Richter, Lecture 3/4, part 2

Invariant subspaces and the index

Let $\mathcal{H} \subseteq Hol(\mathbb{D})$ be a RKHS, $z\mathcal{H} \subseteq \mathcal{H}$

Additionally we will assume

$\forall \lambda \in \mathbb{D}\ \exists\ c_\lambda > 0 \quad \|(z-\lambda)f\| \geq c_\lambda \|f\|\ \forall f \in \mathcal{H}$

Then the range of $M_z - \lambda I$ is closed.

<u>Examples</u>: $\mathcal{H} = H^2, L_a^2, D, D(\mu)$

or any subspace of these.

<u>Goal</u>: <u>Theorem</u>: If $(0) \neq \mathcal{M} \in Lat(M_z, D(\mu))$,

then $\dim \mathcal{M} \ominus z\mathcal{M} = 1$.

<u>Consequence</u>: If $T = M_z|\mathcal{M}$, then

T is u.e. to $(M_z, D(\sigma))$ for

some $\sigma \in M_+(\mathbb{T})$.

What is σ? From the earlier proofs

we know that if $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$,

then $\|D\, q(T)\varphi\|^2 = \int |q|^2 d\sigma$

<u>Note</u>: $D = (T^*T - I)^{\frac{1}{2}}$ on $\mathcal{M}$

But we also have

$$\|D\,q(T)\varphi\|^2_{D(\mu)} = \|T\,q(T)\,\varphi\|^2_{D(\mu)} - \|q(T)\varphi\|^2_{D(\mu)}$$

$$= \|zq\varphi\|^2_{D(\mu)} - \|q\varphi\|^2_{D(\mu)}$$

$$= \int |q\varphi|^2 d\mu \quad \forall q \text{ poly}$$

$$\Rightarrow \quad d\sigma = |\varphi|^2 d\mu$$

Def: $\quad d\mu_\varphi = |\varphi|^2 d\mu$

$$\Rightarrow \quad \|q\varphi\|^2_{D(\mu)} = \|q\|^2_{D(\mu_\varphi)} \quad \forall q \text{ poly}$$

$$\Rightarrow \quad \mathcal{M} = \varphi\, D(\mu_\varphi)$$

Def: $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{H})$

$$\operatorname{ind} \mathcal{M} = \dim \mathcal{M} \ominus z\mathcal{M}$$

Ex 1: $\mathcal{H} = H^2$, $\mathcal{M} \in \mathrm{Lat}(M_z, H^2)$

$\mathcal{M} = \varphi H^2 \Rightarrow \mathcal{M} \ominus z\mathcal{M} = \{c\varphi : c \in \mathbb{C}\}$

φ inner (verify)

Ex2 (see Lecture 1) ABFP

$\forall n \in \mathbb{N} \cup \{\infty\}\ \exists\, \mathcal{M}_n \in \mathrm{Lat}(M_z, L^2_a) \ni \mathrm{ind}\, \mathcal{M}_n = n$

Lemma: If $A \in \mathcal{B}(\mathcal{H})$, if $c > 0$ $\ni$

$$\|Ax\| \geq c\|x\| \quad \forall x \in \mathcal{H}$$

then $\exists\, \varepsilon > 0$ $\ni$

$$\dim \ker A^* = \dim \ker(A^* - \bar{\lambda}) \quad \forall |\lambda| < \varepsilon$$

Hence

$$\dim \mathcal{H} \ominus A\mathcal{H} = \dim \mathcal{H} \ominus (A - \lambda)\mathcal{H}$$

This Lemma follows from more general properties of semi-Fredholm operators. A simple proof of the Lemma follows.

$$A^*A \geq cI \Rightarrow (A^*A)^{-1} \text{ exists}$$

Let $L = (A^*A)^{-1}A^*$ so $LA = I$.

$$\Rightarrow \quad (1-\lambda L)A = A - \lambda$$

$$* \quad \Rightarrow \quad A^*(1-\bar{\lambda}L^*) = A^* - \bar{\lambda}$$

$$\Rightarrow \quad (1-\bar{\lambda}L^*)\ker(A^*-\bar{\lambda}) \subseteq \ker A^*$$

