Richter, Lecture 2

The wandering subspace theorem for 2-isos

Theorem: If $S \in \mathcal{B}(\mathcal{H})$ is a 2-isometry,

i.e. $$\|Sx\|^2 - \|x\|^2 = \|S^2x\|^2 - \|Sx\|^2 \quad \forall x,$$

and if S is analytic, i.e. $\bigcap_{n\geq 0} S^n\mathcal{H} = (0)$,

then $$\mathcal{H} = [\mathcal{K}]_S = \bigvee_{n\geq 0} S^n\mathcal{K}, \text{ where}$$

$$\mathcal{K} = \ker S^* = (\operatorname{ran} S)^\perp = \mathcal{H} \ominus S\mathcal{H}.$$

Recall: $$\|Sx\|^2 - \|x\|^2 \geq 0$$

$$D = (S^*S - I)^{1/2}$$

$$\|Dx\|^2 = \|Sx\|^2 - \|x\|^2$$

$$\|DSx\|^2 = \|Dx\|^2$$

$$\|S^nx\|^2 - \|x\|^2 = \sum_{k=0}^{n-1} \|S^{k+1}x\|^2 - \|S^kx\|^2$$

$$= \sum_{k=0}^{n-1} \|DS^kx\|^2 = n\|Dx\|^2$$

Special case of the proof:

Let $\mathcal{M} \subseteq \mathcal{H} \subseteq Hol(\mathbb{D})$

Think of $\mathcal{H} = D$ the Dirichlet space

$\mathcal{M} \in Lat(M_z, D)$, $\dim \mathcal{M} \ominus z\mathcal{M} = 1$

We'll prove the theorem for $T = M_z | \mathcal{M}$, so

$\mathcal{H}$ in the statement of the theorem will be $\mathcal{M}$ in the proof. $\mathcal{H}$ will disappear.

Let $\varphi \in \underset{\|\varphi\| = 1}{\mathcal{M} \ominus z \mathcal{M}}^{= \mathcal{K}}$, so $\mathcal{K} = \{c\varphi : c \in \mathbb{C}\}$

Let $f \in \mathcal{M}$

<u>to show</u>: $\exists\, p_n$ polys $\ni p_n \varphi \longrightarrow f$

We will also assume that $\varphi(0) \neq 0$

Then $g = \frac{f}{\varphi} \in Hol(\varepsilon \mathbb{D})$, $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ for $|z| < \varepsilon$

So $f = g\varphi$. Will use $p_n(z) = \sum_{\ell=0}^{n} \hat{g}(\ell) z^\ell$, $p_{n_j} \varphi \to f$

$f \mapsto \langle f, \varphi\rangle \varphi$ is the projection onto $\mathcal{K}$ since $\|\varphi\| = 1$

So $f \longmapsto f - \langle f, \varphi \rangle \varphi$ is the projection onto $\mathcal{K}^{\perp} = z\mathcal{M}$. Hence

$$Lf = \frac{f - \langle f, \varphi\rangle \varphi}{z} \text{ takes } \mathcal{M} \to \mathcal{M}$$

Note: Since $\mathcal{M} \subseteq Hol(\mathbb{D})$ we must have

$$f(0) - \langle f, \varphi\rangle \varphi(0) = 0$$

so $\quad g(0) = \frac{f}{\varphi}(0) = \langle f, \varphi\rangle$ and

$$Lf = \frac{g\varphi - g(0)\varphi}{z} = \frac{g - g(0)}{z}\varphi$$

$$Lf(z) = \sum_{k=1}^{\infty} \hat{g}(k) z^{k-1} \varphi(z)$$

We will show that if $p_n(z) = \sum_{\ell=0}^{n} \hat{g}(\ell) z^{\ell}$

then a subsequence $\{p_{n_j}\}$ satisfies

$p_{n_j}\varphi \to f$ weakly.

