Richter: Lecture 5 + extra notes.

Each $D(\mu)$ has a CNP kernel

$|\lambda| \leq 1$

$$D_\lambda(f) = \int_{|z|=1} \left| \frac{f(z) - f(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi}$$

$\mu \in M_+(\overline{\mathbb{D}})$

$$\Rightarrow \quad D(\mu) = \|f\|^2_{H^2} + \int_{|\lambda| \leq 1} D_\lambda(f) \, d\mu(\lambda)$$

extended definition

~~Thm (Aleman)~~

Thm: Aleman: $D(\mu)$ $\mu \in M_+(\overline{\mathbb{D}})$

are the superharmonically weighted Dirichlet spaces

i.e. If $h : \mathbb{D} \to [0, \infty)$, $-h$ subharmonic,

then $\exists\, \mu \in M_+(\overline{\mathbb{D}}) \ni \forall f \in H^2$

$$\int D_\lambda(f) \, d\mu = \int_{|z|<1} |f'|^2 \, h \, \frac{dA}{\pi}$$

In fact, for $|\lambda| < 1$

$$D_\lambda(f) = 2\int |f'(z)|^2 \log\left|\frac{1-\bar{\lambda}z}{z-\lambda}\right| \frac{dA(z)}{(1-|\lambda|^2)\pi}$$

Pf: Littlewood-Paley

If $g \in H^2$, then

$$\|g-g(0)\|_{H^2}^2 = 2\int |g'(z)|^2 \log\frac{1}{|z|} \frac{dA(z)}{\pi}$$

(verify)

Let $\varphi_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$. Then for $f \in H^2$

$$(1-|\lambda|^2)\, D_\lambda(f) = \int P_\lambda(z)\, |f - f(\lambda)|^2 \frac{|dz|}{2\pi}$$

(Poisson-kernel)

$$= \int_{|z|=1} |f\circ\varphi_\lambda(z) - (f\circ\varphi_\lambda)(0)|^2 \frac{|dz|}{2\pi}, \quad g = f\circ\varphi_\lambda$$

$$= 2\int_{|z|<1} |(f\circ\varphi_\lambda)'(z)|^2 \log\frac{1}{|z|} \frac{dA(z)}{\pi} \qquad \text{L-P.}$$

$$= 2\int_{|w|<1} |f'(w)|^2 \log\frac{1}{|\varphi_\lambda^{-1}(w)|} \frac{dA(w)}{\pi}$$

$$= 2\int_{|w|<1} |f'(w)|^2 \log\left|\frac{1-\bar{\lambda}w}{w-\lambda}\right|^2 \frac{dA(w)}{\pi}$$

Thus

$$\int_{|\lambda|<1} D_\lambda(f)\, d\mu = \int_{|z|<1} |f'(z)|^2 \left(2\int_{|\lambda|<1} \log\left|\frac{1-\bar\lambda z}{z-\lambda}\right| \frac{1}{1-|\lambda|^2}\, d\mu(\lambda)\right) \frac{dA(z)}{\pi}$$

$$h_1(z) = U_\mu(z) = 2\int_{|\lambda|<1} \log\left|\frac{1-\bar\lambda z}{z-\lambda}\right| \frac{1}{1-|\lambda|^2}\, d\mu(\lambda)$$

is ≥ 0, superharmonic and $h_1(z) \to 0$ as $|z| \to 1$

The converse follows by a theorem of Littlewood.

Ex: For $0 < \alpha < 1$

$$(1-|z|^2)^{1-\alpha} = -\frac{1}{2\pi}\int \log\left|\frac{1-\bar\lambda z}{z-\lambda}\right| \Delta_\lambda (1-|\lambda|^2)^{1-\alpha}\, dA(\lambda)$$

i.e. $D(\mu)$ includes "weighted Dirichlet" spaces D_α $\quad D \subseteq D_\alpha \subseteq H^2$

<u>Thm</u>: (Shimorin) If $\mu \in M_+(\overline{\mathbb{D}})$, then the reproducing kernel for $D(\mu)$ is a CNP kernel.

