CYCLICITY IN THE DIRICHLET SPACE

THOMAS RANSFORD

Series of five lectures presented at the summer graduate workshop on the Dirichlet space, MSRI, June 2011

Lecture 1: Introduction

Let S be a bounded linear operator on a Banach space X. For $x \in X$, write
$$[x] := \overline{\operatorname{span}\{S^n x : n \geq 0\}}.$$
This is the smallest closed S-invariant subspace containing x. We say x is *cyclic* if $[x] = X$. We shall be interested mainly in the special case $X = \mathcal{D}$ and $S : f(z) \mapsto zf(z)$ (shift). Then
$$[f] = \overline{\{pf : p \text{ is a polynomial}\}},$$
and f is cyclic iff $[f] = \mathcal{D}$.

Examples.

• $[1] = \mathcal{D}$, so 1 is cyclic.
Useful consequence: *f is cyclic iff there exist polynomials (p_n) such that $p_n f \to 1$ in $\mathcal{D}$.*

• $[z] = \{f \in \mathcal{D} : f(0) = 0\}$, so z is not cyclic.
More generally, if f cyclic, then $f(z) \neq 0$ for all $z \in \mathbb{D}$.

• $[z-1] = \mathcal{D}$, so $z-1$ is cyclic (even though it vanishes at $z = 1$).
[*Proof*: Let $f \in \mathcal{D} \ominus [z-1]$. Then $f \perp (z^{n+1} - z^n)$ for all $n \geq 1$, so $(n+1)\widehat{f}(n+1) = n\widehat{f}(n)$ for all $n \geq 1$, so $\widehat{f}(n) = c/n$ for all $n \geq 1$. Since $f \in \mathcal{D}$, we have $\sum_n n|\widehat{f}(n)|^2 < \infty$, so $\sum_n |c|^2/n < \infty$, so $c = 0$ and $f =$ constant. Finally $f \perp (z-1)$ implies $f = 0$.]

Problem. *Which $f \in \mathcal{D}$ are cyclic?*

Why do we care?

- Important step towards classifying the closed shift-invariant subspaces of $\mathcal{D}$. Each such subspace has the form $[f]$ for some $f \in \mathcal{D}$. But what is $[f]$? When is $[f] = \mathcal{D}$?
- There is a nice answer in the case of H^2: f cyclic $\iff$ f is outer.
- Even though $\mathcal{D}$ is not an algebra, in some sense $[f] \leftrightarrow$ closed ideal generated by f, and cyclic $\leftrightarrow$ invertible.

Very brief history

- Brown–Shields (1984): Systematic study of cyclicity in $\mathcal{D}$. Isolated two necessary conditions for cyclicity, and conjectured that they are sufficient.

  Partial versions of the Brown–Shields conjecture obtained in:
- Hedenmalm–Shields (1990)
- Richer–Sundberg (1994)
- El-Fallah–Kellay–Ransford (2006, 2009)

Necessary conditions for cyclicity

Recall that, if $f \in \mathcal{D}$, then the radial limit f^* exists everywhere on $\mathbb{T}$ outside a set of logarithmic capacity zero. We write c for capacity.

Theorem (Brown–Shields, 1984). *If f is cyclic in $\mathcal{D}$, then f is outer and $c(\{f^* = 0\}) = 0$.*

Proof that f is outer. If f is cyclic in $\mathcal{D}$, then f is cyclic in H^2. By Beurling's theorem, f is cyclic in H^2 iff f is outer.

Proof that $c(\{f^* = 0\}) = 0$. Set $E := \{f^* = 0\}$. Let $\mathcal{D}_E := \{g \in \mathcal{D} : g^* = 0 \text{ q.e. on } E\}$. Then $\mathcal{D}_E$ is an invariant subspace containing f (clear) and it is closed in $\mathcal{D}$ (see below). If f is cyclic, then necessarily $\mathcal{D}_E = \mathcal{D}$, in particular $1 \in \mathcal{D}_E$, which implies that $c(E) = 0$.

[To see $\mathcal{D}_E$ is closed in $\mathcal{D}$, let (g_n) be a sequence in $\mathcal{D}_E$ converging to g in $\mathcal{D}$. For each $t > 0$, we have $c(|g_n^* - g^*| > t) \leq A\|g_n - g\|_{\mathcal{D}}^2/t^2$. In particular $c(E \cap \{|g^*| > t\}) \leq A\|g_n - g\|_{\mathcal{D}}^2/t^2$. Let $n \to \infty$ and then $t \to 0$ to get $c(E \cap \{|g^*| > 0\}) = 0$. Thus $g \in \mathcal{D}_E$.] □

Conjecture (Brown–Shields). *If $f \in \mathcal{D}$ is outer and $c(\{f^* = 0\}) = 0$, then f is cyclic.*

Finally, here are two pertinent examples.