If $|\lambda| < \frac{1}{\|L\|}$, then $1-\bar{\lambda}L^*$ is 1-1, so

$$\dim\ker(A^*-\bar{\lambda}) \leq \dim\ker A^*$$

If $|\lambda| < \frac{1}{\|L\|}$, then $(1-\bar{\lambda}L^*)^{-1}$ exists,

apply to $*$ and get

$$A^* = (A^*-\bar{\lambda})(1-\bar{\lambda}L^*)^{-1}$$

and $(1-\bar{\lambda}L^*)^{-1}\ker A^* \subseteq \ker A^*-\bar{\lambda}$

so $\dim\ker A^* \leq \dim\ker(A^*-\bar{\lambda})$

$T \in \mathcal{B}(\mathcal{H})$

<u>Cor</u>: If $\forall \lambda_0 \in \mathbb{D}$ $\exists c_0 > 0$ $\ni$

$$\|(T-\lambda_0)x\| \geq c_0 \|x\| \quad \forall x \in \mathcal{H}$$

then $\dim \ker T^* = \dim \ker(T^* - \bar{\lambda})$ $\forall |\lambda| < 1$

pf: Let $\Omega = \{\lambda \in \mathbb{D} : \dim \ker T^* = \dim \ker(T^* - \bar{\lambda})\}$

Then $0 \in \Omega$ and Ω is open by the previous Lemma ~~with~~ applied with $A = T - \lambda_0$

But $\mathbb{D} \setminus \Omega$ is also open by the same reason.

$\Rightarrow \Omega = \mathbb{D}$ //

<u>Cor</u>: $\mathcal{H} \subseteq Hol(\mathbb{D})$, $\|(z-\lambda)f\| \geq c_\lambda \|f\|$ $\forall f$

$\mathcal{M} \in Lat(M_z, \mathcal{H})$

$\Rightarrow \operatorname{ind} \mathcal{M} = \dim \mathcal{M} \ominus (z-\lambda)\mathcal{M}$ $\quad \forall \lambda \in \mathbb{D}$.

<u>Def</u>: $Z(\mathcal{M}) = \{\lambda \in \mathbb{D} : f(\lambda) = 0 \ \forall f \in \mathcal{M}\}$

the zero set of $\mathcal{M}$

Lemma: Let $\mathcal{H} \subseteq Hol(\mathbb{D})$ be a RKHS, $z\mathcal{H} \subseteq \mathcal{H}$
and $\forall \lambda \in \mathbb{D}\ \exists c_\lambda > 0 \quad \|(z-\lambda)f\| \geq c_\lambda \|f\| \ \forall f \in \mathcal{H}$.

Also assume $Z(\mathcal{H}) = \emptyset$.

~~Thm~~ Let $\mathcal{M} \in Lat(M_z, \mathcal{H})$

TFAE

(a) $\operatorname{ind} \mathcal{M} = 1$

(b) $\exists \lambda \in \mathbb{D} \setminus Z(\mathcal{M}) \ni \left.\begin{array}{l} f \in \mathcal{M} \\ f(\lambda) = 0 \end{array}\right\} \Rightarrow \frac{f}{z-\lambda} \in \mathcal{M}$

(c) $\forall \lambda \in \mathbb{D} \setminus Z(\mathcal{M}) : \left.\begin{array}{l} f \in \mathcal{M} \\ f(\lambda) = 0 \end{array}\right\} \Rightarrow \frac{f}{z-\lambda} \in \mathcal{M}$

Pf: (a) $\Rightarrow$ (c) verify

(c) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (a) Pick $\lambda \in \mathbb{D} \setminus Z(\mathcal{M})$ as in (b).

Since $\lambda \notin Z(\mathcal{M})$ $\exists g \in \mathcal{M}$ with $g(\lambda) \neq 0$

to show: $\dim \mathcal{M} \ominus (z-\lambda)\mathcal{M} = 1$

equivalently $\dim \mathcal{M}/_{(z-\lambda)\mathcal{M}} = 1$

Clearly $g \notin (z-\lambda)\mathcal{M}$, so $\dim \mathcal{M}/(z-\lambda)\mathcal{M} \geq 1$.

