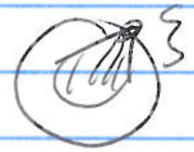
Richter: Lecture 3, part 1. ①

## The local Dirichlet integral

Note: If $f \in H^2$, $\alpha \in \mathbb{C}$, $\zeta \in \mathbb{T}$, and if

$$\int \left| \frac{f(z)-\alpha}{z-\zeta} \right|^2 \frac{|dz|}{2\pi} < \infty$$

then $\text{nt-}\lim_{\lambda \to \zeta} f(\lambda) = \alpha$

pf:  $\lambda \in \Gamma(\zeta) \iff \exists\, c \quad \frac{|\lambda-\zeta|}{1-|\lambda|^2} \le c$

$$g(\lambda) = \frac{f(\lambda)-\alpha}{\lambda-\zeta} \in H^2$$

$$\begin{aligned} \Rightarrow |f(\lambda)-\alpha|^2 &= |\lambda-\zeta|^2 |g(\lambda)|^2 \\ &\le |\lambda-\zeta|^2 \|g\|^2_{H^2} \|k_\lambda\|^2_{H^2} \\ &= \frac{|\lambda-\zeta|^2}{1-|\lambda|^2} \|g\|^2_{H^2} \qquad k_\lambda(z) = \frac{1}{1-\bar{\lambda} z} \\ &\le c\,|\lambda-\zeta|\, \|g\|^2_{H^2} \to 0 \end{aligned}$$

Def: $f \in H^2$, $\zeta \in \mathbb{T}$

$$D_\zeta(f) = \begin{cases} \infty & \text{if } f \text{ does not have a nontangential limit at } \zeta \\ \int \left| \frac{f(z)-f(\zeta)}{z-\zeta} \right|^2 \frac{|dz|}{2\pi} & \text{if } f \text{ has nontangential limit } f(\zeta) \text{ at } \zeta. \end{cases}$$

<u>Exercise</u>: If $f \in H^2$, $D_\zeta(f) < \infty$, then

$$\lim_{\substack{\lambda \to \zeta \\ \lambda \in \Omega_c(\zeta)}} f(\lambda) = f(\zeta)$$

$$\Omega_c(\zeta) = \left\{ \lambda \in \mathbb{D} : \frac{|\lambda - \zeta|^2}{1 - |\lambda|^2} \leq c \right\} \text{ for some } c > 0$$

# A fact about the local Dirichlet integral

<u>Thm</u>: $\zeta \in \overline{\mathbb{D}}$, $f \in H^2$, $r < 1$, $f_r(z) = f(rz)$

$$\Rightarrow \quad D_\zeta(f_r) \le 3\, D_\zeta(f)$$

(actually one can get $D_\zeta(f_r) \le D_\zeta(f)$

D. Sarason )

Pf: By problem 4 we may assume $|\zeta| < 1$.

$f \in H^2 \Rightarrow D_\zeta(f) < \infty$ and

$$g(z) = \frac{f(z) - f(\zeta)}{z - \zeta} \in H^2 \qquad \|g\|_{H^2}^2 = D_\zeta(f)$$

$$f(z) = f(\zeta) + (z - \zeta)\, g(z)$$

$$f_r(z) = f(\zeta) + (rz - \zeta)\, g(rz), \quad f_r(\zeta) = f(\zeta) + (r\zeta - \zeta)\, g_r(\zeta)$$

$$\frac{f_r(z) - f_r(\zeta)}{z - \zeta} = \frac{(rz \overset{\pm r\zeta}{-} \zeta)\, g_r(z) - (r\zeta - \zeta)\, g_r(\zeta)}{z - \zeta}$$

$$= r\, g_r(z) + \zeta(r - 1)\, \frac{g_r(z) - g_r(\zeta)}{z - \zeta}$$

$$\Rightarrow D_\zeta(f_r) \le 2\left(\|g_r\|_{H^2}^2 + (1-r)^2 \left\| \frac{g_r - g_r(\zeta)}{z-\zeta}\right\|_{H^2}^2\right)$$

<u>to show</u>: $(1-r)^2 \left\| \frac{g_r - g_r(\zeta)}{z-\zeta}\right\|_{H^2}^2 \le \frac{1}{2}\|g\|_{H^2}$

$$g(rz) = \frac{1}{2\pi i}\int_{|w|=1} \frac{g(w)}{w-rz}\,dw \quad \text{Cauchy for } g \in H^2$$

$$g(r\zeta) = \frac{1}{2\pi i}\int_{|w|=1} \frac{g(w)}{w-r\zeta}\,dw$$

$$\frac{g(rz)-g(r\zeta)}{z-\zeta} = \frac{1}{2\pi i}\int_{|w|=1} g(w)\left(\frac{(w-r\zeta)-(w-rz)}{(w-rz)(w-r\zeta)(z-\zeta)}\right)$$

$$= \frac{r}{2\pi i}\int \frac{g(w)}{(w-rz)(w-r\zeta)}\,dw$$

$$\Rightarrow \left|\frac{g(rz)-g(r\zeta)}{z-\zeta}\right|^2 \le \frac{r^2}{4\pi^2}\left(\int |g|^2 |dw|\right)\left(\int \frac{|dw|}{|w-rz|^2|w-r\zeta|^2}\right)$$