- Brown–Cohn (1985): Given a compact $E \subset \mathbb{T}$ of capacity zero, there exists a cyclic $f \in \mathcal{D} \cap A(\mathbb{D})$ such that $\{f = 0\} = E$.
  (Thus the capacity zero condition in the theorem cannot be improved.)

- Carleson (1952): Given a compact $E \subset \mathbb{T}$ satisfying
$$\int_{\mathbb{T}} \log(\operatorname{dist}(\zeta, E))\,|d\zeta| > -\infty,$$
  there exists an outer $f \in A^1(\mathbb{D})$ such that $\{f = 0\} = E$.
  (Thus the capacity condition in the theorem is not redundant.)

Lecture 2: Getting started

We begin with two notions of zero set. Let $f \in \mathcal{D}$. We write:

$$Z(f^*) := \{\zeta \in \mathbb{T} : \lim_{r\to 1^-} f(r\zeta) = 0\}$$
$$\underline{Z}(f) := \{\zeta \in \mathbb{T} : \liminf_{z\to\zeta} |f(z)| = 0\}.$$

Note that $Z(f^*) \subset \underline{Z}(f)$, with equality if $|f|$ is continuous on $\overline{\mathbb{D}}$. Also $\underline{Z}(f)$ is closed in $\mathbb{T}$. It can happen that $|Z(f^*)| = 0$ and $\underline{Z}(f) = \mathbb{T}$ (but what if f is outer?).

We shall concentrate our efforts on the following 'weak Brown–Shields conjecture':

Problem. *If $f \in \mathcal{D}$ is outer and $c(\underline{Z}(f)) = 0$, then is f is cyclic?*

The case $\underline{Z}(f) = \emptyset$

Theorem (Brown–Shields, 1984)**.** *If $f \in \mathcal{D}$ is outer and $\underline{Z}(f) = \emptyset$, then f is cyclic.*

The proof is a simple consequence the following lemma.

Lemma (Richter–Sundberg, 1991)**.** *If $f, g \in \mathcal{D}$ and $|f| \le C|g|$ on $\mathbb{D}$, then $[f] \subset [g]$.*

Proof of Theorem. The hypothesis $\underline{Z}(f) = \emptyset$ implies that there exist $a, b > 0$ such that $|f| \ge a$ on $b < |z| < 1$. Since f is outer, it has no zeros in $\mathbb{D}$, so in fact $|f| \ge a' > 0$ on $\mathbb{D}$. By the lemma, $[f] \supset [1]$, so f is cyclic. $\square$

The case $\underline{Z}(f)$ is a singleton

Theorem (Hedenmalm–Shields (1990), Richter–Sundberg (1994))**.** *If $f \in \mathcal{D}$ is outer and $\underline{Z}(f) = \{1\}$, then f is cyclic.*

We sketch a proof due to El-Fallah–Kellay-Ransford, based on a technique of Korenblum.

Lemma 1 (Carleson, 1960)**.** *Let $f \in H^2$ be outer. Then*

$$\mathcal{D}(f) = \frac{1}{4\pi^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(\log|f^*(\lambda)| - \log|f^*(\zeta)|)(|f^*(\lambda)|^2 - |f^*(\zeta)|^2)}{|\lambda - \zeta|^2} \,|d\lambda|\,|d\zeta|.$$

Lemma 2 (Fusion lemma)**.** *Let $f_1, f_2 \in \mathcal{D}$ be outer. Suppose that $|f_j^*(\zeta)| \le Cd(\zeta, E)$, where E is a closed subset of $\mathbb{T}$ of measure zero. Let $\mathbb{T} \setminus E = U_1 \cup U_2$, where U_1, U_2 are disjoint open subsets, and let f be the outer function such that $|f^*| = |f_j^*|$ on U_j. Then $f \in \mathcal{D}$ and*

$$\mathcal{D}(f) \le \mathcal{D}(f_1) + \mathcal{D}(f_2) + \frac{C^2\pi^2}{2} \log\Big(\frac{C\pi}{|f_1(0)|}\Big) + \frac{C^2\pi^2}{2} \log\Big(\frac{C\pi}{|f_2(0)|}\Big).$$