Fact: $h_n \to h$ weakly $\iff \begin{cases} \exists c \quad \|h_n\| \le c \ \forall n \\ h_n(z) \to h(z) \ \forall z \in \mathbb{D} \end{cases}$

~~Since~~ Note $\quad f = g\varphi = g(0)\varphi + z\frac{g - g(0)}{z}\varphi$

$$\|f\|^2 = \|g(0)\varphi + zLf\|^2 = |g(0)|^2 + \|zLf\|^2$$

$$= |g(0)|^2 + \|Lf\|^2 + \|zLf\|^2 - \|Lf\|^2$$

$$= |g(0)|^2 + \|Lf\|^2 + \|DLf\|^2$$

Similarly $Lf(z) = \sum_{\ell=1}^{\infty} \hat{g}(k) z^{\ell-1} \varphi(z)$

So $= \hat{g}(1)\varphi + z \sum_{\ell=2}^{\infty} \hat{g}(k) z^{\ell-2} \varphi(z)$

$$\|Lf\|^2 = |\hat{g}(1)|^2 + \|L^2 f\|^2 + \|DL^2 f\|^2$$

$$\Rightarrow \|f\|^2 = |\hat{g}(0)|^2 + |\hat{g}(1)|^2 + \|L^2 f\|^2 + \|DLf\|^2 + \|DL^2 f\|^2$$

$$\Rightarrow \|f\|^2 \geq \sum_{\ell=0}^{\infty} |\hat{g}(k)|^2 + \sum_{\ell=1}^{\infty} \|DL^\ell f\|^2 \quad (*)$$

$\Rightarrow g \in H^2 \subseteq Hol(\mathbb{D})$

$$f(z) = g\varphi(z) = \left(\sum_{\ell=0}^{n-1} \hat{g}(k) z^\ell\right)\varphi + \left(\sum_{\ell=n}^{\infty} \hat{g}(k) z^\ell\right)\varphi$$

$$f(z) = p_{n-1}\varphi + z^n L^n f$$

where $p_n(z) \to g(z)$ $\forall z \in \mathbb{D}$ (since $g \in H^2$)

by $(*)$

Thus we must show $z^{n_j} L^{n_j} f \to 0$ weakly for some $n_j \to \infty$. Since we know that $p_{n-1}(z) \to g(z)$ $\forall z$ so $z^n L^n f(z) \to 0$ $\forall z$ we only need

$$\|z^{n_j} L^{n_j} f\| \leq C$$

Recall from page 1 that

$$\|z^n L^n f\|^2 - \|L^n f\|^2 = n\|DL^n f\|^2$$

Hence from ⊗

$$\|f\|^2 \geq \|g\|_{H^2}^2 + \sum_{k=1}^{\infty} \frac{1}{k}\left(\|z^k L^k f\|^2 - \|L^k f\|^2\right)$$

$$\Rightarrow \lim_{k \to \infty} \|z^k L^k f\|^2 - \|L^k f\|^2 = 0$$

Note $\|f\|^2 = |\hat{g}(0)|^2 + \|Lf\|^2 + \|DLf\|^2$
$\geq \|Lf\|^2 \quad \forall f$ so

$\|L\| \leq 1 \Rightarrow \|L^k f\| \leq \|f\|$

$$\Rightarrow \varlimsup_{k} \|z^k L^k f\| < \infty$$

$\exists\ k_j \to \infty \ni z^{k_j} L^{k_j} f \to h$ weakly

we already know that $h = 0$ from earlier, but we see that $h \in z^n \mathcal{M}\ \forall n$ so $h \in \bigcap_n z^n \mathcal{M} = (0)$ by the analyticity of $T = M_z|\mathcal{M}$. //

This page was left out in the lecture:

The general case follows along the same lines: We observe that $T^*T \geq I$ implies that $(T^*T)^{-1}$ exists. We set

$L = (T^*T)^{-1}T^*$, so $LT = I$

(L is a left inverse of T)

One verifies that TL is a projection hence $P = I - TL$ is a projection

$\ker TL = \ker T^* = (\mathcal{H} \ominus T\mathcal{H})$

$\operatorname{ran} P = \mathcal{H} \ominus T\mathcal{H} = \mathcal{K}$

$$I = P + TL = P + T(P+TL)L$$
$$I = P + TPL + T^2L^2$$

$$I = \underbrace{\sum_{k=0}^{n-1} T^k P L^k}_{} + T^n L^n$$

$$x = \underbrace{\sum_{k=0}^{n-1} T^k P L^k x}_{\in [\mathcal{K}]} + T^n L^n x$$

Now show that $\|T^{n_j} L^{n_j} x\| \leq C$ for some n_j (similar to above)

Theorem $T \in \mathcal{B}(\mathcal{H})$

TFAE

(a) T is an analytic 2-isometry with

$$\dim \mathcal{H} \ominus T\mathcal{H} = 1$$

(b) T is unitarily equivalent to $(M_z, D(\mu))$

$$\|p\|^2_{D(\mu)} = \|p\|^2_{H^2} + \int D_\zeta(p)\, d\mu(\zeta)$$

$$D_\zeta(p) = \int \left| \frac{p(z) - p(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}$$

pf: (b) $\Rightarrow$ (a) is in a way routine. Part of it is on the exercises and part will follow from a later lecture.

