C_{n,M,\sigma,p} \left(\prod_{j=1}^{M} \left\| M_{g_j} \right\|^p_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n)}\right) \|h\|^p_{B_p^\sigma(\mathbb{B}_n)}.$$

This follows then since the term $\Omega_\mu^{\mu+1} h$ is *essentially* what appears in the left hand side above.

Open Problems and Future Directions

1. Does the algebra $H^{\infty}(\mathbb{B}_n)$ of bounded analytic functions on the ball have a Corona in its maximal ideal space?
2. Does the Corona Theorem for the multiplier algebra of the Drury-Arveson space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ extend to more general domains in $\mathbb{C}^n$?
3. Can we prove a Corona Theorem for *any* algebra in higher dimensions that is not the multiplier algebra of a Hilbert space with the complete Nevanlinna-Pick property? Any $\frac{1}{2} < \sigma \leq \frac{n}{2}$ would be extremely interesting.
4. Can one prove the equivalence between a "weakened" version of the Baby Corona Problem and the Corona Problem when $\frac{1}{2} < \sigma < \frac{n}{2}$? This would be useful to approach the above problem.

Thank You!