$$\Rightarrow (1-r)^2 \left\| \frac{g_r - g_r(\zeta)}{z-\zeta}\right\|^2 \le (1-r)^2 r^2 \|g\|_{H^2}^2$$

$$\int\left(\int \frac{1}{|w-rz|^2}\frac{|dz|}{2\pi}\right)\frac{|dw|/2\pi}{|w-r\zeta|^2}$$

$$= r^2 \frac{(1-r)^2}{(1-r^2)^2}\|g\|_{H^2}^2 = \frac{r^2}{(1+r)^2}\|g\|_{H^2}^2$$

$$\le \frac{1}{4}\|g\|_{H^2}^2 \le \frac{1}{2}\|g\|_{H^2}^2$$

Richter: Lecture 3, part 1. 

## The local Dirichlet integral

<u>Note</u>: If $f \in H^2$, $\alpha \in \mathbb{C}$, $\zeta \in \mathbb{T}$, and if

$$\int \left| \frac{f(z)-\alpha}{z-\zeta} \right|^2 \frac{|dz|}{2\pi} < \infty$$

then $\displaystyle \operatorname{nt-lim}_{\lambda \to \zeta} f(\lambda) = \alpha$

<u>pf</u>:  $\lambda \in \Gamma(\zeta) \iff \exists c \quad \dfrac{|\lambda-\zeta|}{1-|\lambda|^2} \le c$

$$g(\lambda) = \frac{f(\lambda)-\alpha}{\lambda-\zeta} \in H^2$$

$$\begin{aligned}
\Rightarrow |f(\lambda)-\alpha|^2 &= |\lambda-\zeta|^2 |g(\lambda)|^2 \\
&\le |\lambda-\zeta|^2 \|g\|^2_{H^2} \|k_\lambda\|^2_{H^2} \\
&= \frac{|\lambda-\zeta|^2}{1-|\lambda|^2} \|g\|^2_{H^2} \qquad k_\lambda(z) = \frac{1}{1-\bar{\lambda} z} \\
&\le c\,|\lambda-\zeta|\, \|g\|^2_{H^2} \to 0
\end{aligned}$$

<u>Def</u>: $f \in H^2$, $\zeta \in \mathbb{T}$

$$D_\zeta(f) = \begin{cases} \infty & \text{if } f \text{ does not have a nontangential limit at } \zeta \\ \displaystyle\int \left| \frac{f(z)-f(\zeta)}{z-\zeta} \right|^2 \frac{|dz|}{2\pi} & \text{if } f \text{ has nontangential limit } f(\zeta) \text{ at } \zeta. \end{cases}$$

<u>Exercise</u>: If $f \in H^2$, $D_\zeta(f) < \infty$, then

$$\lim_{\substack{\lambda \to \zeta \\ \lambda \in \Omega_c(\zeta)}} f(\lambda) = f(\zeta)$$

$$\Omega_c(\zeta) = \left\{ \lambda \in \mathbb{D} : \frac{|\lambda - \zeta|^2}{1 - |\lambda|^2} \le c \right\} \text{ for some } c > 0$$

# A fact about the local Dirichlet integral

<u>Thm</u>: $\zeta \in \overline{\mathbb{D}}$, $f \in H^2$, $r < 1$, $f_r(z) = f(rz)$

$$\Rightarrow \quad D_\zeta(f_r) \leq 3\, D_\zeta(f)$$

(actually one can get $D_\zeta(f_r) \leq D_\zeta(f)$

D. Sarason )

Pf: By problem 4 we may assume $|\zeta| < 1$.

$f \in H^2 \Rightarrow D_\zeta(f) < \infty$ and

$$g(z) = \frac{f(z) - f(\zeta)}{z - \zeta} \in H^2 \qquad \|g\|_{H^2}^2 = D_\zeta(f)$$

$$f(z) = f(\zeta) + (z - \zeta)\, g(z)$$

$$f_r(z) = f(\zeta) + (rz - \zeta)\, g(rz), \quad f_r(\zeta) = f(\zeta) + (r\zeta - \zeta)\, g_r(\zeta)$$

$$\frac{f_r(z) - f_r(\zeta)}{z - \zeta} = \frac{(rz \overset{\pm r\zeta}{-} \zeta)\, g_r(z) - (r\zeta - \zeta)\, g_r(\zeta)}{z - \zeta}$$

$$= r\, g_r(z) + \zeta (r-1) \frac{g_r(z) - g_r(\zeta)}{z - \zeta}$$

$$\Rightarrow D_\zeta(\rho_r) \le 2\left(\|g_r\|_{H^2}^2 + (1-r)^2 \left\| \frac{g_r - g_r(\zeta)}{z-\zeta} \right\|_{H^2}^2\right)$$