Pf: (1) $\mu = \sum_{i=1}^{n} a_i \delta_{\lambda_i}$ $\quad |\lambda_i| < 1,\ a_i > 0$

(2) approximation

For $|\lambda| < 1$ we have

$$D_\lambda(f) = \int \left| \frac{f(z) - f(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi}$$

$$= \int \frac{|f(z)|^2 - 2\,\mathrm{Re}\, f(z)\overline{f(\lambda)} + |f(\lambda)|^2}{|z-\lambda|^2} \frac{|dz|}{2\pi}$$

$$= \int \frac{|f(z)|^2}{|z-\lambda|^2} \frac{|dz|}{2\pi} - 2\,\mathrm{Re}\, \overline{f(\lambda)} \int \frac{f(z)}{1-\bar{\lambda} z} \overline{\frac{1}{1-\bar{\lambda} z}} \frac{|dz|}{2\pi}$$
$$+ |f(\lambda)|^2 \int \frac{1}{|1-\bar{\lambda} z|^2} \frac{|dz|}{2\pi}$$

$$= \int \frac{|f(z)|^2}{|1-\bar{\lambda} z|^2} \frac{|dz|}{2\pi} - \frac{|f(\lambda)|^2}{1-|\lambda|^2}$$

Thus

$$\|f\|^2_{D(\mu)} = \int |f|^2 \frac{|dz|}{2\pi} + \sum a_i \, D_{\lambda_i}(f)$$

$$= \int |f|^2 \left(1 + \sum \frac{a_i}{|z-\lambda_i|^2}\right) \frac{|dz|}{2\pi}$$

$$- \sum_i \frac{a_i}{1-|\lambda_i|^2} |f(\lambda_i)|^2$$

Let F_n be a rational outer function with $|F_n|^2 = 1 + \sum_{i=1}^{n} \frac{a_i}{|z-\lambda_i|^2}$ for $|z|=1$

and $c_i = \frac{a_i}{1-|\lambda_i|^2}$, then

$$\|f\|^2_{D(\mu)} = \|f F_n\|^2_{H^2} - \sum_{i=1}^{n} c_i |f(\lambda_i)|^2$$

Note: $f(\lambda) = \frac{1}{F(\lambda)} (fF)(\lambda) = \langle fF, \frac{1}{\overline{F(\lambda)}} k_\lambda \rangle_{H^2}$

$$k_\lambda(z) = \frac{1}{1-\bar{\lambda} z}$$

$$= \langle fF, \frac{1}{\overline{F(\lambda)} F(z)} k_\lambda F \rangle_{H^2}$$

$\Rightarrow$ $\frac{1}{\overline{F(\lambda)} F(z)} k_\lambda(z)$ is the reproducing kernel for $\|f\| = \|fF\|_{H^2}$ and this is CNP

Thus the foll. Lemma + induction will finish the proof of 1

<u>Lemma</u>: If k is a CNP kernel and if $\alpha \in \mathbb{D}$, $a > 0$ with

$$(*) \qquad \|f\|_1^2 = \|f\|^2 - a|f(\alpha)|^2 > 0 \quad \forall f \neq 0,\ f \in \mathcal{H}$$

then ~~the map~~ $k'_\alpha(z)$, the reproducing kernel for $\mathcal{H}$ with norm $\|\cdot\|_1$ is a CNP

~~Note. $\alpha \Leftrightarrow 1 - a k_\alpha(\alpha) \neq 0$~~

<u>Pf</u>: $(*) \Rightarrow$

$$\|k_\alpha\|^2 - a|k_\alpha(\alpha)|^2 > 0$$

$$\|k_\alpha\|^2 - a\|k_\alpha\|^4 > 0$$

$$\Rightarrow \quad a\, k_\alpha(\alpha) < 1$$

(we may assume $k_\alpha(\alpha) \neq 0$, otherwise $f(\alpha) = 0\ \forall f \in \mathcal{H}$).