If $f \in \mathcal{M}$, then $f - \frac{f}{g}(\lambda) g \in \mathcal{M}$ and it is 0 at λ. By hypothesis (b)

$$h = \frac{f - \frac{f}{g}(\lambda) g}{z-\lambda} \in \mathcal{M}$$

$$\Rightarrow \quad f - \frac{f}{g}(\lambda) g \; \cancel{\in (z-\lambda)\mathcal{M}} = (z-\lambda) h \in (z-\lambda)\mathcal{M}$$

$$f = \frac{f}{g}(\lambda) g + (z-\lambda) h$$

$$\Rightarrow \dim \frac{\mathcal{M}}{(z-\lambda)\mathcal{M}} = 1 \quad //$$

<u>Cor</u>: $\mathcal{H}$ as above, $Z(\mathcal{H}) = \emptyset$

$$\Rightarrow \operatorname{ind} \mathcal{H} = 1 \iff \left.\begin{matrix} f \in \mathcal{H} \\ f(\lambda) = 0 \end{matrix}\right\} \Rightarrow \frac{f}{z-\lambda} \in \mathcal{H}$$

<u>Cor</u>: $D(\mu) \ominus z D(\mu)$ is 1-dimensional

Def: $M(\mathcal{H}) = \{ \varphi \in Hol(\mathbb{D}) : \varphi f \in \mathcal{H} \ \forall f \in \mathcal{H} \}$

the multipliers of $\mathcal{H}$

Facts: (1) $\varphi \in M(\mathcal{H}) \Rightarrow M_\varphi \ (f \to \varphi f) \in \mathcal{B}(\mathcal{H})$

(2) $M_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$, k_λ reproducing kernel for $\mathcal{H}$

Def : $\varphi \in M(\mathcal{H})$

$$\|\varphi\|_M = \|M_\varphi\|_{\mathcal{B}(\mathcal{H})}$$

Note: If $\mathcal{H}$ is as above $\subseteq Hol(\mathbb{D})$

$\operatorname{ind} \mathcal{H} = 1$, $Z(\mathcal{H}) = \emptyset$,
then $\forall \varphi \in M(\mathcal{H}), \lambda \in \mathbb{D}$ $\quad \frac{\varphi - \varphi(\lambda)}{z - \lambda} \in M(\mathcal{H})$

pf : Let $f \in \mathcal{H}$. Then

$$g = (\varphi - \varphi(\lambda)) f \in \mathcal{H} , \ g(\lambda) = 0$$

$$\Rightarrow \frac{g}{z-\lambda} \in \mathcal{H}$$

$$\Rightarrow \frac{\varphi - \varphi(\lambda)}{z-\lambda} f \in \mathcal{H} \ \forall f \in \mathcal{H}$$

<u>Theorem</u>: If $\mathcal{H} \subseteq Hol(\mathbb{D})$ is as above, $\operatorname{ind} \mathcal{H} = 1$, $Z(\mathcal{H}) = \emptyset$,

and if ~~the~~ $M(\mathcal{H})$ is dense in $\mathcal{H}$,

then $\forall\, \mathcal{M} \in \operatorname{Lat} M(\mathcal{H})$ with $\mathcal{M} \cap M(\mathcal{H}) \neq (0)$

we have $\operatorname{ind} \mathcal{M} = 1$

<u>pf</u>: Let $\varphi \in \mathcal{M} \cap M(\mathcal{H})$, $\varphi \neq 0$. Then $\exists\, \lambda \in \mathbb{D}$

$\ni \varphi(\lambda) \neq 0$ so $\lambda \in \mathbb{D} \setminus Z(\mathcal{M})$

Let $f \in \mathcal{M}$ with $f(\lambda) = 0$

<u>to show</u>: $\dfrac{f}{z-\lambda} \in \mathcal{M}$

We know $\dfrac{f}{z-\lambda} \in \mathcal{H}$ $\quad (\operatorname{ind} \mathcal{H} = 1)$

$$\psi = \frac{\varphi - \varphi(\lambda)}{z-\lambda} \in M(\mathcal{H})$$

$*$ $\Rightarrow \psi f = \dfrac{\varphi - \varphi(\lambda)}{z-\lambda} f \in \mathcal{M}$ (since $\mathcal{M} \in \operatorname{Lat} M(\mathcal{H})$)

If $\psi_n \in M(\mathcal{H})$ with $\psi_n \to \dfrac{f}{z-\lambda}$ in $\mathcal{H}$,

then $\psi_n \varphi \to \dfrac{\varphi f}{z-\lambda}$ in $\mathcal{H}$, hence

$\dfrac{\varphi f}{z-\lambda} \in \mathcal{M}$ (since $\psi_n \varphi \in \mathcal{M}$). By $*$ $\dfrac{\varphi(\lambda) f}{z-\lambda} \in \mathcal{M}$

$\Rightarrow f/_{z-\lambda} \in \mathcal{M}$.