<u>toshow:</u> $(1-r)^2 \left\| \frac{g_r - g_r(\zeta)}{z-\zeta} \right\|_{H^2}^2 \le \frac{1}{2}\|g\|_{H^2}^2$

$$g(rz) = \frac{1}{2\pi i}\int_{|w|=1} \frac{g(w)}{w - rz}\,dw \quad \text{Cauchy for } g \in H^2$$

$$g(r\zeta) = \frac{1}{2\pi i}\int_{|w|=1} \frac{g(w)}{w - r\zeta}\,dw$$

$$\frac{g(rz) - g(r\zeta)}{z-\zeta} = \frac{1}{2\pi i}\int_{|w|=1} g(w)\left(\frac{(w - r\zeta) - (w - rz)}{(w-rz)(w-r\zeta)(z-\zeta)}\right.$$

$$= \frac{r}{2\pi i}\int \frac{g(w)}{(w-rz)(w-r\zeta)}\,dw$$

$$\Rightarrow \left|\frac{g(rz) - g(r\zeta)}{z-\zeta}\right|^2 \le \frac{r^2}{4\pi^2}\left(\int |g|^2 |dw|\right)\left(\int \frac{|dw|}{|w-rz|^2|w-r\zeta|^2}\right)$$

$$\Rightarrow (1-r)^2 \left\| \frac{g_r - g_r(\zeta)}{z-\zeta} \right\|^2 \le (1-r)^2 r^2 \|g\|_{H^2}^2$$

$$\int\left(\int \frac{1}{|w-rz|^2}\frac{|dz|}{2\pi}\right)\frac{\frac{|dw|}{2\pi}}{|w-r\zeta|^2}$$

$$= r^2 \frac{(1-r)^2}{(1-r^2)^2}\|g\|_{H^2}^2 = \frac{r^2}{(1+r)^2}\|g\|_{H^2}^2$$

$$\le \frac{1}{4}\|g\|_{H^2}^2 \le \frac{1}{2}\|g\|_{H^2}^2$$

Theorem: If $\varphi \in H^\infty$ and if $f, \varphi f \in D(\mu)$,

then $\varphi_r f \to \varphi f$ weakly

$\varphi_r(z) = \varphi(rz)$

pf: Note: $f_n \to f$ weakly $\iff$ (a) $\exists C$ $\|f_n\| \le C$
(b) $f_n(z) \to f(z)$ $\forall z \in \mathbb{D}$

(verify)

Hence it suffices to show that $\exists C \ni$

$$\|\varphi_r f\|_{D(\mu)} \le C$$

$$\|\varphi_r f\|^2_{D(\mu)} = \int |\varphi_r(z) f(z)|^2 \frac{|dz|}{2\pi} + \int D_\zeta(\varphi_r f)\, d\mu$$
$$\le \|\varphi_r\|^2_\infty \int |f|^2 \frac{|dz|}{2\pi} + \int D_\zeta(\varphi_r f)\, d\mu$$
$$\le \|\varphi\|^2_\infty \|f\|^2_{H^2} + \int D_\zeta(\varphi_r f)\, d\mu$$

need: $D_\zeta(\varphi_r f) \le C\left(D_\zeta(\varphi f) + D_\zeta(f)\right)$

$$D_\zeta(\varphi_r f) = \int \left|\frac{\varphi_r f(z) - \varphi_r f(\zeta)}{z - \zeta}\right|^2 =$$

$$|a+b|^2 \le 2(|a|^2+|b|^2)$$

$$= \int \left| \varphi_r(z) \frac{f-f(\zeta)}{z-\zeta} + \frac{(\varphi_r(z)-\varphi_r(\zeta))}{z-\zeta} f(\zeta) \right|^2$$

$$\le 2\left(\|\varphi\|_\infty^2 D_\zeta(f) + |f(\zeta)|^2 D_\zeta(\varphi_r)\right)$$

$$\le 2\left(\|\varphi\|_\infty^2 D_\zeta(f) + 3|f(\zeta)|^2 D_\zeta(\varphi)\right)$$

$$= C\left(\|\varphi\|_\infty^2 D_\zeta(f) + \int \left| \frac{\varphi(z)f(\zeta) - \varphi(\zeta)f(\zeta)}{z-\zeta} \right|^2\right)$$

$$= C\left(\|\varphi\|_\infty^2 D_\zeta(f) + \int \left| \frac{\varphi(z)(f(\zeta)-f(z))}{z-\zeta} + \frac{(\varphi f)(z) - \varphi f(\zeta)}{z-\zeta} \right|^2\right)$$

$$\le C\left(\|\varphi\|_\infty^2 D_\zeta(f) + 2\left(\|\varphi\|_\infty^2 D_\zeta(f) + D_\zeta(\varphi f)\right)\right)$$

$$\le C'\left(\|\varphi\|_\infty^2 D_\zeta(f) + D_\zeta(\varphi f)\right)$$

Cor 1: $f, g \in D(\mu)$ $|g| \le |f| \Rightarrow [g] \subseteq [f]$

Pf: $\varphi = \frac{g}{f} \in H^\infty$, $\varphi_r f \to g$ weakly $\Rightarrow g \in [f]$

Cor 2: $\mathcal{M} \in \mathrm{Lat}(M_z, D(\mu)) \Longrightarrow \mathcal{M} \in \mathrm{Lat}\, M(D(\mu))$