Clearly $\|f\|_1 \leq \|f\|$, but also

$$a|f(\alpha)|^2 \leq a\|f\|^2\|k_\alpha\|^2 \Rightarrow \|f\|_1^2 \geq (1 - a\|k_\alpha\|^2)\|f\|^2$$

So $\|f\|$, $\|f\|_1$ are equivalent norms on $\mathcal{H}$.

Calculate k' in terms of k:

$$\langle f, g\rangle_1 = \langle f, g\rangle - a\, f(\alpha)\,\overline{g(\alpha)}$$

$$f = k_\lambda, \quad g = k'_z$$

$$\langle k_\lambda, k'_z\rangle_1 = \langle k_\lambda, k'_z\rangle - a\, k_\lambda(\alpha)\,\overline{k'_z(\alpha)}$$

$$k_\lambda(z) = \overline{k'_z(\lambda)} - a\, k_\lambda(\alpha)\, k'_\alpha(z)$$

(*) $$k_\lambda(z) = k'_\lambda(z) - a\, k_\lambda(\alpha)\, k'_\alpha(z)$$

$$\cancel{k'_\lambda(z) = k_\lambda(z) + a\, k_\lambda(\alpha)\, k'_\alpha(z)}$$

$$\cancel{\lambda = \alpha \quad k'_\alpha(z) = k_\alpha(z) + a\, k_\alpha(\alpha)\, k'_\alpha(z)}$$

(*)

$$\lambda = \alpha \qquad k_\alpha(z) = k'_\alpha(z) - a\, k_\alpha(\alpha)\, k'_\alpha(z)$$

$$\Rightarrow \qquad k'_\alpha(z) = \frac{1}{1 - a\, k_\alpha(\alpha)}\, k_\alpha(z)$$

$$k'_\lambda(z) = k_\lambda(z) + \underbrace{\frac{a\, k_\alpha(\alpha)}{1 - a\, k_\alpha(\alpha)}}_{c}\; \frac{k_\lambda(\alpha)\, k_\alpha(z)}{k_\alpha(\alpha)}$$

$$k_\lambda(z) = \frac{\overline{f(\lambda)}\, f(z)}{1 - u_\lambda(z)}$$

where $f(z) = \frac{k_\alpha(z)}{\|k_\alpha\|}$, $u_\alpha(z) = 0$, $u_\lambda(z) >> 0$

Set

$$u_\lambda'(z) = \frac{u_\lambda(z)}{c+1-c\,u_\lambda(z)} = \frac{1}{c+1} \sum_{n=0}^{\infty} \left(\frac{c}{c+1}\right)^n u_\lambda(z)^{n+1} >> 0$$

by the Schur product theorem

Now calculate

<u>claim</u>: $$k_\lambda'(z) = \frac{k_\lambda'(\alpha)\, k_\alpha'(z)}{k_\alpha'(\alpha)} \quad \frac{1}{1-u_\lambda'(z)}$$

<u>to show</u>: $(1-u_\lambda'(z))\, k_\lambda'(z) = \overline{g(\lambda)}\, g(z)$

since $u_\alpha'(z) = 0$

$$(1-u')k' = \frac{c+1 - c\,u_\lambda(z) - u_\lambda(z)}{c+1 - c\,u_\lambda(z)} \left(k_\lambda(z) + c\,\overline{f(\lambda)}f(z) \right)$$

$$= \frac{(c+1)\,(1-u_\lambda(z))}{c+1-c\,u_\lambda(z)} \overline{f(\lambda)}\, f(z) \left(\frac{1}{1-u_\lambda(z)} + c \right)$$

$$= \frac{c+1}{(c+1) - c\,u_\lambda(z)} \overline{f(\lambda)} f(z) \left(1 + c - c\,u_\lambda(z) \right) = (c+1)\overline{f}f$$