Proof. Apply Lemma 1 to f. The contributions from $U_1 \times U_1$ and $U_2 \times U_2$ are bounded above by $\mathcal{D}(f_1)$ and $\mathcal{D}(f_2)$ respectively. It remains to bound the integral

$$\int_{U_1}\int_{U_2} \frac{(\log|f_1^*(\lambda)| - \log|f_2^*(\zeta)|)(|f_1^*(\lambda|^2 - |f_2^*(\zeta)|^2)}{|\lambda-\zeta|^2}\,|d\lambda|\,|d\zeta|.$$

If $\lambda \in U_1$ and $\zeta \in U_2$, then there is a point of E between them, so $d(\lambda,\zeta) = d(\lambda,E)+d(\zeta,E)$, and consequently

$$\Big|\frac{|f_1^*(\lambda)|^2 - |f_2^*(\zeta)|^2}{d(\lambda,\zeta)^2}\Big| \le \frac{C^2 d(\lambda,E)^2 + C^2 d(\zeta,E)^2}{d(\lambda,E)^2 + d(\zeta,E)^2} = C^2.$$

The required estimate follows easily from this. □

Lemma 3. *Let M be a closed subspace of $\mathcal{D}$ and let f be a holomorphic function on $\mathbb{D}$. Suppose that there exists a sequence (f_n) in M such that:*

- *$f_n(z) \to f(z)$ for each $z \in \mathbb{D}$;*
- *$\sup_n \mathcal{D}(f_n) < \infty$.*

Then $f \in M$.

Proof. Since (f_n) is norm-bounded in M, a subsequence converges weakly to $g \in M$. For each $z \in \mathbb{D}$, evaluation at z is continuous on $\mathcal{D}$, so $g(z) = f(z)$. Thus $f = g \in M$. □

Proof of the theorem. WLOG f is bounded (technical argument). Let $\epsilon_n \to 0^+$. Let $E_n := \{e^{i\epsilon_n}, e^{-i\epsilon_n}\}$. Let f_n be the outer function such that

$$|f_n^*(\zeta)| = \begin{cases} |(\zeta - e^{i\epsilon_n})(\zeta - e^{-i\epsilon_n})f^*(\zeta)| & \text{on the arc of } \mathbb{T}\setminus E \text{ containing } 1 \\ |(\zeta - e^{i\epsilon_n})(\zeta - e^{-i\epsilon_n})| & \text{on the other arc.} \end{cases}$$

Observe that:

- $f_n \in \mathcal{D}$ and $\sup_n \mathcal{D}(f_n) < \infty$ (fusion lemma),
- $f_n(z) \to (z-1)^2$ for each $z \in \mathbb{D}$ (dominated convergence theorem)
- $|f_n| \le C_n|f|$, so $[f_n] \subset [f]$.

By Lemma 3, applied with $M = [f]$, we have $(z-1)^2 \in [f]$. But $(z-1)$ is cyclic, so $(z-1)^2$ is cyclic, so f is cyclic. □

Remark. It seems like it takes a lot of effort to deal with the case where $\underline{Z}(f)$ is a singleton. However, essentially the same technique yields the following much more general result:

Theorem. *Let $f \in \mathcal{D}$ be outer. If $g \in \mathcal{D}$ and $|g(z)| \le C\,\mathrm{dist}(z, \underline{Z}(f))$, then $g \in [f]$.*

Lecture 3: Capacity enters the picture

Let us briefly recall how capacity is defined. The *energy* of a probability measure μ on $\mathbb{T}$ is

$$I(\mu) := \iint \log\frac{2}{|\lambda-\zeta|}\,d\mu(\lambda)\,d\mu(\zeta) = \log 2 + \sum_{n\ge 1}\frac{|\widehat{\mu}(n)|^2}{n}.$$

The *logarithmic capacity* of a compact subset E of $\mathbb{T}$ is

$$c(E) := 1/\inf\{I(\mu) : \mu \text{ is a probability measure on } E\}.$$

Our aim is to prove the following theorem. We write $E_t := \{\zeta \in \mathbb{T} : d(\zeta, E) \le t\}$.