Thus, in order to show that

$$\forall \underset{\mathcal{M}\neq 0}{\mathcal{M}} \in Lat(M_z, D(\mu)) \text{ we have } \operatorname{ind} \mathcal{M} = 1$$

it suffices to show:

(a) $Lat(M_z, D(\mu)) = Lat\ M(D(\mu))$

(b) $\mathcal{M} \cap M(D(\mu)) \neq (0)$

In order to verify (a) it suffices to note the following fact:

If $f, \varphi f \in D(\mu)$, $\|\varphi\|_\infty < \infty$, then

$$\varphi_r f \to \varphi f \text{ weakly in } D(\mu)$$

$$\varphi_r(z) = \varphi(rz)$$

This is shown elsewhere. If $\varphi \in M(D(\mu))$, then $\|\varphi\|_\infty < \infty$ and if $\mathcal{M} \in Lat(M_z, D(\mu))$ $f \in \mathcal{M}$, then $\varphi_r f \in \mathcal{M}$, hence $\varphi f \in \mathcal{M}$, i.e. $\mathcal{M} \in Lat\ M(D(\mu))$.

(b) 2 possible proofs are known

① Aleman, Richter - Sundberg

Step 1: We know for $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$

$$\|p\varphi\|^2_{D(\mu)} = \|p\|^2_{D(\mu_\varphi)} \qquad d\mu_\varphi = |\varphi|^2 d\mu$$

So, if $\|\varphi\|_\infty \le 1$, then

$$\|p\varphi\|^2_{D(\mu)} = \|p\|^2_{D(\mu_\varphi)} =$$

$$= \|p\|^2_{H^2} + \int D_\zeta(p)\, |\varphi|^2 d\mu$$

$$\le \|p\|^2_{H^2} + \int D_\zeta(p)\, d\mu = \|p\|^2_{D(\mu)}$$

$$\Rightarrow \|\varphi\|_{M(D(\mu))} \le 1$$

Thus we only need to show $\|\varphi\|_\infty \le 1$

Step 2: Assume $0 \notin Z(\mathcal{M})$ and

set $\varphi = \dfrac{P_{\mathcal{M}} k_0}{\|P_{\mathcal{M}} k_0\|}$, k_0 = reproducing kernel at 0

then $\|\varphi\| = 1$ and one easily verifies $\varphi \in \mathcal{M} \ominus z\mathcal{M}$. Furthermore, φ satisfies

$$\varphi(0) = \sup\{|f(0)| : f \in \mathcal{M},\ \|f\|_{D(\mu)} \le 1\}$$

$(\varphi(0) = \|P_{\mathcal{M}} k_0\|$ and $|f(0)| = |\langle f, P_{\mathcal{M}} k_0\rangle| \le \|f\|\,\|P_{\mathcal{M}} k_0\|)$

Step 3: If $f \in \mathcal{M}$, $\|f\|_\infty > 1$, then one can ~~construc~~ construct a ~~c~~ cut-off function $f_1 \in \mathcal{M}$ with $\|f_1\|_\infty \le 1$

$$\frac{|f_1(0)|}{\|f_1\|_{D(\mu)}} > \frac{|f(0)|}{\|f\|_{D(\mu)}}$$

so such f could not be the φ from Step 2

hence $\|\varphi\|_\infty \le 1$.

What is f_1? If $f = Sg$ inner-outer, define $|g_1| = \begin{cases} |g| & \text{if } |g| \le 1 \text{ at } z \in \mathbb{T} \\ 1 & \text{if } |g| > 1 \end{cases}$

g_1 outer with boundary values $\mathfrak{I}$, $f_1 = Sg_1$

(2) 2^{nd} proof uses CNP-kernels