Richter Lecture 4/5

CNP - kernels and invariant subspaces

CNP - complete Nevanlinna-Pick

Def: $u : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is positive definite if $\forall a_1, \dots a_n \in \mathbb{C}\ \forall \lambda_1, \dots \lambda_n \in \mathbb{D}$

$$\sum_{i,j} a_i \bar{a}_j\, u(\lambda_i, \lambda_j) \geq 0$$

Write $u >> 0$ if u is positive definite

Let $\mathcal{H} \subseteq Hol(\mathbb{D})$ have reproducing kernel $k_\lambda(z)$, then

$$0 \leq \left\| \sum_{i=1}^n a_i k_{\lambda_i} \right\|^2 = \sum_{i,j} a_i \bar{a}_j\, k_{\lambda_i}(\lambda_j)$$

so $k_\lambda(z) >> 0$.

Conversely, a theorem of E.H. Moore states that if $u >> 0$, then $\exists\, \mathcal{H}$ a Hilbert space of functions with reproducing kernel

$u_\lambda(z) = u(\lambda, z)$

If $\{u_n\} \subseteq \mathcal{C}$ is any o.n. basis for $\mathcal{C}$, then

$$u_\lambda(z) = \sum_{n \geq 1} \overline{u_n(\lambda)}\, u_n(z)$$

<u>Def</u>: Let $\mathcal{H} \subseteq Hol(\mathbb{D})$ have reproducing kernel $k_\lambda(z)$. k is a CNP

$\Leftrightarrow \exists\ \alpha \in \mathbb{D} \setminus Z(\mathcal{H})\ \exists\ u : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$

$$u_\lambda(z) >> 0$$

$$\ni \quad u_\lambda(z)\, k_\lambda(z) = k_\lambda(z) - \frac{k_\lambda(\alpha)\, k_\alpha(z)}{k_\alpha(\alpha)}$$

<u>Note</u>: ① k is a CNP $\Leftrightarrow$ $1 - \dfrac{k_\lambda(\alpha)\, k_\alpha(z)}{k_\alpha(\alpha)\, k_\lambda(z)} >> 0$

② $\lambda = \alpha \implies u_\alpha(z)\, k_\alpha(z) = 0 \ \forall z$, $k_\alpha \neq 0$

$\implies u_\alpha(z) = 0 \quad \forall z$

$$\implies k \text{ is CNP} \Leftrightarrow k_\lambda(z) = \frac{\overline{f(\lambda)}\, f(z)}{1 - u_\lambda(z)}$$

where $u_\lambda(z) >> 0$, $f : \mathbb{D} \to \mathbb{C}$

$u_\alpha(z) = 0 \ \forall z$

for some α with

$f(\alpha) \neq 0$

(verify)

(3) $$k_\lambda(z) - \frac{k_\alpha(z)\, k_\lambda(\alpha)}{k_\alpha(\alpha)} = (I - P_\alpha)\, k_\lambda(z)$$

where $P_\alpha = \frac{k_\alpha}{\|k_\alpha\|} \otimes \frac{k_\alpha}{\|k_\alpha\|}$ projection

<u>Def</u>: $f \otimes g\,(h) = \langle h, g\rangle f$

So $P_\alpha k_\lambda = \langle k_\lambda, \frac{k_\alpha}{\|k_\alpha\|}\rangle \frac{k_\alpha}{\|k_\alpha\|} = \frac{k_\lambda(\alpha)}{k_\alpha(\alpha)} k_\alpha$

$$P_\alpha k_\lambda(z) = \frac{k_\lambda(\alpha)\, k_\alpha(z)}{k_\alpha(\alpha)} \quad //$$

<u>Lemma</u>: If $u_n \in Hol(\mathbb{D})$, $Q \in \mathcal{B}(\mathcal{H})$, $Q \geq 0$

such that $$\left(\sum_{n \geq 1} \overline{u_n(\lambda)}\, u_n(z)\right) k_\lambda(z) = \langle Q k_\lambda, k_z\rangle$$

$\forall \lambda, z \in \mathbb{D}$

then $u_n \in M(\mathcal{H})$ $\forall n$ and

$$Q = \sum_{n \geq 1} M_{u_n} M_{u_n}^* \qquad (SOT)$$

<u>pf</u>: Let $\mathcal{D} = \left\{ \sum_{i=1}^{n} a_i k_{\lambda_i} : a_i \in \mathbb{C} \right\}$. Then $\mathcal{D}$ is dense in $\mathcal{H}$.