Theorem (El-Fallah–Kellay–Ransford, 2006)**.** *Let $f \in \mathcal{D}$ be outer and let $E := \underline{Z}(f)$. Suppose that*

$$\int_0 c(E_t) \frac{\log\log(1/t)}{t\log(1/t)}\, dt < \infty. \tag{1}$$

Then f is cyclic.

Lemma 1. *Let E be a closed subset of $\mathbb{T}$. Suppose that there exists $h \in \mathcal{D}$ such that*

$$\operatorname{Re} h^*(\zeta) \ge \log\log \frac{2}{d(\zeta, E)} \quad \textit{and} \quad |\operatorname{Im} h^*(\zeta)| \le 1.$$

If $f \in \mathcal{D}$ is outer and $\underline{Z}(f) \subset E$, then f is cyclic.

Proof. Let $\Lambda := \{\lambda \in \mathbb{C} : |\arg\lambda| < 1/2\}$. For $\lambda \in \Lambda$, define

$$g_\lambda(z) := \exp(-\lambda e^{h(z)}) \qquad (z \in \mathbb{D}).$$

It is easy to show that:

- $g_\lambda \in \mathcal{D}$ for all $\lambda \in \Lambda$,
- $\lambda \mapsto g_\lambda : \Lambda \to \mathcal{D}$ is holomorphic,
- $|g_\lambda(z)| \le \operatorname{dist}(z, E)^{\lambda\cos(1)}$ $(\lambda > 0)$,
- $\|g_\lambda - 1\|_{\mathcal{D}} \to 0$ as $\lambda \to 0,\ \lambda > 0$.

By the theorem at the end of Lecture 2, if $\lambda > 0$ sufficiently large, then $g_\lambda \in [f]$. By the identity principle, $g_\lambda \in [f]$ for all $\lambda \in \Lambda$. Hence $1 \in [f]$. $\square$

For which sets E does such an h exist? Note that, in general, if $h \in \mathcal{D}$ and

$$|h^*(\zeta)| \ge \phi(d(\zeta, E)) \quad \text{q.e.}$$

where $\phi : (0, \pi] \to \mathbb{R}^+$ is an decreasing function, then $|h^*| \ge \phi(t)$ on E_t, and so by the strong-type inequality for capacity,

$$\int_0 c(E_t)\, |d\phi^2(t)| < \infty. \tag{2}$$

There is a converse.

Lemma 2. *Let E be a closed subset of $\mathbb{T}$, and let $\phi : (0, \pi] \to \mathbb{R}^+$ be decreasing and continuous. If (2) holds, then there exists $h \in \mathcal{D}$ such that, q.e. on $\mathbb{T}$,*

$$\operatorname{Re} h^*(\zeta) \ge \phi(d(\zeta, E)) \quad \textit{and} \quad |\operatorname{Im} h^*(\zeta)| \le 1.$$

We shall need an auxiliary result about Hilbert spaces, whose proof is left as an exercise.

Lemma 3. *Let (h_n) be a sequence in a Hilbert space H such that $(h_m - h_n) \perp h_n$ whenever $m \ge n$. Then $\sum_n h_n/\|h_n\|^2$ converges in H if and only if $\sum_n n/\|h_n\|^2 < \infty$.*

Sketch of proof of Lemma 2. Choose $\delta_n > 0$ so that $\phi(\delta_n) = n$ and set $c_n := c(E_{\delta_n})$. The condition (2) is then equivalent to $\sum_n nc_n < \infty$. By forgetting the first few n, we may also suppose that $\sum_n c_n < 1/2$. Let μ_n be the equilibrium measure on E_{δ_n}. This is the unique probability measure on E_{δ_n} such that $I(\mu_n) = 1/c_n$. Define

$$h_n(z) := -\int \log(1 - ze^{-i\theta})\, d\mu_n(\theta) \qquad (z \in \mathbb{D}).$$

Note that h_n is holomorphic and

$$\|h_n\|_{\mathcal{D}}^2 = \sum_{k\geq 1} k|\widehat{h}_n(k)|^2 = \sum_{k\geq 1} \frac{|\widehat{\mu}_n(k)|^2}{k} = I(\mu_n) - \log 2 = 1/c_n + O(1).$$

Using Lemma 3, it follows that $h := \sum_n c_n h_n$ converges in $\mathcal{D}$. Further,

$$|\operatorname{Im} h| \leq \sum_n c_n \pi/2 < 1$$

and, if $\zeta \in E_{\delta_N}$,

$$\operatorname{Re} h^*(\zeta) \geq \sum_{n=1}^N c_n \int \log \frac{1}{|e^{i\theta} - \zeta|}\, d\mu_n(\theta) = \sum_{n=1}^N c_n I(\mu_n) = N + O(1).$$