This proves 1, Sleminari for $\mu = \sum_i a_i \delta_{\lambda_i}$

2 $k_\lambda(z)$ is CNP

$$\iff u_\lambda(z) = 1 - \frac{k_\alpha(z)\, k_\lambda(\alpha)}{k_\alpha(\alpha)\, k_\lambda(z)} >> 0$$

Exercise: If k is the reprod. kernel for $D(\mu)$

then $k_\lambda(z) \neq 0 \quad \forall \lambda, z \in \mathbb{D}$

Thus, we need $\{k^n\}$ CNP $\ni$

$$k^n_\lambda(z) \to k_\lambda(z) \qquad \text{as } n \to \infty \quad \forall \lambda, z \in \mathbb{D}$$

(a) If $R < 1$ and $\operatorname{supp} \mu \in R\overline{\mathbb{D}}$,

then let $\varepsilon > 0$

Note: $\exists\, \delta > 0 \ \forall F \subseteq R\overline{\mathbb{D}}$, $\operatorname{diam} F < \delta$

$$\Rightarrow |D_{\lambda_1}(f) - D_{\lambda_2}(f)| \leq \varepsilon \|f\|^2_{H^2}$$

$$\forall f \in H^2,\ \lambda_1, \lambda_2 \in F$$

Take a partition $F_1 \ldots F_n$ of $R\overline{\mathbb{D}}$

$$\mu_n = \sum_{i=1}^{n} \mu(F_i)\, \delta_{\lambda_i}, \text{ for } \lambda_i \in F_i$$

Then

$$\left|\int D_\lambda(f)\,d\mu_n - \int D_\lambda(f)\,d\mu\right|$$

$$\le \sum_{i=1}^{n} \int_{F_i} |D_{\lambda_i}(f) - D_\lambda(f)|\,d\mu(\lambda)$$

$$\le \varepsilon \|f\|^2_{H^2}$$

So we have: $\exists\ \mu_n$ finite sums of pts masses

$\exists\ \varepsilon_n \to 0 \ \ni$

$$\left|\,\|f\|^2_{D(\mu_n)} - \|f\|^2_{D(\mu)}\right| \le \varepsilon_n \|f\|_{H^2} \quad \forall f \in H^2$$

Lemma: $\Rightarrow\ k^n_\lambda(z) \to k_\lambda(z)$

$$D(\mu_n) = D(\mu) = H^2$$

Let $i_n : \cancel{H^2 \to D(\mu_n)}\ D(\mu_n) \to H^2$, $i_0 : \cancel{H^2 \to D(\mu)}\ D(\mu) \to H^2$

be the inclusion mappings, then

$$\|i_n(f)\|^2_{D(\mu_n)} \to \|i_0(f)\|^2_{D(\mu)} \quad \forall f$$

$$\Rightarrow \langle i_n^* i_n f, f\rangle \to \langle i_0^* i_0 f, f\rangle \quad \forall f$$

$$\mathcal{H}_n = \mathcal{H}_0 = \mathcal{H}$$

Lemma: If $\exists\, \varepsilon_n \to 0$ $\ni$

$$\big|\, \|f\|_n - \|f\|_0 \,\big| \le \varepsilon_n \|f\|$$

and $\exists\, c, C > 0$ $\quad c\|f\|_n \le \|f\| \le C\|f\|_n$

$\forall f, n$

~~so that~~ then $k_\lambda^n(z) \to k_\lambda^0(z)$

Pf: Let $i_n : \mathcal{H}_n \to \mathcal{H}$ be the inclusion

Then $\forall f \in \mathcal{H} = \mathcal{H}_n$ $\forall \lambda$

$$f(\lambda) = \langle i_n(f), k_\lambda \rangle = \langle f, i_n^* k_\lambda \rangle_n$$