Let $f = \sum a_i k_{\lambda_i} \in \mathcal{D}$

Consider

$$\sum_n \| \sum_i a_i \overline{u_n(\lambda_i)} k_{\lambda_i} \|^2$$

$$= \sum_n \sum_{i,j} a_i \bar{a}_j \overline{u_n(\lambda_i)} u_n(\lambda_j) k_{\lambda_i}(\lambda_j)$$

$$= \sum_{i,j} a_i \bar{a}_j \langle Q k_{\lambda_i}, k_{\lambda_j} \rangle$$

$$= \langle Qf, f \rangle = \| Q^{\frac{1}{2}} f \|^2 \leq \| Q^{\frac{1}{2}} \|^2 \|f\|^2$$

Thus, if $f = 0$, then $\sum_i a_i \overline{u_n(\lambda_i)} k_{\lambda_i} = 0 \; \forall n$

So define $T_n : \mathcal{D} \to \mathcal{H}$ by

$$T_n\left(\sum a_i k_{\lambda_i}\right) = \sum a_i \overline{u_n(\lambda_i)} k_{\lambda_i}$$

this is well-defined and satisfies

$$\|T_n f\|^2 \leq \|Q^{\frac{1}{2}}\|^2 \|f\|^2 \quad \forall f \in \mathcal{D}$$

So T_n extends to be a bounded operator on $\mathcal{H}$

and $\sum_n \|T_n f\|^2 = \langle Qf, f \rangle \quad \forall f \in \mathcal{D}$

$$\sum_n \langle T_n^* T_n f, f \rangle = \langle Qf, f \rangle$$

$\Rightarrow \sum_n T_n^* T_n = Q$ (SOT limit)

$$T_n k_\lambda = \overline{u_n(\lambda)}\, k_\lambda \Rightarrow T_n = M_{u_n}^*$$

$$(T_n^* f)(\lambda) = \langle T_n^* f, k_\lambda \rangle = \langle f, T_n k_\lambda \rangle$$
$$= u_n(\lambda) \langle f, k_\lambda \rangle = u_n(\lambda) f(\lambda)$$

so $T_n^* f = u_n f \Rightarrow u_n \in M(\mathcal{H})$

and $\sum M_{u_n} M_{u_n}^* = Q$ //

Cor: k is a CNP kernel

(*) $\iff \exists\, u_n \in M(\mathcal{H}) \ni \sum_n M_{u_n} M_{u_n}^* = I - P_\alpha$
$\exists\, \alpha \in \mathbb{D} \setminus Z(\mathcal{H})$

Thm: (McCullough-Trent)

If k is a CNP kernel, if $\mathcal{M} \in \operatorname{Lat} M(\mathcal{H})$,

then $\exists\, \varphi_n \in M(\mathcal{H}) \ni$

$$P_{\mathcal{M}} = \sum M_{\varphi_n} M_{\varphi_n}^* \quad (SOT)$$

Note: (a) $I - P_\alpha$ = proj. onto $\{ f \in \mathcal{H} : f(\alpha) = 0 \}$

check: $P_\alpha f = \dfrac{f(\alpha)}{k_\alpha(\alpha)} k_\alpha$

so the $\exists\, \alpha \in \mathbb{D} \setminus Z(\mathcal{H})$ in (*) can be replaced by

$\forall\, \alpha \in \mathbb{D} \setminus Z(\mathcal{H})$.

(b) $P_m = \sum M_{\varphi_n} M_{\varphi_n}^* \Rightarrow \varphi_n \in m \;\forall n$

Define $T : \mathcal{H} \to \bigoplus_{n\geq 1} \mathcal{H}$

$$f \to \{ M_{\varphi_n}^* f \}$$

Then $T^* : \bigoplus_{n\geq 1} \mathcal{H} \to \mathcal{H}$, $T^*\{f_n\} = \sum_n \varphi_n f_n$

and $P_m = \sum M_{\varphi_n} M_{\varphi_n}^* = T^* T$

So T^* is a partial isometry with

ran T^* = ran $P_m = m$

$\Rightarrow T^*\{0, \dots 0, 1, 0, \dots\} = \varphi_n \in m$.