Thus, after a small adjustment, $\operatorname{Re} h^*(\zeta) \geq \phi(d(\zeta, E))$. □

Proof of Theorem. Apply the preceding results with $\phi(t) = \log\log(2/t)$. Condition (2) translates into condition (1). □

Lecture 4: Measure-theoretic criteria

In the previous section we encountered the condition

$$\int_0 c(E_t) \frac{\log\log(1/t)}{t \log(1/t)}\, dt < \infty.$$

This is hard to check in practice, because of the difficulty in estimating $c(E_t)$. It is implied by a stronger condition, expressed in terms of $|E_t|$ (the Lebesgue measure of E_t), namely:

$$\int_0 \frac{|E_t|}{(t \log(1/t))^2}\, dt < \infty.$$

In particular, there exist Cantor-type sets that satisfy this latter condition, thereby providing examples of infinite sets for which the weak Brown–Shields conjecture holds. However, even for Cantor-type sets, this condition is strictly stronger than capacity zero. From this point of view, the following theorem is better.

Theorem (El-Fallah–Kellay–Ransford, 2009)**.** *Let $f \in \mathcal{D}$ be outer and let $E = \underline{Z}(f)$. Suppose that $|E_t| = O(t^\alpha)$ for some $\alpha > 0$, and that*

$$\int_0 \frac{dt}{|E_t|} = \infty.$$

Then f is cyclic.

Remark. For Cantor-type sets, the condition $|E_t| = O(t^\alpha)$ is automatic, and the condition $\int_0 dt/|E_t| = \infty$ is *equivalent* to $c(E) = 0$.

Let E be a closed subset of $\mathbb{T}$ of measure zero. Let $w : [0, \pi] \to \mathbb{R}^+$ be a continuous, increasing function such that $\int_{\mathbb{T}} |\log w(d(\zeta, E))|\,|d\zeta| < \infty$. We write h_w for the outer function satisfying

$$|h_w(\zeta)| = w(d(\zeta, E)) \quad (\zeta \in \mathbb{T}).$$

The first lemma is a simplified form of Carleson's formula for these 'distance functions' h_w.

Lemma 1. *Suppose that $t \mapsto w(t^\gamma)$ is concave, for some $\gamma > 0$. Then*

$$\mathcal{D}(h_w) \le \begin{cases} C_\gamma \int_0^\pi w'(t)^2 |E_t|\,dt & \text{if } \gamma > 2 \\ C_\gamma \int_0^\pi w'(t)^2 t^{1-2/\gamma} \log(\pi/t) |E_t|\,dt & \text{if } \gamma < 2. \end{cases}$$

We shall also need:

Lemma 2 (Richter–Sundberg, 1992)**.** *Let $g \in \mathcal{D}$ be an outer function, let $\beta > 0$ and suppose that $g^\beta \in \mathcal{D}$. Then $[g^\beta] = [g]$.*

Sketch of proof of the theorem. Let $g := h_w$, where $w(t) = t$. By Lemma 1 (with $\gamma = 1$),

$$\mathcal{D}(g) \le C \int_0^\pi t^{-1} \log(\pi/t) |E_t|\,dt \le C' \int_0^\pi \log(\pi/t) t^{\alpha-1}\,dt < \infty,$$

so $g \in \mathcal{D}$. Also $|g(z)| \le C \operatorname{dist}(z, E)$. By the Theorem at the end of Lecture 2, we have $g \in [f]$. So it is enough to prove that g is cyclic.

Now fix $\beta \in (\frac{1-\alpha}{2}, \frac{1}{2})$, and consider g^β. Note that $g^\beta = h_w$, where $w(t) = t^\beta$, so by Lemma 1 (this time with $\gamma = 1/\beta$),

$$\mathcal{D}(g^\beta) \le C \int_0^\pi (\beta t^{\beta-1})^2 |E_t|\,dt \le C' \int_0^\pi t^{2\beta-2+\alpha}\,dt < \infty,$$

so $g^\beta \in \mathcal{D}$. By Lemma 2 we have $[g^\beta] = [g]$, so it is enough to prove that g^β is cyclic.