(a) $\Rightarrow$ (b)

Let $x_0 \in \mathcal{H} \ominus T\mathcal{H}$, $\|x_0\| = 1$

Will show: If q is a polynomial, then

$$\|q(T)x_0\|^2 = \|q\|^2_{H^2} + \int D_\zeta(q)\, d\mu(\zeta)$$

for some $\mu \in M_+(\mathbb{T})$

Then $U: q(T)x_0 \longrightarrow q$,

takes a dense subset of $\mathcal{H}$ (use the wandering subspace theorem to see this) isometrically into $D(\mu)$. Hence U extends to be isometric

The implication (b) $\Rightarrow$ (a) together with the wandering subspace theorem implies that the polys are dense in $D(\mu)$, hence U is unitary and $UT = M_z U$.

For a polynomial q set

$$q_z(\zeta) = \frac{q(z) - q(\zeta)}{z - \zeta}, \text{ so } D_\zeta(q) = \int |q_z(\zeta)|^2 \frac{|dz|}{2\pi}$$

Then

$$\int D_\zeta(q)\, d\mu(\zeta) = \int_{|z|=1} \int_{|\zeta|=1} |q_z(\zeta)|^2 \, d\mu(\zeta) \frac{|dz|}{2\pi}$$

Last time we saw $\exists \mu \in M_+(\mathbb{T})$ such that

$$\| D q(T) x_0 \|^2 = \int |q|^2 d\mu \quad \forall q \text{ poly}$$

So $\int D_{\mathfrak{J}}(q)\, d\mu = \int_{|z|=1} \| D\, q_z(T) x_0 \|^2 \frac{|dz|}{2\pi}$

We calculate for $|z| = 1$

$\| D\, q_z(T) x_0 \|^2 = \| T q_z(T) x_0 \|^2 - \| q_z(T) x_0 \|^2$

$= \| (T - z) q_z(T) x_0 + z q_z(T) x_0 \|^2 - \| q_z(T) x_0 \|^2$

$= \| (q(T) - q(z)) x_0 \|^2$

$\quad + 2 \operatorname{Re} \langle (q(T) - q(z)) x_0 , z q_z(T) x_0 \rangle + 0$

$= \| q(T) x_0 \|^2 - 2 \operatorname{Re} \langle q(T) x_0 , q(z) x_0 \rangle$

$\quad + |q(z)|^2 + 2 \operatorname{Re} \langle q(T) x_0 , z q_z(T) x_0 \rangle$

$\quad - 2 \operatorname{Re} \langle q(z) x_0 , z q_z(T) x_0 \rangle$

<u>Note</u> : $\langle q(T) x_0 , x_0 \rangle = q(0)$ since
$x_0 \in \mathcal{H} \ominus T\mathcal{H}$
$\| x_0 \| = 1$

So
$\int D_{\mathfrak{J}}(q)\, d\mu = \int_{|z|=1} \| D\, q_z(T) x_0 \|^2 \frac{|dz|}{2\pi} =$

$$= \|q(T)x_0\|^2 - 2\,\mathrm{Re}\int_{|z|=1} \overline{q(z)}\; q(0)\,\frac{|dz|}{2\pi} + \|q\|_{H^2}^2$$

$$+ 0 - 2\,\mathrm{Re}\int q(z)\,\overline{z\,q_z(0)}\,\frac{|dz|}{2\pi}$$

$$= \|q(T)x_0\|^2 - 2\,|q(0)|^2 + \|q\|_{H^2}^2$$

$$- 2\,\mathrm{Re}\int q(z)\,\overline{q(z) - q(0)}\,\frac{|dz|}{2\pi}$$

$$= \|q(T)x_0\|^2 - \|q\|_{H^2}^2$$

$$\Rightarrow \|q(T)x_0\|^2 = \|q\|_{H^2}^2 + \int D_\zeta(q)\,d\mu \quad //$$