Lecture 1.

Euler's equations

First well posed mathematical form of modern fluid mechanics was formulated by Euler 1755.

→ Eulerian formulation → „bulk view"

→ Lagrangian formulation → „particle view".

→ when you have fluid quantities (pressure, temp, density) defined at given points in space x, at a given time t then we speak of Eulerian description

→ when the fluid quantities are defined as associated to a moving particle of fluid, followed along its trajectory

— " — " —

Continuum hypothesis: It means that any small volume element in the fluid is always supposed so large that it contains a very large number of particles.

„fluid particle"
„or point in a fluid" } are to be understood in this sense.

- pressure, temperature, density are assumed well defined and they will vary smoothly from one point to another
- Attached is „macroscopic view".

- dimensions (space) $x \in \mathbb{R}^d$, $d = 2, 3, \ldots$

  $x = (x, y, z) \in \mathbb{R}^3$
- $t \in \mathbb{R}$
- ideal fluids = each particle is going to push its neibers equaly in each direction.

$$\begin{cases} \text{incompressibility (density is constant)} \\ \text{irrotational} \to u = \text{fluid velocity} \\ \qquad\qquad \text{curl } u = 0 \\ \text{inviscid} \\ \qquad \to \text{no viscosity (no internal friction)} \end{cases}$$

Derivation of equation underlying the dynamics of ideal fluid are based on 3 principles

i) Conservation of matter. → (continuity equation)

ii) Newton's second law (or balance of momentum) → equs of motions (Euler)

iii) Conservation of energy. → eqs of state.

Euler's equation.

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0. \\ \nabla \cdot u = 0 \text{ (incompressibility conditions)} \end{cases}$$

$\nabla \times u = 0$ (irrotational) comes as a consequence of conservation of mass

$u(t, x)$ = velocity of the fluid.

$u(t, x)$ is a vector valued function : $\mathbb{R}^d \to \mathbb{R}^d$

$p(t, x)$ scalar function $\quad \mathbb{R}^d \to \mathbb{R}$.

→ $(d+1)$ equations.

$$\underline{u \cdot \nabla u} = \sum_{j=1}^{d} u_j \frac{\partial}{\partial x_i} u \quad (\text{notation})$$

$u = (u_1, u_2, \ldots, u_d)$

Material Derivative : $u_t + u \cdot \nabla u$

D_t $\quad \boxed{\partial_t + u \cdot \nabla = D_t}$

Observations

1. Nonlinear eqs. → the advection of the velocity field by itself
   $(u \cdot \nabla u)$ is your nonlinear term. In 3-d → we do not know if sols exists for all time.

2. Symmetries
   - $t \to -t$, $u \to -u$ <u>time reversal</u> (solve both forward and backward in time)
   - scaling law $\boxed{\underline{\tilde{u}(t,x)} = \frac{\tau}{\lambda} u(\tau t, \lambda x)}$, λ, τ positive constant

     ↓

     leads (usually) to "critical Sobolev exponent"

     $\underline{\underline{H}}^{s_c}$ critical Sobolev space. (L^2 based) do perform local-well posedness <u>LWP</u>

     $\tau = 1$ $\quad \tilde{u}(x,t) = \frac{1}{\lambda} u(t, \lambda x)$ → a particular scaling law.

     → $H^{\frac{d}{2}}$, $\frac{d}{2} = s_c$ Not quite

↳ under this scaling ↗ $\nabla \tilde{u} \approx \nabla u \rightarrow D_t \nabla u \simeq |\nabla u|^2$

$\|\nabla u\|_{L^\infty} \rightarrow \underline{\underline{H^{d/2+1}}}$ critical Sobolev space in terms of CWP theory.

- invariance under translations in space and time,

$x \to x + a$
$t \to t \to ta$

and invariance under rotations $x \to Rx$ $R^T R = 1$.

- Galilean transform (on Chane an intrinsic meaning)
- $t \to -\frac{1}{t}$, $x \to \frac{x}{t}$ + time translations + scaling

invariance under fractional linear transfs ↗ $\left.\begin{array}{l} t \to \dfrac{at+b}{ct+d} \\ x \to \dfrac{x}{ct+d} \end{array}\right.$ $ad - bc = 1$. a, b, c, d constants

! used in turbulence theory.

Incompressible flows → Incompressible + irrotational flows → water waves

$\boxed{2D}$ from now on spatial dimensions is two

$\mathbb{R}^2$. $u_x + v_y = 0$ $u = (u_1, u_2)$ - velocity vector field.

$\vec{u} = (u, v)$

$\boxed{\nabla \cdot u = 0}$ $u(t, x) = u(t, x, y)$

incompressibility conds

Two scalar functions that play a role in the analysis of water waves:

(A) → stream function $\psi(t,x,y) \to \mathbb{R}$. (exists regardless if irrot or not)

(B) → velocity potential $\varphi(t,x,t) \to \mathbb{R}$ (exists only irrotation flows)

(A) $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$

$$\underline{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) \quad \to \nabla \cdot u = 0$$

The stream function, at any instant of time, we have that the velocity of the fluid. u is tangent to the streamlines $\psi = \text{constant}$.

(B) $u = \dfrac{\partial \varphi}{\partial x}$, $v = \dfrac{\partial \varphi}{\partial y}$ $\Rightarrow$ $\Delta \varphi = 0$ in $\mathbb{R}^2$ (harmonic in $\mathbb{R}^2$)

It follow φ, ψ satisfy the Cauchy-Riemann eq.

Set, z as $z = x + iy$. $F : \mathbb{C} \to \mathbb{C}$.

$$\underline{F(z,t) := \varphi(x,y) + i\psi(x,y)}$$

used in the study of two dimension irrotational flows.

$\Omega(t) \subset \mathbb{R}^2$. Euler in a domain that is not the whole $\mathbb{R}^2$.

Next we will discuss

tomorrow we continue with this case.