For $\delta \in (0, 1)$, define $w_\delta : [0, \pi] \to [0, 1]$ by

$$w_\delta(t) := \begin{cases} t^\beta, & 0 \le t \le \delta \\ A_\delta - \log \int_t^\pi ds/|E_s|, & \delta \le t \le \eta_\delta \\ 1, & \eta_\delta \le t \le \pi. \end{cases}$$

Here A_δ and η_δ are constants chosen to make w_δ continuous. It is easy to check that $\eta_\delta \to 0$ as $\delta \to 0$. By Lemma 1 again,

$$\begin{aligned}\mathcal{D}(h_{w_\delta}) &\le C_\gamma \int_0^\pi w_\delta'(t)^2 |E_t|\,dt \\ &\le C_\gamma \int_0^\delta (\beta t^{\beta-1})^2 |E_t|\,dt + C_\gamma \int_\delta^{\eta_\delta} \frac{dt}{|E_t|(\int_t^\pi ds/|E_s|)^2} \\ &\le C \int_0^\delta t^{2\beta-2+\alpha}\,dt + C\Big(\int_\delta^\pi \frac{ds}{|E_s|}\Big)^{-1} \\ &\to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

Thus $h_{w_\delta} \to 1$ in $\mathcal{D}$ as $\delta \to 0$. Also, for each $\delta > 0$, the quotient $w_\delta(t)/t^\beta$ is bounded, so $[h_{w_\delta}] \subset [g^\beta]$. Hence $1 \in [g^\beta]$, as required.

Actually, we cheated, because $t \mapsto w_\delta(t^\gamma)$ is not obviously a concave function for any $\gamma > 2$. We need to modify the definition of w_δ, replacing $s \mapsto |E_s|$ by a regularized function of s. The details are omitted. □

Remark. The regularization mentioned above proceeds via the following lemma, which may be of independent interest.

Lemma. *Let $u : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $u(x) - x$ is decreasing. Define*

$$\widetilde{u}(x) := \inf\{u(y) : y \ge x\}.$$

Then $\widetilde{u} = u$ on a set of lower density at least $\liminf_{x\to\infty} u(x)/x$.

Lecture 5: Approach via duality

Recall that

$$\mathcal{D} = \Big\{ f(z) = \sum_{k\ge 0} a_k z^k : \sum_{k\ge 0} (k+1)|a_k|^2 < \infty \Big\}.$$

Its dual may be identified with

$$\mathcal{B}_e := \Big\{ \phi(z) = \sum_{k\ge 0} b_k / z^{k+1} : \sum_{k\ge 0} |b_k|^2/(k+1) < \infty \Big\},$$

the duality being given by

$$\langle f, \phi \rangle := \sum_{k\ge 0} a_k b_k.$$

Theorem (Hedenmalm–Shields (1990), Richter–Sundberg (1994))**.** *If $f \in \mathcal{D}$ is outer and $\phi \in [f]^\perp$, then ϕ extends to be holomorphic on $\mathbb{C} \setminus \underline{Z}(f)$, and $\phi|_{\mathbb{D}}$ belongs to the Smirnov class $\mathcal{N}^+$.*

Proof. We sketch the proof in the case when f continuous on $\overline{\mathbb{D}}$. Consider the Banach algebra $A := \mathcal{D} \cap A(\mathbb{D})$, and let I be the closed ideal generated by f. Note that $I \subset [f]$, so $\phi(I) = 0$. Thus ϕ induces a continuous linear functional $\widetilde{\phi}$ on the quotient algebra A/I. The character space of A can be identified with $\overline{\mathbb{D}}$, and that of A/I with $Z(f)$. In particular, the spectrum of z in A/I is $Z(f)$. Define $\psi : \mathbb{C} \setminus Z(f) \to \mathbb{C}$ by

$$\psi(w) := \langle (w-z)^{-1}, \widetilde{\phi} \rangle_{A/I} \qquad (w \in \mathbb{C} \setminus Z(f)).$$

Then ψ is holomorphic in $\mathbb{C} \setminus Z(f)$. Further, if $|w| > 1$, then

$$\psi(w) = \langle (z-w)^{-1}, \phi \rangle_A = \Big\langle \sum_{k\geq 0} z^k/w^{k+1}, \phi \Big\rangle_A = \sum_{k \geq 0} \langle z^k, \phi \rangle_A / w^{k+1} = \phi(w),$$

so ψ is an analytic continuation of ϕ. Also, for $|w| < 1$, we have

$$\Big\langle \frac{f(w) - f(z)}{w-z}, \phi \Big\rangle_A = \langle f(w)(w-z)^{-1}, \widetilde{\phi} \rangle_{A/I} = f(w)\psi(w).$$