Proof: (McCullough-Trent theorem)

It suffices to show that

$$\frac{P_m k_\lambda(z)}{k_\lambda(z)} >> 0$$

Because, then $\exists\, \varphi_n \in Hol(\mathbb{D}) \ni$

$$\sum_n \overline{\varphi_n(\lambda)}\, \varphi_n(z) = \frac{P_m k_\lambda(z)}{k_\lambda(z)}$$

($\{\varphi_n\}$ = o.n.b. of auxiliary Hilbert space $\mathcal{E}$)

<u>Thm</u> (McCullough-Trent)

If k is a CNP kernel, if $m \in Lat(M(\mathcal{H}))$

$*$ then $$\frac{P_m k_\lambda(z)}{k_\lambda(z)} >> 0$$

Consequently, $\exists\ \varphi_n \in Hol(\mathbb{D}) \ni$

$$\frac{P_m k_\lambda(z)}{k_\lambda(z)} = \sum_n \overline{\varphi_n(\lambda)}\, \varphi_n(z)$$

$$\Rightarrow \langle P_m k_\lambda, k_z \rangle = \sum \overline{\varphi_n(\lambda)}\, \varphi_n(z)\, k_\lambda(z)$$

$$\Rightarrow \quad P_m = \sum_n M_{\varphi_n} M_{\varphi_n}^* \quad \text{SOT}$$

by the Lemma

<u>pf</u> of $*$ <u>Spse</u> $\sum M_{u_n} M_{u_n}^* = 1 - P_\alpha$

① Note that $\forall n \quad P_m M_{u_n} P_m = M_{u_n} P_m$

$$\Rightarrow P_m M_{u_n}^* P_m = P_m M_{u_n}^*$$

$$\Rightarrow P_m M_{u_n} P_m M_{u_n}^* P_m = M_{u_n} P_m M_{u_n}^*$$

So $\sum_n \overline{\varphi_n(w)}\varphi_n(z)\, k_1(z) = \langle P_m k_1, k_2 \rangle$

Thus we are done by the Lemma.

k is a CNP kernel

$\Rightarrow \exists\, u_n \in M(\mathcal{H}) \ni$

$$\sum_n M_{u_n} M_{u_n}^* = I - P_\alpha$$

(1) Note that $u_n m \subseteq m$

$$\Rightarrow \quad P_m M_{u_n} P_m = M_{u_n} P_m$$

$$P_m M_{u_n}^* P_m = P_m M_{u_n}^*$$

$$\Rightarrow P_m \left(M_{u_n} P_m M_{u_n}^* \right) P_m = M_{u_n} P_m M_{u_n}^*$$

(2) Set

$$Q = P_m - \sum_n M_{u_n} P_m M_{u_n}^*$$

$$= P_m \left(I - \sum_n M_{u_n} (I - P_{m^\perp}) M_{u_n}^* \right) P_m$$

$$= P_m \left(\left(I - \sum_n M_{u_n} M_{u_n}^* \right) + \sum_n M_{u_n} P_{m^\perp} M_{u_n}^* \right) P_m$$

$$= P_m P_\alpha P_m + P_m \left(\sum_n M_{u_n} P_{m^\perp} M_{u_n}^* \right) P_m \geq 0$$

$\Rightarrow \langle Q k_\lambda, k_z \rangle >> 0$

$$\langle Q k_\lambda, k_z \rangle = (P_m k_\lambda)(z) - \langle \sum_n M_{u_n} P_m M_{u_n}^* k_\lambda, k_z \rangle$$

$$= P_m k_\lambda (z) - \sum_n u_n(z) \overline{u_n(\lambda)} P_m k_\lambda (z)$$

$$= (1 - u_\lambda(z)) P_m k_\lambda (z)$$

$$= \overline{f(\lambda)} f(z) \frac{P_m k_\lambda (z)}{k_\lambda(z)} \quad \text{(see page 2)}$$

$$\Rightarrow \frac{P_m k_\lambda (z)}{k_\lambda(z)} >> 0$$

<u>Cor</u>: ~~...~~ If $k_0(z) = 1$ $\forall z \in \mathbb{D}$, if $m \in \operatorname{Lat} M(\mathcal{H})$, $0 \notin z(m)$, then

if $\varphi \in m \ominus zm$, $\|\varphi\| \le 1 \Rightarrow \|\varphi\|_{M(\mathcal{H})} \le 1$

Pf: $\varphi = \dfrac{P_m k_0}{\|P_m k_0\|} = \dfrac{P_m k_0}{k_0}(z) \sqrt{\left(\dfrac{k_0}{P_m k_0}\right)(0)}$

So, if $v_\lambda(z) = \dfrac{P_m k_\lambda(z)}{k_\lambda(z)}$, then $\varphi(z) = \dfrac{v_0(z)}{\sqrt{v_0(0)}}$,

so φ is a unit vector in that auxiliary Hilbert space $\mathcal{E}$. Any unit vector can be part of an o.n.b., so choose $\varphi_n \ni \varphi_1 = \varphi$.