$\Rightarrow$ $i_n^* k_\lambda = k_\lambda^n$ and ~~it suffices~~

$$(i_n i_n^* k_\lambda)(z) = k_\lambda^n(z)$$

So it suffices to show

$$i_n i_n^* \to i_0 i_0^* \quad \text{WOT}$$

We have

$$\|f\|_n^2 = \|i_n^{-1}(f)\|_n^2 = \langle i_n^{-1*} i_n^{-1}(f), f \rangle$$

$$= \langle (i_n i_n^*)^{-1} f, f \rangle$$

So & with $A_n = i_n i_n^*$ we have

$$|\langle A_n^{-1} f, f\rangle - \langle A_0^{-1} f, f\rangle| \leq \varepsilon_n \|f\|$$

and this implies $A_n \to A_0$ WOT
(A_n s.a.)

pf: $|\langle A_n - A_0 f, g\rangle|$

$= |\langle A_0(A_0^{-1} - A_n^{-1}) A_n f, g\rangle|$

$= |\langle (A_0^{-1} - A_n^{-1}) A_n f, A_0 g\rangle|$

polarization

$= \left|\frac{1}{4}\left(\sum_{j=0}^{3} i^j \langle (A_0^{-1} - A_n^{-1})(A_n f + i^j A_0 g), (A_n f + i^j A_0 g)\rangle\right)\right|$

$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i$

$\leq \frac{1}{4} \sum_{j=0}^{3} \varepsilon_n \|A_n f + i^j A_0 g\|^2$

$\leq \varepsilon_n \left(\|A_n\|\|f\| + \|A_0\|\|g\|\right)^2, \quad \|A_n\| \leq C$

$\to 0$

If $\mu(\mathbb{T}) = 0$, let $\mu_R(A) = \mu(A \cap r\overline{\mathbb{D}})$

then $\|f\|^2_{D(\mu_R)} \leq \|f\|^2_{D(\mu)}$

and $\longrightarrow$ as $R \to 1$

So $D(\mu) \subseteq D(\mu_R)$

$k_\lambda(z) << k^R_\lambda(z)$

and the result follows by a theorem of

Aronszajn.

Remaining case: $\mu = \mu_{\mathbb{D}} + \mu_{\mathbb{T}}$

Only do $\mu_{\mathbb{D}} = 0$

Define $V_r(\lambda) = \int_{z \in \mathbb{T}} r^2 \frac{(1-|\lambda|^2)}{|z - r\lambda|^2} \, d\mu(z) \qquad |\lambda| \leq 1$

$$\varphi_\alpha(\lambda) = \frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda}, \text{ so } 1 - |\varphi_\alpha(\lambda)|^2 = \frac{(1-|\alpha|^2)(1-|\lambda|^2)}{|1-\bar{\alpha}\lambda|^2}$$

Hence

$$V_r(\lambda) = \frac{r^2}{1-r^2} \int (1 - |\varphi_{rz}(\lambda)|^2) \, d\mu(z)$$

is superharmonic in $\mathbb{D}$, $\Delta V_r(\lambda) \geq 0$, $|\lambda| < 1$

and $V_r(\lambda) = 0$, $|\lambda| = 1$, $V_r \in C^{\infty}(\overline{\mathbb{D}})$

~~$\Leftrightarrow V_r(\lambda) = 2\int \log\left|\frac{1-\bar{\lambda}w}{w-\lambda}\right| (\Delta V_r)(w) \frac{dA(w)}{\pi}$~~

So by Aleman's theorem (or calculate explicitly

$$\|f\|^2_{H^2} + \int |f'(z)|^2 V_r(z) \frac{dA}{\pi} = \|f\|^2_{D(\mu_r)}, \quad \mu_r(\mathbb{T}) = 0$$

and $V_r(\lambda) \nearrow V(\lambda) = \displaystyle\int_{|z|=1} \frac{1-|\lambda|^2}{|1-\lambda\bar{z}|^2} d\mu(z)$

So $\|f\|^2_{D(\mu_r)} \nearrow \|f\|^2_{D(\mu)}$ as $r \to 1$

So $D(\mu) \subseteq D(\mu_r)$ $\quad \forall r < 1$

and one has another theorem of Aronszajn