The left-hand side can be expressed as the Cauchy transform of a finite measure on $\mathbb{T}$. As such, it belongs to $\cap_{p<1} H^p$, and in particular to $\mathcal{N}^+$. Since $\phi|_{\mathbb{D}}$ is the quotient of the left-hand side by the bounded outer function f, it follows that $\phi|_{\mathbb{D}} \in \mathcal{N}^+$. □

A closed subset E of $\mathbb{T}$ is called a *Bergman–Smirnov exceptional set* if

$$\left.\begin{array}{l} \phi \in \mathrm{Hol}(\mathbb{C} \setminus E) \\ \phi|_{\mathbb{D}_e} \in \mathcal{B}_e \\ \phi|_{\mathbb{D}} \in \mathcal{N}^+ \end{array}\right\} \Rightarrow \phi \equiv 0.$$

Corollary. *If $f \in \mathcal{D}$ is outer and $\underline{Z}(f)$ is a Bergman–Smirnov exceptional set, then f is cyclic.*

Proof. Just combine the theorem above with the Hahn–Banach theorem. □

Problem. *Which subsets E of $\mathbb{T}$ are Bergman–Smirnov exceptional sets?*

Obviously the empty set is one. The next step is:

Theorem. *A singleton is a Bergman–Smirnov exceptional set.*

For this we use the following generalized maximum principle.

Lemma (Solomjak, 1983)**.** *Let E be a closed subset of $\mathbb{T}$ and let ϕ be holomorphic on $\mathbb{C} \setminus E$. Suppose that*

$$\log|\phi(z)| \leq \rho(\mathrm{dist}(z, \mathbb{T})) \qquad (z \in \mathbb{C} \setminus E),$$

where $\rho : (0,\infty) \to (0,\infty)$ is a decreasing function with $\sup_{t>0} \rho(t)/\rho(2t) < \infty$. Then there exists a constant C such that

$$\log|\phi(z)| \leq C\rho(\mathrm{dist}(z, E)) \qquad (z \in \mathbb{C} \setminus E).$$

Proof of the theorem. Let $\phi \in \mathrm{Hol}(\mathbb{C} \setminus \{1\})$ with $\phi|_{\mathbb{D}_e} \in \mathcal{B}_e$ and $\phi|_{\mathbb{D}} \in \mathcal{N}^+$. Since $\phi|_{\mathbb{D}_e} \in \mathcal{B}_e$, we have

$$|\phi(z)| \leq \frac{C}{|z| - 1} \qquad (|z| > 1).$$

Also, since $\phi|_{\mathbb{D}} \in \mathcal{N}^+$, we have

$$\log|\phi(z)| \leq \frac{C}{1-|z|} \qquad (|z|<1).$$

Using the lemma, it follows that

$$\log|\phi(z)| \leq \frac{C'}{|z-1|} \qquad (z \in \mathbb{C}\setminus\{1\}).$$

Combining this with the first estimate on ϕ, we deduce that

$$|\phi(z)| \leq \frac{C''}{|z-1|^2} \qquad (1<|z|<2).$$

In particular $(z-1)^2\phi(z)$ is bounded on $\mathbb{T}\setminus\{1\}$. As $(z-1)^2\phi(z)$ is Smirnov on $\mathbb{D}$, it is also bounded inside $\mathbb{D}$. Thus, at worst, 1 is a pole of ϕ. Moreover, as $\phi|_{\mathbb{D}_e} \in \mathcal{B}_e$, it cannot have a pole at 1. So 1 is a removable singularity. Thus ϕ is entire and hence $\phi \equiv 0$. $\square$

From this, we can deduce the result for countable sets.

Theorem (Hedenmalm–Shields, 1990)**.** *Every countable closed subset E of $\mathbb{T}$ is a Bergman–Smirnov exceptional set.*

Proof. Let $\phi \in \mathrm{Hol}(\mathbb{C}\setminus E)$ with $\phi|_{\mathbb{D}_e} \in \mathcal{B}_e$ and $\phi|_{\mathbb{D}} \in \mathcal{N}^+$. Let E_1 be the (closed) subset of E consisting of those points across which ϕ cannot be continued analytically. If $E_1 \neq \emptyset$, then it contains an isolated point ζ. Using the Cauchy integral, we can decompose ϕ as $\phi_1+\phi_2$, where ϕ_1 is holomorphic in $\mathbb{C}\setminus(E\setminus\{\zeta\})$ and ϕ_2 is holomorphic in $\mathbb{C}\setminus\{\zeta\}$. Outside $\overline{\mathbb{D}}$ and near ζ, both ϕ and ϕ_1 are square-integrable, whence so is ϕ_2. It follows that $\phi_2|_{\mathbb{D}_e} \in \mathcal{B}_e$. Likewise, on $\mathbb{T}$ and near ζ, both $(\log^+|\phi(re^{i\theta})|)_{r<1}$ and $(\log^+|\phi_1(re^{i\theta})|)_{r<1}$ are uniformly integrable, whence so is $(\log^+|\phi_2(re^{i\theta})|)_{r<1}$. It follows that $\phi_2|_{\mathbb{D}} \in \mathcal{N}^+$. By the preceding theorem, $\phi_2 \equiv 0$. Hence $\phi = \phi_1$, which is holomorphic at ζ, contradicting the fact that $\zeta \in E_1$. We conclude that E_1 is empty, that ϕ is entire and hence that $\phi \equiv 0$. $\square$

To prove the weak Brown–Shields conjecture, it would suffice to show that every compact subset E of $\mathbb{T}$ of capacity zero is a Bergman–Smirnov exceptional set. This is still an open problem. Carleson has shown that the Bergman–Bergman exceptional sets are precisely the sets of capacity zero.

References

[1] L. Brown, W. Cohn, 'Some examples of cyclic vectors in the Dirichlet space', *Proc. Amer. Math. Soc.* **95** (1985), 42–46.

[2] L. Brown, A. Shields, 'Cyclic vectors in the Dirichlet space', *Trans. Amer. Math. Soc.* **285** (1984), 269–304.

[3] L. Carleson, 'Sets of uniqueness for functions regular in the unit circle', *Acta Math.* **87** (1952), 325–345.

[4] L. Carleson, 'A representation formula for the Dirichlet integral', *Math. Z.* **73** (1960), 190–196.

[5] O. El-Fallah, K. Kellay, T. Ransford, 'Cyclicity in the Dirichlet space', *Ark. Mat.* **44** (2006), 61–86.

[6] O. El-Fallah, K. Kellay, T. Ransford, On the Brown–Shields conjecture for cyclicity in the Dirichlet space, *Adv. Math.* 222 (2009), 2196–2214.

[7] H. Hedenmalm, A. Shields, 'Invariant subspaces in Banach spaces of analytic functions', *Michigan Math. J.* **37** (1990), 91–104.
[8] S. Richter, C. Sundberg, 'A formula for the local Dirichlet integral', *Michigan Math. J.* **38** (1991), 355–379.
[9] S. Richter, C. Sundberg, 'Multipliers and invariant subspaces in the Dirichlet space', *J. Operator Theory* **28** (1992), 167–186.
[10] S. Richter, C. Sundberg, 'Invariant subspaces of the Dirichlet shift and pseudocontinuations', *Trans. Amer. Math. Soc.* **341** (1994), 863–879.
[11] B. Solomjak, 'Calculuses, annihilators and hyperinvariant subspaces', *J. Operator Theory* **9** (1983), 341–370.

Exercises

1. Let S be a bounded linear operator on a Banach space X. A vector $x \in X$ is called *hypercyclic* for S if the set $\{S^n x : n \geq 0\}$ is dense in X (without taking the span). Which $f \in \mathcal{D}$ are hypercyclic for the shift?

2. Give an example of a function $f \in \mathcal{D}$ such that $f^2 \notin \mathcal{D}$.

3. Let $f \in A^1(\mathbb{D})$ with $f \not\equiv 0$. Let $E := \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$. Prove that

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E)\, |d\zeta| > -\infty.$$

4. Show that there exists $f \in \mathcal{D}$ such that $|Z(f^*)| = 0$ and $\underline{Z}(f) = \mathbb{T}$. Can f be chosen to be outer?

5. Prove the following lemma, used in Lecture 3. Let (h_n) be a sequence in a Hilbert space H such that $(h_m - h_n) \perp h_n$ whenever $m \geq n$. Then $\sum_n h_n/\|h_n\|^2$ converges in H if and only if $\sum_n n/\|h_n\|^2 < \infty$.